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# *APPROXIMATE SMOOTHNESS OF CONTINUOUS FUNCTIONS*

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There are numerous examples in the literature of continuous functions on the real line $R$ which have a derivative at no point of $R$. Filipczak [4] constructed one of the more interesting such functions in that her function $f$ has a symmetric derivative nowhere; i.e., at each point $x$, $\lim_{h\to 0^+} D^1 f(x, h)$ fails to exist, where

$$D^1 f(x, h) = [f(x+h) - f(x-h)]/2h.$$

In [2] it was further shown that Filipczak's construction leads to a function which has an approximate symmetric derivative nowhere; i.e., $\lim_{h\to 0^+} \operatorname{ap} D^1 f(x, h)$ fails to exist at each $x \in R$.

An alternate, or complementary, symmetry criterion is obtained by considering the divided difference

$$D^2 f(x, h) = [f(x+h) + f(x-h) - 2f(x)]/2h.$$

A function is said to be *smooth* (*approximately smooth*) at $x$ if

$$\lim_{h\to 0^+} D^2 f(x, h) = 0 \quad (\lim_{h\to 0^+} \operatorname{ap} D^2 f(x, h) = 0).$$

For properties of smooth and approximately smooth functions, the interested reader is referred to [9], [5]–[7], [3]. Clearly, a function which is (approximately) differentiable at $x$ is both (approximately) symmetrically differentiable and (approximately) smooth at $x$. Conversely, if $f$ is both (approximately) symmetrically differentiable and (approximately) smooth at $x$, then $f$ is (approximately) differentiable at $x$.

The purpose of the present paper is to show that a continuous function can fail to behave in either of these symmetric senses at each point in $R$. We shall construct a continuous function $f$ having the property that at each $x \in R$,

$$\limsup_{h\to 0^+} \operatorname{ap} D^1 f(x, h) = +\infty, \quad \liminf_{h\to 0^+} \operatorname{ap} D^1 f(x, h) = -\infty,$$

and

$$\limsup_{h\to 0^+} \mathrm{ap}\,|D^2 f(x, h)| = +\infty .$$

Indeed, if $C$ denotes the metric space of all continuous real-valued functions on $[0, 1]$ with the $L_\infty$-metric,

$$\varrho(f, g) = \max\{|f(x)-g(x)|:\ x\in[0, 1]\},$$

then the collection of functions having the property described above will be shown to be residual in $C$.

Definitions of terms used in this paper are as found in [8]. The notation $|G|$ will be used for the Lebesgue measure of a measurable set $G$.

We begin by letting $\alpha$ and $\beta$ be any two positive numbers and defining the basic function $f_{\alpha,\beta}$ on $[0, 7\alpha]$ by

$$f_{\alpha,\beta}(x) = \begin{cases} \beta x/\alpha & \text{for } x\in[0, \alpha], \\ \beta(2\alpha-x)/\alpha & \text{for } x\in[\alpha, 2\alpha], \\ 2\beta(2\alpha-x)/\alpha & \text{for } x\in[2\alpha, 3\alpha], \\ 2\beta(x-4\alpha)/\alpha & \text{for } x\in[3\alpha, 4\alpha], \\ 0 & \text{for } x\in[4\alpha, 7\alpha]. \end{cases}$$

We then extend $f_{\alpha,\beta}$ to $R$ by $f_{\alpha,\beta}(x+7n\alpha) = f_{\alpha,\beta}(x)$, yielding a function of period $7\alpha$.

LEMMA. *The function $f_{\alpha,\beta}$ defined on $R$ as above satisfies the following conditions*:

(1) *$f_{\alpha,\beta}$ is continuous and periodic of period $7\alpha$.*

(2) $|f_{\alpha,\beta}(x)| \leqslant 2\beta$ *for all $x$.*

(3) $|f_{\alpha,\beta}(x_1)-f_{\alpha,\beta}(x_2)| \leqslant (2\beta/\alpha)|x_1-x_2|$ *for all $x_1$, $x_2$.*

(4) *For each $x$ there exists a closed interval $J(x)$ such that*

(4a) $J(x) \subseteq [.9\alpha, 4.1\alpha]$;

(4b) $|J(x)| = .1\alpha$;

(4c) *if $t\in J(x)$, then* $|D^2 f_{\alpha,\beta}(x, t)| \geqslant \beta/25\alpha$.

Proof. Conditions (1), (2), and (3) are clearly satisfied. To verify (4) we shall consider separately the three cases:

$$x\in[0, 2.1\alpha],\quad x\in[2.1\alpha, 3.9\alpha],\quad \text{and}\quad x\in[3.9\alpha, 7\alpha].$$

Then the periodicity of $f_{\alpha,\beta}$ will yield (4) for any other $x$. (Before turning to consideration of these cases, let us denote $f_{\alpha,\beta}$ by $f$ for simplicity of notation in the remainder of this proof.)

First, suppose $x \in [0, 2.1\alpha]$. If we let

$$J(x) = [3\alpha - x, 3.1\alpha - x],$$

then (4a) and (4b) are obvious. Let $t \in J(x)$. Then $x+t \in [3\alpha, 3.1\alpha]$, implying that

$$f(x+t) \leqslant f(3.1\alpha) = -1.8\beta.$$

Consequently,

$$(5) \qquad D^2 f(x, t) \leqslant \frac{-1.8\beta + f(x-t) - 2f(x)}{2t}$$

$$\leqslant \frac{-1.8\beta + \beta - 2f(2.1\alpha)}{6.2\alpha} = \frac{-2\beta}{31\alpha} < \frac{-\beta}{25\alpha}.$$

Next, if $x \in [2.1\alpha, 3.9\alpha]$, we set

$$J(x) = [6\alpha - x, 6.1\alpha - x],$$

yielding (4a) and (4b). Let $t \in J(x)$. Then $x+t \in [6\alpha, 6.1\alpha]$, implying $f(x+t) = 0$; $x-t \in [-1.9\alpha, 1.8\alpha]$, implying $f(x-t) \geqslant 0$; and

$$f(x) \leqslant f(2.1\alpha) = f(3.9\alpha) = -.2\beta.$$

Hence

$$(6) \qquad D^2 f(x, t) \geqslant \frac{0+0-2(-.2\beta)}{8\alpha} = \frac{\beta}{20\alpha} > \frac{\beta}{25\alpha}.$$

Finally, if $x \in [3.9\alpha, 7\alpha]$, we set

$$J(x) = [x - 3\alpha, x - 2.9\alpha],$$

again giving (4a) and (4b) immediately. If $t \in J(x)$, then $x-t \in [2.9\alpha, 3\alpha]$, implying that $f(x-t) \leqslant -1.8\beta$. Furthermore,

$$0 \geqslant f(x) \geqslant f(3.9) = -.2\beta.$$

Consequently,

$$(7) \qquad D^2 f(x, t) \leqslant \frac{\beta + (-1.8\beta) - 2(-.2\beta)}{8.2\alpha} = \frac{-2\beta}{41\alpha} < \frac{-\beta}{25\alpha}.$$

From (5)–(7) we see that (4c) is satisfied and the proof is complete.

Remark. The function $f_{\alpha,\beta}$ defined here satisfies the properties listed in the lemma in [2] with the exceptions that properties (2) and (3) of that paper are now replaced by properties (2) and (3) of this present work. In fact, the proof given in that paper applied to the present $f_{\alpha,\beta}$ need not be altered at all to prove that the stated properties hold.

THEOREM 1. *There is a continuous function $f$ defined on $R$ which fails to* *be approximately smooth and fails to have an approximate symmetric derivative (finite or infinite) at each and every point in $R$. More precisely, at each $x \in R$*

(i)
$$\limsup_{h\to 0^+} \operatorname{ap} D^1 f(x, y) = +\infty,$$

(ii)
$$\liminf_{h\to 0^+} \operatorname{ap} D^1 f(x, h) = -\infty,$$

(iii)
$$\limsup_{h\to 0^+} \operatorname{ap} |D^2 f(x, h)| = +\infty.$$

Proof. As in [2] and [4] we let $0 < a < b < 1$, where the actual values of $a$ and $b$ will be fixed later, and set

$$f_n(x) = f_{a^n, b^n}(x)$$

for each natural number $n$, where $f_{a^n,b^n}$ is the function from the lemma with $\alpha = a^n$ and $\beta = b^n$. Then the desired continuous function $f$ is given by

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

We shall verify that (iii) holds. Let $x \in R$ and let $n$ be a natural number. From the lemma we know there is an interval

$$J_n(x) \subseteq [.9a^n, 4.1a^n]$$

of length $.1a^n$ such that

$$|D^2 f_n(x, t)| \geqslant b^n/25a^n \quad \text{for } t \in J_n(x).$$

So, if $t \in J_n(x)$, then

$$\begin{aligned}
|D^2 f(x, t)| &= \Big|\sum_{m=1}^{\infty} D^2 f_m(x, t)\Big| \\
&\geqslant |D^2 f_n(x, t)| - \sum_{m=1}^{n-1} |D^2 f_m(x, t)| - \sum_{m=n+1}^{\infty} |D^2 f_m(x, t)| \\
&\geqslant \frac{1}{25}\left(\frac{b}{a}\right)^n - \sum_{m=1}^{n-1}\left[\frac{|f_m(x+t) - f_m(x)|}{2t} + \frac{|f_m(x-t) - f_m(x)|}{2t}\right] \\
&\quad - \frac{1}{2t}\sum_{m=n+1}^{\infty}\left[|f_m(x+t)| + |f_m(x-t)| + 2|f_m(x)|\right] \\
&\geqslant \frac{1}{25}\left(\frac{b}{a}\right)^n - 2\sum_{m=1}^{n-1}\left(\frac{b}{a}\right)^m - \frac{1}{1.8a^n}\sum_{m=n+1}^{\infty} 8b^m \\
&= \frac{1}{25}\left(\frac{b}{a}\right)^n - 2\frac{(b/a)^n - b/a}{b/a - 1} - \frac{40}{9a^n}\frac{b^{n+1}}{1-b} \\
&\geqslant \left(\frac{b}{a}\right)^n\left(\frac{1}{25} - \frac{2a}{b-a} - \frac{40}{9}\frac{b}{1-b}\right).
\end{aligned}$$

If we now set $a = 10^{-6}$ and $b = 10^{-3}$, then this last expression is clearly greater than $10^n$. If $p$ is any positive number, let $N$ be a natural number greater than the logarithm base 10 of $p$. Then, for any $n > N$ and $t \in J_n(x)$, we have

$$|D^2 f(x, t)| > p \quad \text{and} \quad |J_n(x) \cap [0, 4.1\alpha]| = .1\alpha,$$

and hence the set $\{t\colon |D^2 f(x, t)| > p\}$ has upper right density at least 1/41 at zero, implying (iii).

In light of the remark preceding this theorem, (i) and (ii) are verified precisely as in the proof of Theorem 1 in [2], using, of course, the now fixed values of $a = 10^{-6}$ and $b = 10^{-3}$.

THEOREM 2. *Let $P$ be the collection of all functions $f$ in $C$ having the following three properties at each point $x$ in* (0, 1):

(i) $$\limsup_{h\to 0^+} \operatorname{ap} D^1 f(x, h) = +\infty,$$

(ii) $$\liminf_{h\to 0^+} \operatorname{ap} D^1 f(x, h) = -\infty,$$

(iii) $$\limsup_{h\to 0^+} \operatorname{ap} |D^2 f(x, h)| = +\infty.$$

*Then $P$ is residual in $C$.*

Proof. Let $N = C \setminus P$ and set

$$N_1 = \{f \in C\colon \exists x \in (0, 1) \text{ such that } \limsup_{h\to 0^+} \operatorname{ap} D^1 f(x, h) < +\infty\},$$

$$N_2 = \{f \in C\colon \exists x \in (0, 1) \text{ such that } \liminf_{h\to 0^+} \operatorname{ap} D^1 f(x, h) > -\infty\},$$

$$N_3 = \{f \in C\colon \exists x \in (0, 1) \text{ such that } \limsup_{h\to 0^+} \operatorname{ap} |D^2 f(x, h)| < +\infty\}.$$

Then $N = N_1 \cup N_2 \cup N_3$. Both $N_1$ and $N_2$ were shown to be of the first category in the proof of Theorem 2 in [2]. It remains to show the same for $N_3$.

For each $f \in C$, $x \in (0, 1)$, and $d > 0$, let

$$E(f, x, d) = \{h > 0\colon |D^2 f(x, h)| < d\};$$

and for each $n = 2, 3, \ldots$, set

$$S_n = \{f \in C\colon \exists x \in [1/n, 1 - 1/n] \text{ such that}$$
$$|E(f, x, n) \cap [0, t]| \geqslant 41t/42 \text{ whenever } t < 1/n\},$$

Then $N_3 \subseteq \bigcup_{n=2}^{\infty} S_n$. We shall show that each $S_n$ is closed and nowhere dense in $C$.

Fix an $n$ and let $f$ be a limit point of $S_n$. Let $\{f_k\}$ be a sequence of functions in $S_n$ such that $\varrho(f_k, f) \to 0$. For each $k = 1, 2, \ldots$, there is an

$x_k \in [1/n, 1-1/n]$ such that

$$|E(f_k, x_k, n) \cap [0, t]| \geqslant 42t/42 \text{ whenever } t < 1/n,$$

and we may assume that $\{x_k\}$ converges to some point $x_0 \in [1/n, 1-1/n]$. Let

$$E_0 = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E(f_k, x_k, n).$$

Then, for $0 < t < 1/n$,

$$(8) \qquad \begin{aligned} |E_0 \cap [0, t]| &= \lim_{m\to\infty} \Big|\bigcup_{k=m}^{\infty} E(f_k, x_k, n) \cap [0, t]\Big| \\ &\geqslant \liminf_{m\to\infty} |E(f_m, x_m, n) \cap [0, t]| \geqslant 41t/42. \end{aligned}$$

Next we show that $E_0 \subseteq E(f, x_0, n)$. Let $h \in E_0$ and suppose $\varepsilon > 0$. By the Arzela–Ascoli Theorem (see, e.g., [1], p. 191) there is a $\delta > 0$ such that $|x-x'| < \delta$ implies

$$|f_k(x)-f_k(x')| < \varepsilon h/2 \quad \text{for all } k = 1, 2, \ldots$$

Clearly,

$$\lim_{k\to\infty} D^2 f_k(x_0, h) = D^2 f(x_0, h),$$

and, consequently, there is a natural number $m > 1$ for which $k \geqslant m$ implies both

$$|D^2 f_k(x_0, h) - D^2 f(x_0, h)| < \varepsilon/2 \quad \text{and} \quad |x_k - x_0| < \delta.$$

Since $h \in E_0$, there is a $k \geqslant m$ for which $h \in E(f_k, x_k, n)$. For this $k$ it follows that $|D^2 f_k(x_k, h)| \leqslant n$ and

$$\begin{aligned} &|D^2 f(x_0, h) - D^2 f_k(x_k, h)| \\ &\qquad \leqslant |D^2 f(x_0, h) - D^2 f_k(x_0, h)| + |D^2 f_k(x_0, h) - D^2 f_k(x_k, h)| \\ &\qquad < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So $|D^2 f(x_0, h)| < n+\varepsilon$, and since this holds for each $\varepsilon > 0$, it follows that $h \in E(f, x_0, n)$. Hence $E_0 \subseteq E(f, x_0, n)$ and from (8) we infer that $S_n$ is closed.

Next we show that $S_n$ is nowhere dense in $C$. To this end, let $p$ be a polynomial, $\varepsilon > 0$, and

$$B(p, \varepsilon) = \{g \in C\colon\ \varrho(g, p) < \varepsilon\}.$$

We shall show that $B(p, \varepsilon) \cap (C \setminus S_n) \neq \emptyset$. Let

$$L = \max\{|p'(x)|\colon\ x \in [0, 1]\}$$

and let $f$ be the function constructed in Theorem 1. Further let

$$\zeta = \varepsilon/2\|f\|, \quad \text{where } \|f\| = \max\{|f(x)|: x\in[0, 1]\}.$$

As shown in the proof of Theorem 1, for each $x\in(0, 1)$, there is a positive number $t$ (which depends on $x$) such that

$$|E(f, x, (L+n)/\zeta)\cap[0, t]| \leqslant (1-1/41)t < 41t/42.$$

Now, $p+\zeta f\in B(p, \varepsilon)$ and for each $h\in E(p+\zeta f, x, n)\cap[0, t]$ we have

$$n \geqslant |D^2(p+\zeta f)(x, h)| = |D^2 p(x, h)+\zeta D^2 f(x, h)| \geqslant |\zeta D^2 f(x, h)|-|D^2 p(x, h)| \geqslant \zeta|D^2 f(x, h)|-L;$$

i.e., $|D^2 f(x, h)| \leqslant (L+n)/\zeta$, implying $h\in E(f, x, (L+n)/\zeta)\cap[0, t]$. Consequently,

$$E(p+\zeta f, x, n)\cap[0, t] \subseteq E(f, x, (L+n)/\zeta)\cap[0, t],$$

and so

$$|E(p+\zeta f, x, n)\cap[0, t]| < 41t/42.$$

Consequently, $p+\zeta f\in(C\setminus S_n)\cap B(p, \varepsilon)$ and $S_n$ is nowhere dense, completing the proof that $N_3$ is of the first category.

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