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Remark on hyperbolic embeddability of relatively compact subspaces of complex spaces

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Abstract. The characterization of hyperbolic embeddability of relatively compact subspaces of a complex space in terms of extension of holomorphic maps from the punctured disc and of limit complex lines is given.

We assume that complex spaces are connected and have a countable topology. Put $D = \{z \in \mathbf{C}: |z| < 1\}$ and $D^* = D \setminus \{0\}$.

Let M be a subspace of a complex space X. We say that M has the *D^*-extension property* for X if every holomorphic map of D^* into M can be extended to a holomorphic map of D into X.

Recall that M is *hyperbolically embedded* in X if for $x, y \in \bar{M}$, $x \neq y$, there exist open neighbourhoods U of x and V of y such that $d_M(U \cap M, V \cap M) > 0$, where for each complex space Z we denote by d_Z the Kobayashi semidistance on Z. It is known [2] that if M is hyperbolically embedded in X then M has the D^*-extension property for X. Some criterions for hyperbolic embedding can be found in [3]. In a particular case when M and X are complex manifolds, M is locally complete hyperbolic and relatively compact Zaidenberg has proved [4] that M is hyperbolically embedded in X if M contains no complex lines and ∂M contains no limit complex lines. Here by a *limit complex line* of ∂M we mean a non-constant holomorphic map of $\mathbf{C}$ into X which can be approximated on each $D_r = rD$ by holomorphic maps of D_r into M. In case where X is a complex space and M is a complement of a hypersurface the result has been established in [3].

The aim of this note is to prove the following

THEOREM. *Let M be a locally complete hyperbolic and relatively compact subspace of a complex space X. Then the following are equivalent:*

(i) *M is hyperbolically embedded in X.*

(ii) *M has the D^*-extension property for X and ∂M contains no limit complex lines.*

(iii) *M contains no complex lines and ∂M contains no limit complex lines.*

Moreover, if one of the above conditions holds, then M is complete hyperbolic.

Proof. It is known [3] that M is hyperbolically embedded in X if and only if

$$c(M) := \sup\{\|df(0)\| : f \in H(D, M)\} < \infty$$

where $H(D, M)$ denotes the space of holomorphic maps from D into M equipped with the compact-open topology.

(i)$\Rightarrow$(ii). By [2], M has the D^*-extension property for X. Thus it remains to show that ∂M contains no limit complex lines. This is an immediate consequence of the inequality $c(M) < \infty$.

(ii)$\Rightarrow$(iii). It suffices to prove that M contains no complex lines. Assume that there exists a non-constant holomorphic map f: $\mathbf{C} \to M$. Observe that the map

$$g\colon (\mathbf{C}\setminus\{0\}) \cup \{\infty\} = \mathbf{C}P^1 \ni z \mapsto \begin{cases} 1/z & \text{for } z \in \mathbf{C}\setminus\{0\}, \\ 0 & \text{for } z = \infty, \end{cases}$$

is holomorphic.

By (ii), the map $(f \circ g)|D^*$ can be extended to a holomorphic map on D. Hence $f \circ g$ can be extended to a holomorphic map θ: $\mathbf{C}P^1 \to X$. Since $f \circ g \neq$ const, it follows that θ is finite. Take a holomorphic map σ: $D^* \to \mathbf{C} \subset \mathbf{C}P^1$ which cannot be extended to a holomorphic map of D into $\mathbf{C}P^1$. By hypothesis $\beta = \theta\sigma$ can be extended to a holomorphic map $\hat{\beta}$: $D \to X$. Since θ is finite, there exists a neighbourhood U of $\hat{\beta}(0)$ such that $\theta^{-1}(U)$ is isomorphic to a bounded domain in $\mathbf{C}$. Thus for sufficiently small $\varepsilon > 0$, $\sigma|D_\varepsilon^*$ can be extended to a holomorphic map of D_ε into $\theta^{-1}(U)$. This is impossible.

(iii)$\Rightarrow$(i). It suffices to show that $c(M) < \infty$. Assume that $c(M) = \infty$. Then there exists a sequence $\{f_n\} \subset H(D, M)$ such that $\|df_n(0)\| = r_n \to \infty$. For each $n \geq 1$ consider the map g_n: $D_{r_n} \to M$, $z \mapsto f_n(z/r_n)$. Then $\|dg_n(0)\| = 1$. As in [3] there exists a sequence of holomorphic maps φ_n: $D_{r_n} \to M$ which is uniformly convergent on every compact subset of every fixed disc in $\mathbf{C}$ to a holomorphic map φ: $\mathbf{C} \to X$ and $\|d\varphi_n(0)\| = 1$ for every $n \geq 1$. Clearly φ is not constant since $\|d\varphi(0)\| = \lim\|d\varphi_n(0)\| = 1$. Since ∂M contains no limit complex lines, $\varphi(\mathbf{C}) \cap M \neq \emptyset$. Put

$$Z = \{z \in \mathbf{C}\colon \text{there exists an open neighbourhood } U \text{ of } z \text{ such that } \varphi(U) \subset \partial M\}.$$

Assume that $Z \neq \emptyset$. Let $z_0 \in \partial Z$. Take an open neighbourhood U of $\varphi(z_0)$ such that $U \cap M$ is complete hyperbolic. Since $\{\varphi_n\}$ uniformly converges to φ on every compact set in $\mathbf{C}$, there exists an open neighbourhood W of z_0 such that $\varphi_n(W) \subset U \cap M$ for $n \geq l$ and $\varphi(W) \subset U \cap \bar{M}$. From the complete hyperbolicity of $U \cap M$, it follows that $H(W, U \cap M)$ is normal [3].

Assume that the sequence $\{\varphi_n|W\}$ contains a subsequence which is compactly divergent. Without loss of generality we may assume that the sequence $\{\varphi_n | W\}$ is compactly divergent. Choose $z_1 \in W$ such that $\varphi(z_1) \in U \cap M$. Then for every compact neighbourhood K' of $\varphi(z_1)$ in $U \cap M$, there exists j such that $\varphi_n(z_1) \notin K'$ $\forall n \geqslant j$. Hence $\varphi_n(z_1) \not\to \varphi(z_1)$. This is a contradiction.

Thus we may assume that $\{\varphi_n | W\}$ converges to a holomorphic map in $H(W, U \cap M)$. Hence $\lim \varphi_n(z_0) \in U \cap M$. Thus $\varphi(z_0) \in U \cap M$, i.e. $\varphi(z_0) \notin \partial M$. This is a contradiction. Therefore $Z = \emptyset$.

Since M contains no complex lines, $\varphi(\mathbf{C}) \cap \partial M \neq \emptyset$. Let $z_0 \in \mathbf{C}$ such that $\varphi(z_0) \in \partial M$. Reasoning as above, there exists an open neighbourhood U of $\varphi(z_0)$ such that $U \cap M$ is complete hyperbolic and an open neighbourhood W of z_0 such that $\varphi_n(W) \subset U \cap M$ $\forall n \geqslant l$ and $\varphi(W) \subset U \cap \bar{M}$. By $Z = \emptyset$ there exists $z_1 \in W$ such that $\varphi(z_1) \in U \cap M$. Reasoning as above, we may assume that the sequence $\{\varphi_n | W\}$ is convergent in $H(W, U \cap M)$. This, as above, leads to a contradiction. Hence $c(M)$ is finite and the implication (iii) $\Rightarrow$ (i) is proved.

Finally, by [3] if M is as in the theorem and M is hyperbolically embedded in X, then M is complete hyperbolic.

The following is an immediate consequence of the above theorem.

COROLLARY. *Let M be a compact complex space. Then the following are equivalent:*

(i) *M is hyperbolic.*
(ii) *M has the D^*-extension property for M.*
(iii) *M contains no complex lines.*

References

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