THE ADDITIVITY OF THE VOLUME
AND SPERNER'S LEMMA

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We shall prove here a theorem according to which if \( T \) is a triangulation of an \( n \)-dimensional simplex

\[
p = \langle p_0, \ldots, p_n \rangle \subset E^n
\]

and \( f \) is a map of the set of vertices of \( T \) into \( E^n \) such that the set of vertices of \( T \) lying on a given \((n-1)\)-dimensional face of \( p \) is mapped into the \((n-1)\)-dimensional hyperplane generated by that face, then the oriented volume \( V(p_0, \ldots, p_n) \) of \( p \) is equal to the sum of oriented volumes \( V(f(s_0), \ldots, f(s_n)) \) of simplices \( \langle f(s_0), \ldots, f(s_n) \rangle \), where \( \langle s_0, \ldots, s_n \rangle \) run over all \( n \)-dimensional simplices of \( T \). This theorem, which establishes the additivity of the oriented volume, implies the oriented version of Sperner's lemma when we restrict our consideration to maps \( f \) having values in the set \( \{p_0, \ldots, p_n\} \) of vertices of \( p \). We regard the connection between these two theorems as interesting, and this is the reason for writing this note.

1. Let \( a_0, \ldots, a_n \) be points of the Euclidean space \( E^n \) which are affine independent, which means that

\[
\begin{vmatrix}
1 & \ldots & 1 \\
a_{01} & \cdots & a_{n1} \\
\vdots & \ddots & \vdots \\
a_{0n} & \cdots & a_{nn}
\end{vmatrix} \neq 0,
\]

(1)

\( a_{ij} \) denoting the \( j \)-th coordinate of \( a_i \).

Let \( \langle a_0, \ldots, a_n \rangle \) be the simplex with vertices \( a_0, \ldots, a_n \). We shall regard the orientation of \( \langle a_0, \ldots, a_n \rangle \) to be equal to 1 or \(-1\) depending on the positive or negative value of the determinant in (1).

The oriented volume \( V(a_0, \ldots, a_n) \) of \( \langle a_0, \ldots, a_n \rangle \) is meant as the \((n!)^{-1}\)-th of the determinant in (1). If the points \( a_0, \ldots, a_n \) are not affine independent, we let \( V(a_0, \ldots, a_n) = 0 \), understanding that the orientation of \( \langle a_0, \ldots, a_n \rangle \) is 0. In the case where inequality (1) holds we call the simplex \( \langle a_0, \ldots, a_n \rangle \) \( n \)-dimensional.
Let us note the equality
\[(2) \quad (-1)^0 V(b, a_0, \ldots, a_n) + (-1)^1 V(b, \hat{a}_0, \ldots, a_n) + \cdots
+ (-1)^n V(b, a_0, \ldots, \hat{a}_n) = 0\]
for points \(b, a_0, \ldots, a_n\) of \(E^n\) (the sign \(\hat{\ })\) means that the point is deleted), obtained by developing along the first row the determinant
\[
\begin{vmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & a_{01} & \ldots & a_{n1} \\
\vdots & \vdots & \ddots & \vdots \\
b_n & a_{0n} & \ldots & a_{nn}
\end{vmatrix}
\]
which is equal to 0.

For every \(j\) we can write equality (2) in the form
\[(3) \quad V(a_0, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_n) - V(a_0, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n)
= \sum_{i \neq j} (-1)^i V(b, a_0, \ldots, \hat{a}_i, \ldots, a_n).\]

2. Let \(p = \langle p_0, \ldots, p_n \rangle\) be a positively oriented \(n\)-dimensional simplex in \(E^n\), i.e., such that \(V(p_0, \ldots, p_n) > 0\). Let \(T\) be a simplicial subdivision of \(p\). Let \(W\) be the set of vertices of \(T\). The geometrical background for further reasoning is concentrated on the following two properties of \(T\):

(4) If points \(a\) and \(b\) lie on the opposite half-spaces into which the space \(E^n\) is disconnected by the \((n-1)\)-hyperplane generated by the simplex \(\langle u_0, \ldots, u_{n-1} \rangle\), then the simplices \(\langle a, u_0, \ldots, u_{n-1} \rangle\) and \(\langle b, u_0, \ldots, u_{n-1} \rangle\) have different orientations.

(5) Each \((n-1)\)-dimensional simplex of \(T\) is the common face of exactly two \(n\)-dimensional simplices of \(T\) if it does not lie on the boundary of \(p\), and is the face of exactly one \(n\)-dimensional simplex of \(T\) if it lies on the boundary of \(p\).

Let \(f: W \to E^n\) be a map. Let
\[V(f) = \sum_{s \in T} V(f(s_0), \ldots, f(s_n)),\]
where \(s = \langle s_0, \ldots, s_n \rangle\) run over all \(n\)-dimensional simplices of \(T\) (which are assumed to be positively oriented).

We call maps \(f\) and \(g\) from \(W\) into \(E^n\) compatible if for every \((n-1)\)-dimensional simplex \(u\) of \(T\) lying on the boundary of \(p\) they take the vertices of \(u\) into the same \((n-1)\)-dimensional hyperplane.

**Lemma 1.** If \(f\) and \(g\) are compatible, then \(V(f) = V(g)\).

**Proof.** Consider the special case where \(f\) and \(g\) differ on a single element \(w\) of \(W\).
In the expression

\[ V(f) - V(g) = \sum_{s \in T} (V(f(s_0, \ldots, f(s_n)) - V(g(s_0, \ldots, g(s_n))) \right)

the non-vanishing differences on the right-hand side can occur only if \( w \) is one of \( s_0, \ldots, s_n \). So, let \( w = s_j \) and fix one such difference

\[ V(f(s_0), \ldots, f(s_n)) - V(g(s_0), \ldots, g(s_n)). \]

According to formula (3), we have then

\[ V(f(s_0), \ldots, f(s_n)) - V(g(s_0), \ldots, g(s_n)) \]

\[ = V(f(s_0), \ldots, f(s_j), \ldots, f(s_n)) - V(f(s_0), \ldots, g(s_j), \ldots, f(s_n)) \]

\[ = \sum_{i \neq j} (-1)^i V(g(s_j), f(s_0), \ldots, \hat{f}(s_i), \ldots, f(s_n)). \]

If the simplex \( \langle s_0, \ldots, \hat{s}_i, \ldots, s_n \rangle \) lies on the boundary of \( p \), then all the points \( f(s_r) \) and \( g(s_r) \), \( r \neq i \), lie on the same \((n-1)\)-dimensional hyperplane. Thus, the summands of that kind on the right-hand side of (7) vanish.

If the simplex \( \langle s_0, \ldots, \hat{s}_i, \ldots, s_n \rangle \) does not lie on the boundary of \( p \), then, by (5), it is a common face of \( \langle s_0, \ldots, s_n \rangle \) and another simplex \( t = \langle t_0, \ldots, t_n \rangle \) of \( T \), where the vertices of \( t \) are assumed to be ordered in such a way that

\[ V(t_0, \ldots, t_n) > 0. \]

Let \( t_k \) be the vertex of \( t \) opposite to the common face of \( s \) and \( t \). So this common face of \( s \) and \( t \), expressed in \( s \) as \( \langle s_0, \ldots, \hat{s}_i, \ldots, s_n \rangle \), is expressed as a face of \( t \) as \( \langle t_0, \ldots, t_k, \ldots, t_n \rangle \). The simplices

\[ \langle s_0, \ldots, \hat{s}_i, \ldots, s_n \rangle \quad \text{and} \quad \langle t_0, \ldots, t_k, \ldots, t_n \rangle \]

differ only in ordering of vertices.

An easy calculation shows that

\[ \text{sgn } V(s_i, s_0, \ldots, \hat{s}_i, \ldots, s_n) = (-1)^i \text{sgn } V(s_0, \ldots, s_i, \ldots, s_n) = (-1)^i \]

\[ = (-1)^i \text{sgn } V(t_0, \ldots, t_k, \ldots, t_n) = (-1)^{i+k} \text{sgn } V(t_k, t_0, \ldots, \hat{t}_k, \ldots, t_n) \]

\[ = (-1)^{i+k+1} \text{sgn } V(s_i, t_0, \ldots, \hat{t}_k, \ldots, t_n), \]

the last equality being obtained by means of (4). This means that the sign of permutation of vertices \( s_0, \ldots, \hat{s}_i, \ldots, s_n \) with respect to \( t_0, \ldots, t_k, \ldots, t_n \) is \((-1)^{i+k+1}\).

Let \( t_l \) be that vertex of \( t \) which is equal to \( w \). We have \( s_j = w = t_i \); clearly, \( l \neq k \). From our calculation it follows that

\[ (-1)^i V(a, g(s_0), \ldots, \hat{g}(s_i), \ldots, g(s_n)) = (-1)^{k+1} V(a, g(t_0), \ldots, \hat{g}(t_k), \ldots, g(t_n)). \]
Hence, substituting $a = f(s_j)$ we get
\[ (-1)^i V(f(s_j), g(s_0), \ldots, \hat{g(s_i)}, \ldots, g(s_n)) = (-1)^{k+1} V(f(s_j), g(t_0), \ldots, \hat{g(t_k)}, \ldots, g(t_n)). \]

But $s_j = t_i$, and we have
\[ (-1)^i V(f(s_j), g(s_0), \ldots, \hat{g(s_i)}, \ldots, g(s_n)) + (-1)^k V(f(t_i), g(t_0), \ldots, \hat{g(t_k)}, \ldots, g(t_n)) = 0. \]

Thus, for every $i$ there exists a $k$ such that the expression above vanishes. The correspondence between $i$ and $k$ is involutary by (5). So, the summands on the right-hand side of (7) cancel.

We can obtain $g$ from $f$ by successive substituting the values of $g$ at elements of $W$ for the values of $f$ at these elements. At each step the change concerns only one value and each two of functions modified in such a way are compatible, as $f$ and $g$ are. Thus, the special case can be successively applied, and the lemma follows in full generality.

3. Assign to every element of $W$ the vertex $\varphi(w) = p_j$ of $p$ such that
\[ j = \min \{ i \mid w \in \langle p_0, \ldots, p_i \rangle \} . \tag{8} \]

Clearly, $\varphi(p_i) = p_i$.

If $w \in \langle p_0, \ldots, p_i \rangle$, then in formula (8) for $\varphi(w)$ the vertices $p_j, j > i$, are inessential. Thus, if $w \in \langle p_0, \ldots, p_i \rangle$, then
\[ \varphi(w) \in \{ p_0, \ldots, p_i \} . \]

Moreover, the function $\varphi$ restricted to $W \cap \langle p_0, \ldots, p_i \rangle$ is equal to the function defined by (8) for the simplex $\langle p_0, \ldots, p_i \rangle$ equipped with the simplicial subdivision induced from that of $\langle p_0, \ldots, p_n \rangle$.

**Lemma 2.** There exists exactly one $n$-dimensional simplex $s$ of $T$ such that $\varphi$ assumes at the vertices of $s$ all the values $p_0, \ldots, p_n$. In addition,
\[ V(s_0, \ldots, s_n) > 0 \]

assuming $\varphi(s_j) = p_i$.

**Proof.** We proceed by induction on $n$. The case $n = 0$ is obvious. Assume $n > 0$ and consider the simplex $\langle p_0, \ldots, p_n \rangle$ and its face $\langle p_0, \ldots, p_{n-1} \rangle$. The function $\varphi$ restricted to $W \cap \langle p_0, \ldots, p_{n-1} \rangle$ is, according to the comment made before the proof, the same as the function defined by (8) for the simplex $\langle p_0, \ldots, p_{n-1} \rangle$ equipped with the simplicial subdivision induced from that of $\langle p_0, \ldots, p_n \rangle$. Thus, by the induction hypothesis, there exists a uniquely determined $(n - 1)$-dimensional simplex $t$ of $T$ lying on $\langle p_0, \ldots, p_{n-1} \rangle$ such that $\varphi$ assumes on the set of vertices all the values $p_0, \ldots, p_{n-1}$.

The simplex $t$, lying on the boundary of $p$, is by (5) a face of exactly one $n$-dimensional simplex of $T$; call that simplex $s$. The function $\varphi$ assumes the
value \( p_n \) at the vertex of \( s \) not belonging to \( t \), as that vertex does not belong to the face \( \langle p_0, \ldots, p_{n-1} \rangle \). Thus \( \varphi \) assumes all the values \( p_0, \ldots, p_n \) on the set of vertices of \( s \).

It remains to prove that such a simplex is unique. Let \( v \) be an \( n \)-dimensional simplex of \( T \) such that \( \varphi \) assumes all the values \( p_0, \ldots, p_n \) on the set of vertices of \( v \). The \((n-1)\)-dimensional face of \( v \) of whose set of vertices the function \( \varphi \) assumes all the values \( p_0, \ldots, p_{n-1} \) lies on \( \langle p_0, \ldots, p_{n-1} \rangle \). By uniqueness, implied by the induction hypothesis, this face coincides with \( t \). But the simplex \( s \) is the only \( n \)-dimensional simplex of \( T \) having \( t \) as a face. Thus \( v = s \).

To prove the second part of the conclusion, assume \( \varphi(s_i) = p_i \).

We have \( s_i \in \langle p_0, \ldots, p_i \rangle \) and \( s_i \notin \langle p_0, \ldots, p_{i-1} \rangle \). Thus, for barycentric coordinates \( \beta_{i0}, \ldots, \beta_{in} \) of \( s_i \) with respect to \( p_0, \ldots, p_n \) we have

\[
\beta_{i0} + \ldots + \beta_{in} = 1, \quad \beta_{ik} \geq 0, \\
\beta_{ik} = 0 \text{ for } k > i \quad \text{and} \quad \beta_{ii} > 0 \text{ for all } i.
\]

Taking into account that

\[
s_i = \sum_{k=0}^{n} \beta_{ik} \cdot p_k \quad \text{and} \quad 1 = \sum_{k=0}^{n} \beta_{ik} \quad \text{for each } i,
\]

we get

\[
\begin{bmatrix}
1 & \ldots & 1 \\
p_{01} & \ldots & p_{n1} \\
p_{0n} & \ldots & p_{nn}
\end{bmatrix}
\begin{bmatrix}
\beta_{00} & \ldots & \beta_{n0} \\
\beta_{0n} & \ldots & \beta_{nn}
\end{bmatrix}
= 
\begin{bmatrix}
1 & \ldots & 1 \\
s_{01} & \ldots & s_{n1} \\
s_{0n} & \ldots & s_{nn}
\end{bmatrix}.
\]

The determinant of the second matrix on the left-hand side equals \( \beta_{00} \cdot \ldots \cdot \beta_{nn} \) since that matrix is triangular (\( \beta_{ik} = 0 \) for \( k > i \)). Since the determinant of the product of matrices is equal to the product of their determinants, we get

\[
V(s_0, \ldots, s_n) = V(p_0, \ldots, p_n) \cdot \beta_{00} \cdot \ldots \cdot \beta_{nn}.
\]

The last expression is positive as \( \beta_{ii} > 0 \), and \( V(p_0, \ldots, p_n) > 0 \) by assumption.

Note that

**Lemma 3.** \( \varphi(w) \neq p_i \) whenever \( w \in \langle p_0, \ldots, \hat{p_i}, \ldots, p_n \rangle \).

**Proof.** In fact, \( \varphi(w) = p_i \) would imply \( w \in \langle p_0, \ldots, p_i \rangle \). Thus we would have \( w \in \langle p_0, \ldots, p_{i-1} \rangle \), contrary to \( \varphi(w) = p_i \).

Satisfying Lemma 3, the function \( \varphi \) is a Sperner function, and Lemma 2 can be considered as a special case of Sperner’s lemma.

**Theorem.** Let \( f : W \rightarrow E^* \) be a map taking vertices of a given \((n-1)\)-dimensional face into the \((n-1)\)-dimensional hyperplane generated by this face. Then

\[
V(f) = V(f(p_0), \ldots, f(p_n)).
\]
Proof. Under the assumption on \( f \) we made, the functions \( f \) and \( f \circ \varphi \) are compatible. In fact, by Lemma 3, if \( s \) is an \((n-1)\)-dimensional simplex lying on \( \langle p_0, \ldots, \hat{p}_i, \ldots, p_n \rangle \), then the vertices \( s_0, \ldots, s_{n-1} \) of \( s \) are mapped by \( \varphi \) into \( \{ p_0, \ldots, \hat{p}_i, \ldots, p_n \} \). But \( f \) maps all the vertices lying on \( \langle p_0, \ldots, \hat{p}_i, \ldots, p_n \rangle \) into an \((n-1)\)-dimensional hyperplane, and this means that \( f \) and \( f \circ \varphi \) are compatible.

We have

\[
V(f \circ \varphi) = \sum_{s \in T} V(f(\varphi(s_0)), \ldots, f(\varphi(s_n))).
\]

By Lemma 2, there exists exactly one \( n \)-dimensional simplex \( s \) of \( T \) such that \( \varphi \) admits all the values \( p_0, \ldots, p_n \) on its vertices. Thus the summands corresponding to the other simplices vanish. We can assume that \( \varphi(s_i) = p_i \). Thus, by Lemma 2, formula (9) can be rewritten in the form

\[
V(f \circ \varphi) = V(f(\varphi(s_0)), \ldots, f(\varphi(s_n))) = V(f(p_0), \ldots, f(p_n)),
\]

which completes the proof, as \( V(f \circ \varphi) = V(f) \) by Lemma 1, \( f \) and \( f \circ \varphi \) being compatible.

Corollary (Sperner's lemma \(^1\); oriented version). Let \( T \) be a simplicial subdivision of \( p = \langle p_0, \ldots, p_n \rangle \) and let \( W \) be the set of vertices of \( T \). Let

\[
f: W \to \{ p_0, \ldots, p_n \}
\]

be a map such that

\[
f(w) \in \{ p_0, \ldots, \hat{p}_i, \ldots, p_n \} \quad \text{whenever} \quad w \in \langle p_0, \ldots, \hat{p}_i, \ldots, p_n \rangle.
\]

Then the difference between the numbers of positively and negatively oriented \( n \)-dimensional simplices \( \langle f(s_0), \ldots, f(s_n) \rangle \) is equal to 1.

Proof. We have \( f(p_i) = p_i \) by assumption. Hence

\[
V(p_0, \ldots, p_n) = V(f(p_0), \ldots, f(p_n)) = V(f) = \sum_{s \in T} V(f(s_0), \ldots, f(s_n)).
\]

The non-vanishing volumes in the last sum differ only in sign, as the absolute values of these volumes are the same and are equal to \( V(p_0, \ldots, p_n) \). Thus we obtain 1 as the sum of

\[
\sum_{s \in T} \text{sgn} \ V(f(s_0), \ldots, f(s_n)),
\]

i.e., the required conclusion.

Taking $f$ to be the identity, we get from the Theorem the following

**Corollary.** $V(p_0, \ldots, p_n) = \sum_{s \in T} V(s_0, \ldots, s_n)$.

This means the additivity of volume mentioned in the introductory part of the paper.

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