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FREE PRODUCTS OF TOPOLOGICAL GROUPS WITH EQUAL UNIFORMITIES, I

BY

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1. Introduction. It is widely known that free products and free products with amalgamation (coproducts and push-outs) exist in the category of topological groups. It is natural to ask if the underlying abstract group of such a product is the corresponding product of abstract groups, and it is not difficult to see that this is the case (Theorems 1 and 2). Another logical question is the following:

(Q) If F is the free product of Hausdorff topological groups G_a, $a \in A$ (perhaps with an amalgamated subgroup H), is F necessarily Hausdorff?

In [3], Graev settles (Q) affirmatively for free products. He employs a rather long and delicate argument. An independent proof is given by Hulanicki [4] for free products of compact Hausdorff groups. More recently Morris [8] announced a simpler proof for all free products. His proof used pseudometrics to construct a topology τ_1 in the algebraic free product; unfortunately, the topology he constructs makes the group operation discontinuous. In this paper, using a technique similar in principle to that of Morris (if somewhat more complex) and simpler than Graev's technique, we establish that a free product of Hausdorff groups is Hausdorff provided that the original groups are *locally invariant*, i.e. every neighborhood of the identity contains a neighborhood invariant under inner automorphisms. This is equivalent to requiring a group to have equal right and left uniformities, or to requiring each group to have a topology determined by the collection of continuous (two-sided) invariant pseudometrics on the group (see [5] and [6]). This class of groups includes all Hausdorff groups which are compact, abelian, or have an invariant metric.

We also establish an affirmative answer to (Q) for a free product of topological groups G_a with an amalgamated subgroup H, provided that each G_a is locally invariant, H is closed in each G_a and that, for every pair $\alpha, \beta \in A$, there is a continuous homomorphism $m_\beta^\alpha\colon G_\beta \to G_\alpha$ which is a homeomorphic isomorphism of $H \subset G_\beta$ on $H \subset G_\alpha$. The last requirement is met, for instance, if all G_α are equivalent (by homeomorphic isomorphisms preserving H) or if H is a retract of each G_α. We do not know if any of these conditions are necessary. (**P 901**)

The difficulty in the proof of Morris can be clarified by realizing that if a topology on a free product is to make the group operation continuous, every neighborhood of the identity must be "big" in the following sense:

PROPOSITION 1. *Suppose that F is a topological group which is not discrete, F is algebraically the free product of certain subgroups G_α, $\alpha \in A$ (A not a singleton), and that $g_1 g_2 \dots g_n$ is the reduced form of some $g \in F$. Then every neighborhood N of the identity e of F contains an element whose reduced form is $g_1 \dots g_n g_0 g_n^{-1} \dots g_1^{-1}$.*

Proof. Since $geg^{-1} = e$ and the group operation is continuous, there is a neighborhood N_0 of e such that $gN_0g^{-1} \subset N$. Let $g_n \in G_\alpha$, and let β be distinct from α. Then $N_0 \cap (G_\beta \setminus \{e\})$ is non-empty and the conclusion follows by selecting g_0 from it.

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2. Results. The following theorem is special case of a widely-known construction, another special case of which occurs, e. g., in [1].

THEOREM 1. *Let G_α be a topological group for each α in an index set A. Let H be a topological group, and, for each $\alpha \in A$, suppose $h_\alpha: H \to G_\alpha$ is a homeomorphic isomorphism onto a subgroup of G_α. Then there are a topological group F and maps $i_\alpha: G_\alpha \to F$ such that*

(1.1) *Each $i_\alpha: G_\alpha \to F$ is a continuous isomorphism onto $i_\alpha(G_\alpha)$.*

(1.2) *As an abstract group, F is isomorphic to $*(G_\alpha: H)$, the free product of the abstract groups G_α with H amalgamated. In particular, $i_\alpha h_\alpha = i_\beta h_\beta$ for all $\alpha, \beta \in A$.*

(1.3) *For every topological group K and every system of continuous homomorphisms $k_\alpha: G_\alpha \to K$ such that $k_\alpha h_\alpha = k_\beta h_\beta$ for all $\alpha, \beta \in A$, there is a unique continuous homomorphism $k: F \to K$ such that $ki_\alpha = k_\alpha$ for each α.*

Proof. Let $\{K_\gamma;\ \gamma \in \Gamma\}$ be the set of all distinct (up to homeomorphic isomorphism) topological groups of cardinality not exceeding that of the disjoint union of the G_α (include groups of countable cardinality, if the G_α are finite). For each γ, let $\{\{k^\alpha_{\gamma\delta};\ \delta \in \Delta_\gamma\}$ be the collection of all systems of continuous homomorphisms $k^\alpha_{\gamma\delta}: G_\alpha \to K_{\gamma\delta}$, where $\alpha \in A$ and $K_{\gamma\delta} = K_\gamma$, for which $k^\alpha_{\gamma\delta} h_\alpha = k^\beta_{\gamma\delta} h_\beta$ for all $\alpha, \beta \in A$. Let P denote the direct product of the $K_{\gamma\delta}$ for all γ, δ. Then P is a topological group. We now map each G_α into P by the map

$$i_\alpha(g) = \{k^\alpha_{\gamma\delta}(g);\ \gamma \in \Gamma,\ \delta \in \Delta_\gamma\} \quad \text{for } \alpha \in A,\ g \in G_\alpha.$$

It can be readily checked that each i_α: $G_\alpha \to P$ is a continuous homomorphism, and, clearly, $i_\beta h_\beta = i_\alpha h_\alpha$. Now let F denote the subgroup of P generated by $\bigcup_\alpha i_\alpha(G_\alpha)$ with the relative topology.

We now establish the existence of the map k of (1.3). Let K and $\{k_\alpha\}$ be as in (1.3). Without loss of generality, we may suppose K is generated by $\bigcup_\alpha k_\alpha(G_\alpha)$. Thus, for some γ and δ, we have $K = K_{\gamma\delta}$ and $k_\alpha = k^\alpha_{\gamma\delta}$, and the desired map k: $F \to K$ is just the projection of $F \subset P$ on the γ, δ-coordinate.

We next complete (1.1) and (1.2). Let K be $\ast(G_\alpha: H)$ as an abstract group, and give K the indiscrete topology. Now each natural injection k_α: $G_\alpha \to K$ is a continuous homomorphism, so k: $F \to K$ is a continuous homomorphism. Since $ki_\alpha = k_\alpha$, i_α is an isomorphism of G_α on $i_\alpha(G_\alpha)$. F as an abstract group is generated by the $i_\alpha(G_\alpha)$ and maps onto $\ast(G_\alpha: H)$; hence k is an isomorphism of F with $\ast(G_\alpha: H)$ as abstract groups.

Finally, in view of (1.2), the map k of (1.3) is, clearly, unique. This completes the proof of Theorem 1.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if* F_0 *and* F_1 *satisfy* (1.1), (1.2) *and* (1.3), *then* F_0 *and* F_1 *are homeomorphically isomorphic by a map preserving the* G_α.

Proof. Suppose given maps i^r_α: $G_\alpha \to F_r$, $r = 0, 1$. Applying (1.3) twice, we get maps i^r: $F_{1-r} \to F_r$, each a continuous homomorphism. By (1.2), we see these maps are inverses, completing the proof.

THEOREM 3. *Given the hypotheses of Theorem* 1, *suppose also that, for each* $\alpha, \beta \in A$, *there is a continuous homomorphism* m^α_β: $G_\beta \to G_\alpha$ *such that* m^α_β *restricted to* $h_\beta(H)$ *is* $h_\alpha h_\beta^{-1}$. *Then*

(1.1′) *Each* i_α: $G_\alpha \to F$ *is a homeomorphic isomorphism onto* $i_\alpha(G_\alpha)$.

Proof. Fix $\alpha \in A$. Let $K = G_\alpha$ and let $k_\beta = m^\alpha_\beta$: $G_\beta \to G_\alpha$ for $\beta \neq \alpha$, letting k_α be the identity map on G_α. Applying (1.3) we get a map k: $F \to G_\alpha$ such that, in particular, ki_α: $G_\alpha \to G_\alpha$ is the identity map. Hence i_α has an inverse, and is a homeomorphism.

Remark. No additional hypothesis is thus needed for (1.1′) in the case of a free product. For the amalgamated case, it would be of interest to know if (1.1′) must hold without the added hypothesis of Theorem 3.

We now proceed to state our main theorem.

THEOREM 4. *Given the hypotheses of Theorems* 1 *and* 3, *suppose also that, for each* $\alpha \in A$,

(4.1) G_α *is Hausdorff*;

(4.2) $h_\alpha(H)$ *is a closed subgroup of* G_α; *and*

(4.3) G_α *is locally invariant, i.e. has a topology determined by invariant pseudometrics.*

Then F *is Hausdorff.*

The proof of Theorem 4 occupies the next section. We observe that in the free product case, $H = \{e\}$ and (4.1) implies (4.2). Hence, in this case, our theorem is distinguished from that of Morris only by (4.3). Our proof is close enough to his in concept that the other theorems in [8] can be established as they are there, with the added requirement of (4.3) for those depending on the Hausdorff property.

3. Proof of Theorem 4. Let $\bar{F}$ denote the abstract group $*(G_a\colon H)$ underlying F. We will confuse $h \in H$, $g \in G_a$, with $h_a(h) \in G_a$ and $i_a(g) \in \bar{F}$, as convenient. We shall introduce a topology τ on $\bar{F}$ which will make $\bar{F}$ a Hausdorff topological group and which will induce the original topology on each $G_a \subset \bar{F}$. Then, applying (1.3) (using $\bar{F}$ with topology τ as K), there is a continuous isomorphism $i\colon F \to \bar{F}$; that is, the topology of F is finer than τ and is, therefore, Hausdorff.

Like Morris, we employ pseudometrics to introduce τ. Let $\{\varrho_a\}_{a \in A}$ be a collection of invariant continuous pseudometrics $\varrho_a\colon G_a \times G_a \to R$ such that if $h \in H$, then $\varrho_a(h, e) = \varrho_\beta(h, e)$ for all $a, \beta \in A$. We describe a pseudometric $\varrho\colon \bar{F} \times \bar{F} \to R$ derived from the ϱ_a.

For $g \in \bar{F}$, let $\boldsymbol{G}$ denote some word $g_1 g_2 \ldots g_n$ such that $g_i \in G_{a_i}$ and $g = g_1 g_2 \ldots g_n$ in $\bar{F}$. Given $\boldsymbol{G}$, let $\boldsymbol{H}$ denote a word $e_1 e_2 \ldots e_n$ such that $e_1 e_2 \ldots e_n = e$, the identity of F, and e_i lies in the same G_a as g_i does (e_i is not necessarily the identity of G_{a_i}).

Now let

$$f(\boldsymbol{G}, \boldsymbol{H}) = \sum_i \varrho_{a_i}(g_i, e_i) \quad \text{and} \quad f(g, e) = \inf f(\boldsymbol{G}, \boldsymbol{H}),$$

where the infimum is taken over all appropriate pairs of words $\boldsymbol{G}, \boldsymbol{H}$. Finally, let

$$\varrho(g, h) = f(gh^{-1}, e).$$

LEMMA 0. *In computing $f(g, e)$, we need only consider pairs $\boldsymbol{G}, \boldsymbol{H}$ for which $g_i \in H$ implies $e_i \in H$.*

Proof. Suppose $\boldsymbol{G} = g_1 \ldots g_n$ and $\boldsymbol{H} = e_1 \ldots e_n$ have some $g_j \in H$ and $e_j \in G_{a_j} \setminus H$. Consideration of the word problem for amalgamated products [7] shows that there is some $r \neq j$ (for concreteness, we take $r < j$) such that $e_r \in G_{a_j} \setminus H$ and $e_{r+1} \ldots e_{j-1} = h \in H$; otherwise, $e_1 \ldots e_n \neq e$. Now alter $\boldsymbol{H}$ by replacing e_j with $e_j^* = g_j \in H$ and replacing e_r with $e_r^* = e_r h e_j g_j^{-1} h^{-1} \in G_{a_j}$. We still have

$$\boldsymbol{H} = e_1 \ldots e_r^* e_{r+1} \ldots e_{j-1} e_j^* \ldots e_n = e_1 \ldots e_{r-1} (e_r h e_j g_j^{-1} h^{-1}) h g_j e_{j+1} \ldots e_n$$

$$= e_1 \ldots e_r h e_j \ldots e_n = e_1 \ldots e_n = e$$

and $f(\boldsymbol{G}, \boldsymbol{H})$ has not increased since, because ϱ_{a_j} is invariant,

$$\varrho_{a_j}(g_r, e_r^*) + \varrho_{a_j}(g_j, e_j^*) = \varrho_{a_j}(g_r, e_r h e_j g_j^{-1} h^{-1})$$
$$\leqslant \varrho_{a_j}(g_r, e_r) + \varrho_{a_j}(e, h e_j g_j^{-1} h^{-1}) = \varrho_{a_j}(g_r, e_r) + \varrho_{a_j}(g_j, e_j)$$

which completes the proof.

LEMMA 1. *$f(g, e) = \inf f(\boldsymbol{G}, \boldsymbol{H})$, where the infimum need be taken only over pairs $(\boldsymbol{G}, \boldsymbol{H})$ for which $\boldsymbol{G}$ is reduced in that either $n = 1$ or else each $g_i \in G_{a_i} \setminus H$ and $a_i \neq a_{i+1}$.*

Proof. If $\boldsymbol{G}$ is not reduced, we can find a pair $(\boldsymbol{G}^*, \boldsymbol{H}^*)$ of shorter words with $f(\boldsymbol{G}^*, \boldsymbol{H}^*) \leqslant f(\boldsymbol{G}, \boldsymbol{H})$. For suppose $g_i, g_{i+1} \in G_{a_i}$ (note: if $g_i \in H$, this is always true), and $g_i g_{i+1} = g_i^* \in G_{a_i}$. Then let $e_i e_{i+1} = e_i^* \in G_{a_i}$. Let

$$\boldsymbol{G}^* = g_1 \dots g_i^* g_{i+2} \dots g_n \quad \text{and} \quad \boldsymbol{H}^* = e_1 \dots e_i^* e_{i+2} \dots e_n.$$

By the invariance of ϱ_{a_i},

$$\varrho_{a_i}(g_i g_{i+1}, e_i e_{i+1}) \leqslant \varrho_{a_i}(g_i, e_i) + \varrho_{a_i}(g_{i+1}, e_{i+1})$$

and the lemma follows.

Hence, in view of the standard results on free products with an amalgamated subgroup [7], we need consider only $\boldsymbol{G}$ of some fixed length n (depending on g) to determine $\varrho(g, e)$.

LEMMA 2. *ϱ is an invariant pseudometric.*

Proof. It is immediate that $\varrho(g, g) = 0$ and $\varrho(gk, hk) = \varrho(g, h)$. Letting $\boldsymbol{K}$ represent k, we see that $\varrho(kgk^{-1}, e)$ is bounded from above by $f(\boldsymbol{K}\boldsymbol{G}\boldsymbol{K}^{-1}, \boldsymbol{K}\boldsymbol{H}\boldsymbol{K}^{-1}) = f(\boldsymbol{G}, \boldsymbol{H})$ and thus

$$\varrho(kgk^{-1}, e) \leqslant \varrho(g, e).$$

Similarly,

$$\varrho(k^{-1}kgk^{-1}k, e) \leqslant \varrho(kgk^{-1}, e)$$

and we see that

$$\varrho(kg, kh) = \varrho\big(k(gh^{-1})k^{-1}, e\big) = \varrho(gh^{-1}, e) = \varrho(g, h).$$

We note that $\varrho(g, e) = \varrho(g^{-1}, e)$, since $f(\boldsymbol{G}, \boldsymbol{H})$ approximates $\varrho(g, e)$ if and only if $f(\boldsymbol{G}^{-1}, \boldsymbol{H}^{-1})$ approximates $\varrho(g^{-1}, e)$. Hence

$$\varrho(g, h) = \varrho(gh^{-1}, e) = \varrho(hg^{-1}, e) = \varrho(h, g).$$

Finally, the triangle inequality holds, since if $f(\boldsymbol{G}_1, \boldsymbol{H}_1)$ approximates $\varrho(g_1, e)$ and $f(\boldsymbol{G}_2, \boldsymbol{H}_2)$ approximates $\varrho(g_2, e)$, then $f(\boldsymbol{G}_1\boldsymbol{G}_2, \boldsymbol{H}_1\boldsymbol{H}_2)$ is an upper bound for $\varrho(g_1 g_2, e)$; hence

$$\varrho(g_1, e) + \varrho(g_2, e) \geqslant \varrho(g_1 g_2, e),$$

and

$$\varrho(g, h) + \varrho(h, k) = \varrho(gh^{-1}, e) + \varrho(hk^{-1}, e) \geqslant \varrho(gh^{-1}hk^{-1}, e) = \varrho(g, k).$$

Now let τ be the topology induced on $\bar{F}$ by the collection of all ϱ constructed in this manner [5].

LEMMA 3. *$\bar{F}$ with topology τ is a topological group.*

Proof. If we recall that sets $N_\varrho(g_0, \varepsilon) = \{g \in \bar{F} \mid \varrho(g, g_0) < \varepsilon\}$ form a subbase, then $g \mapsto g^{-1}$ is readily continuous, since $\varrho(g_1, g_2) = \varrho(g_1^{-1}, g_2^{-1})$, and the group operation is jointly continuous, since $\varrho(a_1 a_2, b_1 b_2) \leqslant \varrho(a_1, b_1) + \varrho(a_2, b_2)$ for each ϱ.

LEMMA 4. *Let ϱ_a be any invariant pseudometric on G_a. Then there is a ϱ such that $\varrho \mid G_a = \varrho_a$, and thus the topology induced on G_a by τ is the original topology of G_a.*

Proof. Fix a and ϱ_a. For each $\beta \in A$, let $m_\beta^a \colon G_\beta \to G_a$ be the map of Theorem 3 (m_a^a is the identity map). Write $\varrho_\beta(g_1, g_2) = \varrho_a\big(m_\beta^a(g_1), m_\beta^a(g_2)\big)$. It is easy to check that ϱ_β is a continuous invariant pseudometric on G_β and $\varrho_\beta(h, e) = \varrho_a(h, e)$ for all $h \in H$ and $\beta \in A$. Let ϱ be the pseudometric on $\bar{F}$ derived from $\{\varrho_\beta\}_{\beta \in A}$. If $g \in G_a$, clearly $\varrho(g, e) = \varrho_a(g, e)$ as desired, completing the proof of the lemma.

LEMMA 5. *In the topology τ, $\bar{F}$ is Hausdorff.*

Proof. It will suffice, given any $g \in \bar{F}$, $g \neq e$, to find a pseudometric ϱ such that $\varrho(g, e) \neq 0$. If $g \in G_a$, choose ϱ_a such that $\varrho_a(g, e) \neq 0$ (G_a is Hausdorff) and construct ϱ as in Lemma 4. Otherwise, let $G = g_1 g_2 \dots g_n$ be a word of minimal length representing g. Now [7] the only elements of the G_a that occur in any reduced word representing g are $h_1 g_i h_2$, where $1 \leqslant i \leqslant n$ and $h_1, h_2 \in H$.

We observe next that if $g \in G_a \setminus H$, there is a continuous invariant pseudometric on G_a with $\varrho(g, h) = 1$ for all $h \in H$. For $e \notin g^{-1}H$ (a closed set since H is closed) and using the customary Urysohn-motivated argument, with invariant neighborhoods of e, we can construct a continuous invariant pseudometric $\varrho_a \colon G_a \times G_a \to [0, 1]$ such that $\varrho_a(e, g^{-1}h) = 1$ for all $g^{-1}h \in g^{-1}H$. Then $\varrho_a(g, h) = 1$ also.

For each g_i, construct such a ϱ_{a_i}. Note

$$\inf\{\varrho_{a_i}(h_1 g_i h_2, h_3) \mid h_1, h_2, h_3 \in H\} = \inf\{\varrho_{a_i}(g_i, h_1^{-1} h_3 h_2^{-1})\} = 1.$$

Now, on each G_a we introduce n pseudometrics; for $a = a_i$, the pseudometric ϱ_{a_i} constructed as above, and for $a \neq a_i$, the pseudometric induced on G_a from the pseudometric ϱ_{a_i} on G_{a_i} by the method of Lemma 4. Call these n pseudometrics $\varrho_{a(i)}$, $1 \leqslant i \leqslant n$, for each G_a. Let

$$\hat{\varrho}_a(g_1, g_2) = \max\{\varrho_{a(i)}(g_1, g_2) \mid 1 \leqslant i \leqslant n\}$$

and let ϱ be constructed from $\{\hat{\varrho}_a\}_{a \in A}$.

Finally, look at any $f(G^*, H)$. By Lemma 1, we can suppose $g_1^* g_2^* \dots g_n^*$ reduced; $e = e_1 e_2 \dots e_n$ and $e_i \in G_{a_i}$, $a_i \neq a_{i+1}$; so, by [7], some $e_i \in H$. Thus

$$f(G^*, H) = \sum_i \hat{\varrho}_{a_i}(g_i^*, e_i) \geqslant \inf\{\varrho_{a_i}(g_i^*, h) \mid 1 \leqslant i \leqslant n;\ h \in H\}$$
$$\geqslant \inf\{\varrho_{a_i}(h_1 g_i h_2, h_3 \mid 1 \leqslant i \leqslant n;\ h_1, h_2, h_3 \in H\} = 1.$$

Hence $\varrho(g, e) = \inf f(G^*, H) \geqslant 1$ as desired. This completes our proof of Theorem 4.

REFERENCES

[1] K. Golema, *Free products of compact general algebras*, Colloquium Mathematicum 13 (1965), p. 165-166.

[2] М. И. Граев, *Свободные топологические группы*, Известия Академии наук СССР, серия математическая, 12 (1948), p. 279-324; English translation: American Mathematical Society Translations (1) 8 (1962), p. 305-364.

[3] — *О свободных продуктах топологических групп*, ibidem 14 (1950), p. 343-354.

[4] A. Hulanicki, *Isomorphic embeddings of free products of compact groups*, Colloquium Mathematicum 16 (1967), p. 235-241.

[5] J. L. Kelley, *General topology*, New York 1955.

[6] V. L. Klee, *Invariant metrics in groups (solution of a problem of Banach)*, Proceedings of the American Mathematical Society 3 (1953), p. 484-487.

[7] A. G. Kurosh, *The theory of groups*, Vol. 2 (2-nd English edition), New York 1960.

[8] S. A. Morris, *Free products of topological groups*, Bulletin of the Australian Mathematical Society 4 (1971), p. 17-29.

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