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THE ADDITIVITY OF THE VOLUME AND SPERNER'S LEMMA

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We shall prove here a theorem according to which if T is a triangulation of an n-dimensional simplex

$$p = \langle p_0, \ldots, p_n \rangle \subset E^n$$

and f is a map of the set of vertices of T into E^n such that the set of vertices of T lying on a given $(n-1)$-dimensional face of p is mapped into the $(n-1)$-dimensional hyperplane generated by that face, then the oriented volume $V(p_0, \ldots, p_n)$ of p is equal to the sum of oriented volumes $V(f(s_0), \ldots, f(s_n))$ of simplices $\langle f(s_0), \ldots, f(s_n)\rangle$, where $\langle s_0, \ldots, s_n\rangle$ run over all n-dimensional simplices of T. This theorem, which establishes the additivity of the oriented volume, implies the oriented version of Sperner's lemma when we restrict our consideration to maps f having values in the set $\{p_0, \ldots, p_n\}$ of vertices of p. We regard the connection between these two theorems as interesting, and this is the reason for writing this note.

1. Let $a_0, \ldots, a_n$ be points of the Euclidean space E^n which are *affine independent*, which means that

$$(1) \qquad \begin{vmatrix} 1 & \ldots & 1 \\ a_{01} & \ldots & a_{n1} \\ \ldots & \ldots & \ldots \\ a_{0n} & \ldots & a_{nn} \end{vmatrix} \neq 0,$$

a_{ij} denoting the j-th coordinate of a_i.

Let $\langle a_0, \ldots, a_n\rangle$ be the *simplex* with vertices $a_0, \ldots, a_n$. We shall regard the *orientation* of $\langle a_0, \ldots, a_n\rangle$ to be equal to 1 or -1 depending on the positive or negative value of the determinant in (1).

The *oriented volume* $V(a_0, \ldots, a_n)$ of $\langle a_0, \ldots, a_n\rangle$ is meant as the $(n!)^{-1}$-th of the determinant in (1). If the points $a_0, \ldots, a_n$ are not affine independent, we let $V(a_0, \ldots, a_n) = 0$, understanding that the orientation of $\langle a_0, \ldots, a_n\rangle$ is 0. In the case where inequality (1) holds we call the simplex $\langle a_0, \ldots, a_n\rangle$ *n-dimensional.*

Let us note the equality

$$(2) \quad (-1)^0 V(\hat{b}, a_0, \ldots, a_n) + (-1)^1 V(b, \hat{a}_0, \ldots, a_n) + \ldots + (-1)^{n+1} V(b, a_0, \ldots, \hat{a}_n) = 0$$

for points $b, a_0, \ldots, a_n$ of E^n (the sign ^ means that the point is deleted), obtained by developing along the first row the determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ b_1 & a_{01} & \ldots & a_{n1} \\ \ldots & \ldots & \ldots & \ldots \\ b_n & a_{0n} & \ldots & a_{nn} \end{vmatrix}$$

which is equal to 0.

For every j we can write equality (2) in the form

$$(3) \quad V(a_0, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n) - V(a_0, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) = \sum_{i \neq j} (-1)^i V(b, a_0, \ldots, \hat{a}_i, \ldots, a_n).$$

2. Let $p = \langle p_0, \ldots, p_n \rangle$ be a *positively oriented* n-dimensional simplex in E^n, i.e., such that $V(p_0, \ldots, p_n) > 0$. Let T be a simplicial subdivision of p. Let W be the set of vertices of T. The geometrical background for further reasoning is concentrated on the following two properties of T:

(4) If points a and b lie on the opposite half-spaces into which the space E^n is disconnected by the $(n-1)$-hyperplane generated by the simplex $\langle u_0, \ldots, u_{n-1} \rangle$, then the simplices $\langle a, u_0, \ldots, u_{n-1} \rangle$ and $\langle b, u_0, \ldots, u_{n-1} \rangle$ have different orientations.

(5) Each $(n-1)$-dimensional simplex of T is the common face of exactly two n-dimensional simplices of T if it does not lie on the boundary of p, and is the face of exactly one n-dimensional simplex of T if it lies on the boundary of p.

Let $f\colon W \to E^n$ be a map. Let

$$V(f) = \sum_{s \in T} V(f(s_0), \ldots, f(s_n)),$$

where $s = \langle s_0, \ldots, s_n \rangle$ run over all n-dimensional simplices of T (which are assumed to be positively oriented).

We call maps f and g from W into E^n *compatible* if for every $(n-1)$-dimensional simplex u of T lying on the boundary of p they take the vertices of u into the same $(n-1)$-dimensional hyperplane.

LEMMA 1. *If f and g are compatible, then $V(f) = V(g)$.*

Proof. Consider the special case where f and g differ on a single element w of W.

In the expression

(6) $$V(f)-V(g)=\sum_{s\in T}\big(V(f(s_0),\dots,f(s_n))-V(g(s_0),\dots,g(s_n))\big)$$

the non-vanishing differences on the right-hand side can occur only if w is one of $s_0,\dots,s_n$. So, let $w=s_j$ and fix one such difference

$$V(f(s_0),\dots,f(s_n))-V(g(s_0),\dots,g(s_n)).$$

According to formula (3), we have then

(7) $$\begin{aligned}V(f(s_0),&\dots,f(s_n))-V(g(s_0),\dots,g(s_n))\\&=V(f(s_0),\dots,f(s_j),\dots,f(s_n))-V(f(s_0),\dots,g(s_j),\dots,f(s_n))\\&=\sum_{i\neq j}(-1)^iV(g(s_j),f(s_0),\dots,\hat{f}(s_i),\dots,f(s_n)).\end{aligned}$$

If the simplex $\langle s_0,\dots,\hat{s}_i,\dots,s_n\rangle$ lies on the boundary of p, then all the points $f(s_r)$ and $g(s_r)$, $r\neq i$, lie on the same $(n-1)$-dimensional hyperplane. Thus, the summands of that kind on the right-hand side of (7) vanish.

If the simplex $\langle s_0,\dots,\hat{s}_i,\dots,s_n\rangle$ does not lie on the boundary of p, then, by (5), it is a common face of $\langle s_0,\dots,s_n\rangle$ and another simplex $t=\langle t_0,\dots,t_n\rangle$ of T, where the vertices of t are assumed to be ordered in such a way that

$$V(t_0,\dots,t_n)>0.$$

Let t_k be the vertex of t opposite to the common face of s and t. So this common face of s and t, expressed in s as $\langle s_0,\dots,\hat{s}_i,\dots,s_n\rangle$, is expressed as a face of t as $\langle t_0,\dots,\hat{t}_k,\dots,t_n\rangle$. The simplices

$$\langle s_0,\dots,\hat{s}_i,\dots,s_n\rangle \quad\text{and}\quad \langle t_0,\dots,\hat{t}_k,\dots,t_n\rangle$$

differ only in ordering of vertices.

An easy calculation shows that

$$\begin{aligned}\operatorname{sgn}V(s_i,s_0,\dots,\hat{s}_i,\dots,s_n)&=(-1)^i\operatorname{sgn}V(s_0,\dots,s_i,\dots,s_n)=(-1)^i\\&=(-1)^i\operatorname{sgn}V(t_0,\dots,t_k,\dots,t_n)=(-1)^{i+k}\operatorname{sgn}V(t_k,t_0,\dots,\hat{t}_k,\dots,t_n)\\&=(-1)^{i+k+1}\operatorname{sgn}V(s_i,t_0,\dots,\hat{t}_k,\dots,t_n),\end{aligned}$$

the last equality being obtained by means of (4). This means that the sign of permutation of vertices $s_0,\dots,\hat{s}_i,\dots,s_n$ with respect to $t_0,\dots,\hat{t}_k,\dots,t_n$ is $(-1)^{i+k+1}$.

Let t_l be that vertex of t which is equal to w. We have $s_j=w=t_l$; clearly, $l\neq k$. From our calculation it follows that

$$(-1)^iV(a,g(s_0),\dots,\widehat{g(s_i)},\dots,g(s_n))=(-1)^{k+1}V(a,g(t_0),\dots,\widehat{g(t_k)},\dots,g(t_n)).$$

Hence, substituting $a = f(s_j)$ we get

$$(-1)^i V(f(s_j), g(s_0), \ldots, \widehat{g(s_i)}, \ldots, g(s_n)) = (-1)^{k+1} V(f(s_j), g(t_0), \ldots, \widehat{g(t_k)}, \ldots, g(t_n)).$$

But $s_j = t_l$, and we have

$$(-1)^i V(f(s_j), g(s_0), \ldots, \widehat{g(s_i)}, \ldots, g(s_n)) + (-1)^k V(f(t_l), g(t_0), \ldots, \widehat{g(t_k)}, \ldots, g(t_n)) = 0.$$

Thus, for every i there exists a k such that the expression above vanishes. The correspondence between i and k is involutary by (5). So, the summands on the right-hand side of (7) cancel.

We can obtain g from f by successive substituting the values of g at elements of W for the values of f at these elements. At each step the change concerns only one value and each two of functions modified in such a way are compatible, as f and g are. Thus, the special case can be successively applied, and the lemma follows in full generality.

3. Assign to every element of W the vertex $\varphi(w) = p_j$ of p such that

$$j = \min\{i\colon w \in \langle p_0, \ldots, p_i\rangle\}. \tag{8}$$

Clearly, $\varphi(p_i) = p_i$.

If $w \in \langle p_0, \ldots, p_i\rangle$, then in formula (8) for $\varphi(w)$ the vertices $p_j, j > i$, are inessential. Thus, if $w \in \langle p_0, \ldots, p_i\rangle$, then

$$\varphi(w) \in \{p_0, \ldots, p_i\}.$$

Moreover, the function φ restricted to $W \cap \langle p_0, \ldots, p_i\rangle$ is equal to the function defined by (8) for the simplex $\langle p_0, \ldots, p_i\rangle$ equipped with the simplicial subdivision induced from that of $\langle p_0, \ldots, p_n\rangle$.

LEMMA 2. *There exists exactly one n-dimensional simplex s of T such that φ assumes at the vertices of s all the values $p_0, \ldots, p_n$. In addition,*

$$V(s_0, \ldots, s_n) > 0$$

assuming $\varphi(s_i) = p_i$.

Proof. We proceed by induction on n. The case $n = 0$ is obvious. Assume $n > 0$ and consider the simplex $\langle p_0, \ldots, p_n\rangle$ and its face $\langle p_0, \ldots, p_{n-1}\rangle$. The function φ restricted to $W \cap \langle p_0, \ldots, p_{n-1}\rangle$ is, according to the comment made before the proof, the same as the function defined by (8) for the simplex $\langle p_0, \ldots, p_{n-1}\rangle$ equipped with the simplicial subdivision induced from that of $\langle p_0, \ldots, p_n\rangle$. Thus, by the induction hypothesis, there exists a uniquely determined $(n-1)$-dimensional simplex t of T lying on $\langle p_0, \ldots, p_{n-1}\rangle$ such that φ assumes on the set of vertices all the values $p_0, \ldots, p_{n-1}$.

The simplex t, lying on the boundary of p, is by (5), a face of exactly one n-dimensional simplex of T; call that simplex s. The function φ assumes the value p_n at the vertex of s not belonging to t, as that vertex does not belong to the face $\langle p_0, \dots, p_{n-1}\rangle$. Thus φ assumes all the values $p_0, \dots, p_n$ on the set of vertices of s.

It remains to prove that such a simplex is unique. Let v be an n-dimensional simplex of T such that φ assumes all the values $p_0, \dots, p_n$ on the set of vertices of v. The $(n-1)$-dimensional face of v of whose set of vertices the function φ assumes all the values $p_0, \dots, p_{n-1}$ lies on $\langle p_0, \dots, p_{n-1}\rangle$. By uniqueness, implied by the induction hypothesis, this face coincides with t. But the simplex s is the only n-dimensional simplex of T having t as a face. Thus $v = s$.

To prove the second part of the conclusion, assume $\varphi(s_i) = p_i$.

We have $s_i \in \langle p_0, \dots, p_i\rangle$ and $s_i \notin \langle p_0, \dots, p_{i-1}\rangle$. Thus, for barycentric coordinates $\beta_{i0}, \dots, \beta_{in}$ of s_i with respect to $p_0, \dots, p_n$ we have

$$\beta_{i0} + \dots + \beta_{in} = 1, \quad \beta_{ik} \geqslant 0,$$

$$\beta_{ik} = 0 \text{ for } k > i \quad \text{and} \quad \beta_{ii} > 0 \text{ for all } i.$$

Taking into account that

$$s_i = \sum_{k=0}^{n} \beta_{ik} \cdot p_k \quad \text{and} \quad 1 = \sum_{k=0}^{n} \beta_{ik} \quad \text{for each } i,$$

we get

$$\begin{bmatrix} 1 & \dots & 1 \\ p_{01} & \dots & p_{n1} \\ \dots & \dots & \dots \\ p_{0n} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} \beta_{00} & \dots & \beta_{n0} \\ & & \\ \dots & \dots & \dots \\ \beta_{0n} & \dots & \beta_{nn} \end{bmatrix} = \begin{bmatrix} 1 & \dots & 1 \\ s_{01} & \dots & s_{n1} \\ \dots & \dots & \dots \\ s_{0n} & \dots & s_{nn} \end{bmatrix}.$$

The determinant of the second matrix on the left-hand side equals $\beta_{00} \cdot \ldots \cdot \beta_{nn}$ since that matrix is triangular ($\beta_{ik} = 0$ for $k > i$). Since the determinant of the product of matrices is equal to the product of their determinants, we get

$$V(s_0, \dots, s_n) = V(p_0, \dots, p_n) \cdot \beta_{00} \cdot \ldots \cdot \beta_{nn}.$$

The last expression is positive as $\beta_{ii} > 0$, and $V(p_0, \dots, p_n) > 0$ by assumption.

Note that

LEMMA 3. *$\varphi(w) \neq p_i$ whenever $w \in \langle p_0, \dots, \hat{p}_i, \dots, p_n\rangle$.*

Proof. In fact, $\varphi(w) = p_i$ would imply $w \in \langle p_0, \dots, p_i\rangle$. Thus we would have $w \in \langle p_0, \dots, p_{i-1}\rangle$, contrary to $\varphi(w) = p_i$.

Satisfying Lemma 3, the function φ is a Sperner function, and Lemma 2 can be considered as a special case of Sperner's lemma.

THEOREM. *Let $f\colon W \to E^n$ be a map taking vertices of a given $(n-1)$-dimensional face into the $(n-1)$-dimensional hyperplane generated by this face. Then*

$$V(f) = V\big(f(p_0), \dots, f(p_n)\big).$$

Proof. Under the assumption on f we made, the functions f and $f\circ\varphi$ are compatible. In fact, by Lemma 3, if s is an $(n-1)$-dimensional simplex lying on $\langle p_0, \ldots, \hat{p}_i, \ldots, p_n\rangle$, then the vertices $s_0, \ldots, s_{n-1}$ of s are mapped by φ into $\{p_0, \ldots, \hat{p}_i, \ldots, p_n\}$. But f maps all the vertices lying on $\langle p_0, \ldots, \hat{p}_i, \ldots, p_n\rangle$ into an $(n-1)$-dimensional hyperplane, and this means that f and $f\circ\varphi$ are compatible.

We have

$$V(f\circ\varphi) = \sum_{s\in T} V\big(f(\varphi(s_0)), \ldots, f(\varphi(s_n))\big). \tag{9}$$

By Lemma 2, there exists exactly one n-dimensional simplex s of T such that φ admits all the values $p_0, \ldots, p_n$ on its vertices. Thus the summands corresponding to the other simplices vanish. We can assume that $\varphi(s_i) = p_i$. Thus, by Lemma 2, formula (9) can be rewritten in the form

$$V(f\circ\varphi) = V\big(f(\varphi(s_0)), \ldots, f(\varphi(s_n))\big) = V\big(f(p_0), \ldots, f(p_n)\big),$$

which completes the proof, as $V(f\circ\varphi) = V(f)$ by Lemma 1, f and $f\circ\varphi$ being compatible.

COROLLARY (Sperner's lemma ([1]); oriented version). *Let T be a simplicial subdivision of $p = \langle p_0, \ldots, p_n\rangle$ and let W be the set of vertices of T. Let*

$$f\colon W \to \{p_0, \ldots, p_n\}$$

be a map such that

$$f(w) \in \{p_0, \ldots, \hat{p}_i, \ldots, p_n\} \quad \textit{whenever} \quad w \in \langle p_0, \ldots, \hat{p}_i, \ldots, p_n\rangle.$$

Then the difference between the numbers of positively and negatively oriented n-dimensional simplices $\langle f(s_0), \ldots, f(s_n)\rangle$ is equal to 1.

Proof. We have $f(p_i) = p_i$ by assumption. Hence

$$V(p_0, \ldots, p_n) = V\big(f(p_0), \ldots, f(p_n)\big) = V(f) = \sum_{s\in T} V\big(f(s_0), \ldots, f(s_n)\big).$$

The non-vanishing volumes in the last sum differ only in sign, as the absolute values of these volumes are the same and are equal to $V(p_0, \ldots, p_n)$. Thus we obtain 1 as the sum of

$$\sum_{s\in T} \operatorname{sgn} V\big(f(s_0), \ldots, f(s_n)\big),$$

i.e., the required conclusion.

([1]) E. Sperner, *Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes*, Abh. Math. Sem. Univ. Hamburg 6 (1928), pp. 265–272.

Taking f to be the identity, we get from the Theorem the following

COROLLARY. $V(p_0, \ldots, p_n) = \sum_{s \in T} V(s_0, \ldots, s_n)$.

This means the additivity of volume mentioned in the introductory part of the paper.

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