FREE PRODUCTS OF TOPOLOGICAL GROUPS
WITH EQUAL UNIFORMITIES, I

BY

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1. Introduction. It is widely known that free products and free products with amalgamation (coproducts and push-outs) exist in the category of topological groups. It is natural to ask if the underlying abstract group of such a product is the corresponding product of abstract groups, and it is not difficult to see that this is the case (Theorems 1 and 2). Another logical question is the following:

(Q) If $F$ is the free product of Hausdorff topological groups $G_\alpha$, $\alpha \in \mathcal{A}$ (perhaps with an amalgamated subgroup $H$), is $F$ necessarily Hausdorff?

In [3], Graev settles (Q) affirmatively for free products. He employs a rather long and delicate argument. An independent proof is given by Hulanicki [4] for free products of compact Hausdorff groups. More recently Morris [8] announced a simpler proof for all free products. His proof used pseudometrics to construct a topology $\tau_1$ in the algebraic free product; unfortunately, the topology he constructs makes the group operation discontinuous. In this paper, using a technique similar in principle to that of Morris (if somewhat more complex) and simpler than Graev’s technique, we establish that a free product of Hausdorff groups is Hausdorff provided that the original groups are locally invariant, i.e. every neighborhood of the identity contains a neighborhood invariant under inner automorphisms. This is equivalent to requiring a group to have equal right and left uniformities, or to requiring each group to have a topology determined by the collection of continuous (two-sided) invariant pseudometrics on the group (see [5] and [6]). This class of groups includes all Hausdorff groups which are compact, abelian, or have an invariant metric.

We also establish an affirmative answer to (Q) for a free product of topological groups $G_\alpha$ with an amalgamated subgroup $H$, provided that each $G_\alpha$ is locally invariant, $H$ is closed in each $G_\alpha$ and that, for every pair $\alpha, \beta \in \mathcal{A}$, there is a continuous homomorphism $m_\beta^\alpha: G_\beta \to G_\alpha$ which is
a homeomorphic isomorphism of $H \cong G_\beta$ on $H \cong G_\alpha$. The last requirement is met, for instance, if all $G_\alpha$ are equivalent (by homeomorphic isomorphisms preserving $H$) or if $H$ is a retract of each $G_\alpha$. We do not know if any of these conditions are necessary. (P 901)

The difficulty in the proof of Morris can be clarified by realizing that if a topology on a free product is to make the group operation continuous, every neighborhood of the identity must be "big" in the following sense:

**Proposition 1.** Suppose that $F$ is a topological group which is not discrete, $F$ is algebraically the free product of certain subgroups $G_\alpha$, $\alpha \in A$ ($A$ not a singleton), and that $g_1 g_2 \ldots g_n$ is the reduced form of some $g \in F$. Then every neighborhood $N$ of the identity $e$ of $F$ contains an element whose reduced form is $g_1 \ldots g_n g \in G_\alpha g^{-1} \ldots g^{-1}$.

**Proof.** Since $g e g^{-1} = e$ and the group operation is continuous, there is a neighborhood $N_\alpha$ of $e$ such that $g N_\alpha g^{-1} \subset N$. Let $g_\alpha \in G_\alpha$, and let $\beta$ be distinct from $\alpha$. Then $N_\alpha \cap (G_\beta \setminus \{e\})$ is non-empty and the conclusion follows by selecting $g_\alpha$ from it.

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**2. Results.** The following theorem is special case of a widely-known construction, another special case of which occurs, e.g., in [1].

**Theorem 1.** Let $G_\alpha$ be a topological group for each $\alpha$ in an index set $A$. Let $H$ be a topological group, and, for each $\alpha \in A$, suppose $h_\alpha : H \to G_\alpha$ is a homomorphic isomorphism onto a subgroup of $G_\alpha$. Then there are a topological group $F$ and maps $i_\alpha : G_\alpha \to F$ such that

1. Each $i_\alpha : G_\alpha \to F$ is a continuous isomorphism onto $i_\alpha(G_\alpha)$.

2. As an abstract group, $F$ is isomorphic to $\times (G_\alpha; H)$, the free product of the abstract groups $G_\alpha$ with $H$ amalgamated. In particular, $i_\alpha h_\alpha = i_\beta h_\beta$ for all $\alpha, \beta \in A$.

3. For every topological group $K$ and every system of continuous homomorphisms $k_\alpha : G_\alpha \to K$ such that $k_\alpha h_\alpha = k_\beta h_\beta$ for all $\alpha, \beta \in A$, there is a unique continuous homomorphism $k : F \to K$ such that $k i_\alpha = k_\alpha$ for each $\alpha$.

**Proof.** Let $\{K_\gamma; \gamma \in \Gamma\}$ be the set of all distinct (up to homeomorphic isomorphism) topological groups of cardinality not exceeding that of the disjoint union of the $G_\alpha$ (include groups of countable cardinality, if the $G_\alpha$ are finite). For each $\gamma$, let $\{k_\delta^\alpha; \delta \in \Delta_\gamma\}$ be the collection of all systems of continuous homomorphisms $k_\delta^\alpha : G_\alpha \to K_\delta$, where $\alpha \in A$ and $K_\delta = k_\gamma$, for which $k_\delta^\alpha h_\alpha = k_\delta^\beta h_\beta$ for all $\alpha, \beta \in A$. Let $P$ denote the direct product of the $K_\gamma$ for all $\gamma, \delta$. Then $P$ is a topological group. We now map each $G_\alpha$ into $P$ by the map

$$i_\alpha(g) = \{k_\delta^\alpha(g); \gamma \in \Gamma, \delta \in \Delta_\gamma\} \quad \text{for} \quad \alpha \in A, \ \gamma \in \Gamma.$$
It can be readily checked that each $i_a: G_a \to P$ is a continuous homomorphism, and, clearly, $i_\beta h_\beta = i_a h_a$. Now let $F$ denote the subgroup of $P$ generated by $\bigcup \{i_a(G_a)\}$ with the relative topology.

We now establish the existence of the map $k$ of (1.3). Let $K$ and \{\(k_a\)} be as in (1.3). Without loss of generality, we may suppose $K$ is generated by $\bigcup k_a(G_a)$. Thus, for some $\gamma$ and $\delta$, we have $K = K_{\gamma, \delta}$ and $k_a = k_{\gamma, \delta}^a$, and the desired map $k: F \to K$ is just the projection of $F < P$ on the $\gamma, \delta$-coordinate.

We next complete (1.1) and (1.2). Let $K$ be $\ast (G_a; H)$ as an abstract group, and give $K$ the indiscrete topology. Now each natural injection $k_a: G_a \to K$ is a continuous homomorphism, so $k: F \to K$ is a continuous homomorphism. Since $ki_a = k_a$, $i_a$ is an isomorphism of $G_a$ on $i_a(G_a)$. $F$ as an abstract group is generated by the $i_a(G_a)$ and maps onto $\ast (G_a; H)$; hence $k$ is an isomorphism of $F$ with $\ast (G_a; H)$ as abstract groups.

Finally, in view of (1.2), the map $k$ of (1.3) is, clearly, unique. This completes the proof of Theorem 1.

**Theorem 2.** Under the hypotheses of Theorem 1, if $F_0$ and $F_1$ satisfy (1.1), (1.2) and (1.3), then $F_0$ and $F_1$ are homeomorphically isomorphic by a map preserving the $G_a$.

**Proof.** Suppose given maps $i_a^r: G_a \to F_r$, $r = 0, 1$. Applying (1.3) twice, we get maps $i^r: F_{1-r} \to F_r$, each a continuous homomorphism. By (1.2), we see these maps are inverses, completing the proof.

**Theorem 3.** Given the hypotheses of Theorem 1, suppose also that, for each $\alpha, \beta \in A$, there is a continuous homomorphism $m_\beta^\alpha: G_\beta \to G_\alpha$ such that $m_\beta^\alpha$ restricted to $h_\beta(H)$ is $h_\alpha h_\beta^{-1}$. Then

\[(1.1') \text{ Each } i_a: G_a \to F \text{ is a homeomorphic isomorphism onto } i_a(G_a).\]

**Proof.** Fix $\alpha \in A$. Let $K = G_a$ and let $k_\beta = m_\beta^\alpha: G_\beta \to G_\alpha$ for $\beta \neq \alpha$, letting $k_\alpha$ be the identity map on $G_\alpha$. Applying (1.3) we get a map $k: F \to G_a$ such that, in particular, $ki_a: G_a \to G_a$ is the identity map. Hence $i_a$ has an inverse, and is a homeomorphism.

**Remark.** No additional hypothesis is thus needed for (1.1') in the case of a free product. For the amalgamated case, it would be of interest to know if (1.1') must hold without the added hypothesis of Theorem 3.

We now proceed to state our main theorem.

**Theorem 4.** Given the hypotheses of Theorems 1 and 3, suppose also that, for each $\alpha \in A$,

\[(4.1) \text{ $G_\alpha$ is Hausdorff;}\]
\[(4.2) h_\alpha(H) \text{ is a closed subgroup of $G_\alpha$;} \text{ and}\]
\[(4.3) G_\alpha \text{ is locally invariant, i.e. has a topology determined by invariant pseudometrics.}\]

Then $F$ is Hausdorff.
The proof of Theorem 4 occupies the next section. We observe that in the free product case, $H = \{e\}$ and (4.1) implies (4.2). Hence, in this case, our theorem is distinguished from that of Morris only by (4.3). Our proof is close enough to his in concept that the other theorems in [8] can be established as they are there, with the added requirement of (4.3) for those depending on the Hausdorff property.

3. Proof of Theorem 4. Let $\bar{F}$ denote the abstract group $\star (G_a: H)$ underlying $F$. We will confuse $h \in H$, $g \in G_a$, with $h(a) \in G_a$ and $i_a(g) \in \bar{F}$, as convenient. We shall introduce a topology $\tau$ on $\bar{F}$ which will make $\bar{F}$ a Hausdorff topological group and which will induce the original topology on each $G_a \subset \bar{F}$. Then, applying (1.3) (using $\bar{F}$ with topology $\tau$ as $K$), there is a continuous isomorphism $i: F \to \bar{F}$; that is, the topology of $F$ is finer than $\tau$ and is, therefore, Hausdorff.

Like Morris, we employ pseudometrics to introduce $\tau$. Let $\{q_{\alpha}\}_{\alpha \in A}$ be a collection of invariant continuous pseudometrics $q_{\alpha}: G_a \times G_a \to \mathbb{R}$ such that if $h \in H$, then $q_{\alpha}(h, e) = q_{\beta}(h, e)$ for all $\alpha, \beta \in A$. We describe a pseudometric $q: \bar{F} \times \bar{F} \to \mathbb{R}$ derived from the $q_{\alpha}$.

For $g \in \bar{F}$, let $G$ denote some word $g_1 g_2 \ldots g_n$ such that $g_i \in G_{a_i}$ and $g = g_1 g_2 \ldots g_n$ in $F$. Given $G$, let $H$ denote a word $e_1 e_2 \ldots e_n$ such that $e_1 e_2 \ldots e_n = e$, the identity of $F$, and $e_i$ lies in the same $G_a$ as $g_i$ does ($e_i$ is not necessarily the identity of $G_{a_i}$).

Now let

$$f(G, H) = \sum_{i} q_{a_i}(g_i, e_i) \quad \text{and} \quad f(g, e) = \inf f(G, H),$$

where the infimum is taken over all appropriate pairs of words $G, H$. Finally, let

$$q(g, h) = f(gh^{-1}, e).$$

Lemma 0. In computing $f(g, e)$, we need only consider pairs $G, H$ for which $g_i \in H$ implies $e_i \in H$.

Proof. Suppose $G = g_1 \ldots g_n$ and $H = e_1 \ldots e_n$ have some $g_j \in H$ and $e_j \in G_{a_j} \setminus H$. Consideration of the word problem for amalgamated products [7] shows that there is some $r \neq j$ (for concreteness, we take $r < j$) such that $e_r \in G_{a_j} \setminus H$ and $e_{r+1} \ldots e_{j-1} = h \in H$; otherwise, $e_1 \ldots e_n \neq e$. Now alter $H$ by replacing $e_j$ with $e_j^* = g_j \in H$ and replacing $e_r$ with $e_r^* = e_r h e_j g_j^{-1} h^{-1} \in G_{a_j}$. We still have

$$H = e_1 \ldots e^*_r e_{r+1} \ldots e_{j-1} e^*_j \ldots e_n = e_1 \ldots e_{r-1} (e_r h e_j g_j^{-1} h^{-1}) h g_j e_{j+1} \ldots e_n$$

$$= e_1 \ldots e_r h e_j \ldots e_n = e_1 \ldots e_n = e$$
and \( f(G, H) \) has not increased since, because \( \varrho_{aj} \) is invariant,

\[
\varrho_{aj}(g_r, e_r^*) + \varrho_{aj}(g_j, e_j^*) = \varrho_{aj}(g_r, e_r^* h e_j g_j^{-1} h^{-1}) \\
\leq \varrho_{aj}(g_r, e_r) + \varrho_{aj}(e, h e_j g_j^{-1} h^{-1}) = \varrho_{aj}(g_r, e_r) + \varrho_{aj}(g_j, e_j)
\]

which completes the proof.

**Lemma 1.** \( f(g, e) = \inf f(G, H) \), where the infimum need be taken only over pairs \((G, H)\) for which \( G \) is reduced in that either \( n = 1 \) or else each \( g_i \in G_{ai} \setminus H \) and \( a_i \neq a_{i+1} \).

**Proof.** If \( G \) is not reduced, we can find a pair \((G^*, H^*)\) of shorter words with \( f(G^*, H^*) \leq f(G, H) \). For suppose \( g_i, g_{i+1} \in G_{ai} \) (note: if \( g_i \in H \), this is always true), and \( g_i g_{i+1} = g_i^* \in G_{ai} \). Then let \( e_i e_{i+1} = e_i^* \in G_{ai} \). Let

\[
G^* = g_1 \ldots g_i g_{i+2} \ldots g_n \quad \text{and} \quad H^* = e_1 \ldots e_i^* e_{i+2} \ldots e_n.
\]

By the invariance of \( \varrho_{ai} \),

\[
\varrho_{ai}(g_i, e_{i+1}) \leq \varrho_{ai}(g_i, e_i) + \varrho_{ai}(g_{i+1}, e_{i+1})
\]

and the lemma follows.

Hence, in view of the standard results on free products with an amalgamated subgroup [7], we need consider only \( G \) of some fixed length \( n \) (depending on \( g \)) to determine \( \varrho(g, e) \).

**Lemma 2.** \( \varrho \) is an invariant pseudometric.

**Proof.** It is immediate that \( \varrho(g, g) = 0 \) and \( \varrho(gk, hk) = \varrho(g, h) \). Letting \( K \) represent \( k \), we see that \( \varrho(kgk^{-1}, e) \) is bounded from above by \( f(KGK^{-1}, KHK^{-1}) = f(G, H) \) and thus

\[
\varrho(kgk^{-1}, e) \leq \varrho(g, e).
\]

Similarly,

\[
\varrho(k^{-1}kgk^{-1}k, e) \leq \varrho(kgk^{-1}, e)
\]

and we see that

\[
\varrho(kg, kh) = \varrho(kgh^{-1}k^{-1}, e) = \varrho(gh^{-1}, e) = \varrho(g, h).
\]

We note that \( \varrho(g, e) = \varrho(g^{-1}, e) \), since \( f(G, H) \) approximates \( \varrho(g, e) \) if and only if \( f(G^{-1}, H^{-1}) \) approximates \( \varrho(g^{-1}, e) \). Hence

\[
\varrho(g, h) = \varrho(gh^{-1}, e) = \varrho(hg^{-1}, e) = \varrho(h, g).
\]

Finally, the triangle inequality holds, since if \( f(G_1, H_1) \) approximates \( \varrho(g_1, e) \) and \( f(G_2, H_2) \) approximates \( \varrho(g_2, e) \), then \( f(G_1 G_2, H_1 H_2) \) is an upper bound for \( \varrho(g_1 g_2, e) \); hence

\[
\varrho(g_1, e) + \varrho(g_2, e) \geq \varrho(g_1 g_2, e),
\]

and

\[
\varrho(g, h) + \varrho(h, k) = \varrho(gh^{-1}, e) + \varrho(hk^{-1}, e) \geq \varrho(gh^{-1} hk^{-1}, e) = \varrho(g, k).
\]
Now let \( \tau \) be the topology induced on \( \tilde{F} \) by the collection of all \( \varrho \) constructed in this manner [5].

**Lemma 3.** \( \tilde{F} \) with topology \( \tau \) is a topological group.

**Proof.** If we recall that sets \( N_{\varrho}(g_0, \varepsilon) = \{g \in \tilde{F} | \varrho(g, g_0) < \varepsilon \} \) form a subbase, then \( g \mapsto g^{-1} \) is readily continuous, since \( \varrho(g_1, g_2) = \varrho(g_1^{-1}, g_2^{-1}) \), and the group operation is jointly continuous, since \( \varrho(a_1a_2, b_1b_2) \leq \varrho(a_1, b_1) + \varrho(a_2, b_2) \) for each \( \varrho \).

**Lemma 4.** Let \( \varrho_a \) be any invariant pseudometric on \( G_a \). Then there is a \( \varrho \) such that \( \varrho|G_a = \varrho_a \), and thus the topology induced on \( G_a \) by \( \tau \) is the original topology of \( G_a \).

**Proof.** Fix \( a \) and \( \varrho_a \). For each \( \beta \in A \), let \( m^a_\beta: G_\beta \to G_a \) be the map of Theorem 3 \((m^a_\beta \) is the identity map). Write \( \varrho_\beta(g_1, g_2) = \varrho_a(m^a_\beta(g_1), m^a_\beta(g_2)) \). It is easy to check that \( \varrho_\beta \) is a continuous invariant pseudometric on \( G_\beta \) and \( \varrho_\beta(h, e) = \varrho_a(h, e) \) for all \( h \in H \) and \( \beta \in A \). Let \( \varrho \) be the pseudometric on \( \tilde{F} \) derived from \( \{\varrho_\beta\}_{\beta \in A} \). If \( g \in G_a \), clearly \( \varrho(g, e) = \varrho_a(g, e) \) as desired, completing the proof of the lemma.

**Lemma 5.** In the topology \( \tau \), \( \tilde{F} \) is Hausdorff.

**Proof.** It will suffice, given any \( g \in \tilde{F}, g \neq e \), to find a pseudometric \( \varrho \) such that \( \varrho(g, e) \neq 0 \). If \( g \in G_a \), choose \( \varrho_a \) such that \( \varrho_a(g, e) \neq 0 \) (\( G_a \) is Hausdorff) and construct \( \varrho \) as in Lemma 4. Otherwise, let \( G = g_1g_2 \cdots g_n \) be a word of minimal length representing \( g \). Now [7] the only elements of the \( G_a \) that occur in any reduced word representing \( g \) are \( h_1, h_2, \ldots, h_i \), where \( 1 \leq i \leq n \) and \( h_1, h_2 \in H \).

We observe next that if \( g \in G_a \setminus H \), there is a continuous invariant pseudometric on \( G_a \) with \( \varrho(g, h) = 1 \) for all \( h \subseteq H \). For \( e \in g^{-1}H \) (a closed set since \( H \) is closed) and using the customary Urysohn-motivated argument, with invariant neighborhoods of \( e \), we can construct a continuous invariant pseudometric \( \varrho_a: G_a \times G_a \to [0, 1] \) such that \( \varrho_a(e, g^{-1}h) = 1 \) for all \( g^{-1}h \subseteq g^{-1}H \). Then \( \varrho_a(g, h) = 1 \) also.

For each \( g_i \), construct such a \( \varrho_{a_i} \). Note

\[
\inf\{\varrho_{a_i}(h_1g_ih_2, h_3) \mid h_1, h_2, h_3 \subseteq H\} = \inf\{\varrho_{a_i}(g_i, h_1^{-1}h_2h_3^{-1})\} = 1.
\]

Now, on each \( G_a \) we introduce \( n \) pseudometrics; for \( a = a_i \), the pseudometric \( \varrho_{a_i} \) constructed as above, and for \( a \neq a_i \), the pseudometric induced on \( G_a \) from the pseudometric \( \varrho_{a_i} \) on \( G_{a_i} \) by the method of Lemma 4. Call these \( n \) pseudometrics \( \varrho_{a(i)}, 1 \leq i \leq n \), for each \( G_a \). Let

\[
\hat{\varrho}_a(g_1, g_2) = \max\{\varrho_{a(i)}(g_1, g_2) \mid 1 \leq i \leq n\}
\]

and let \( \varrho \) be constructed from \( \{\hat{\varrho}_a\}_{a \in A} \).
Finally, look at any \( f(G^*, H) \). By Lemma 1, we can suppose \( g_1^* g_2^* \ldots g_n^* \) reduced; \( e = e_1 e_2 \ldots e_n \) and \( e_i \in G_{a_i}^* \), \( a_i \neq a_{i+1} \); so, by [7], some \( e_i \in H \). Thus

\[
f(G^*, H) = \sum \hat{e}_{a_i}(g_i^*, e_i) > \inf \{ e_{a_i}(g_i^*, h) \mid 1 \leq i \leq n; \ h \in H \}
\]

\[
> \inf \{ e_{a_1}(h_1 g_i h_2, h_3) \mid 1 \leq i \leq n; \ h_1, h_2, h_3 \in H \} = 1.
\]

Hence \( e(g, e) = \inf f(G^*, H) > 1 \) as desired. This completes our proof of Theorem 4.

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