ANNALES
POLONICI MATHEMATICI
XXVII (1972)

Some structures on an f-structure manifold

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The idea of f-structure on a differentiable manifold was initiated and developed by Yano [3, 5]. Koto [4] defined and studied certain structures on almost Hermitian manifold, some of which were reformulated by Gray [1] in terms of exterior and co-derivatives. In the present paper we define and study some structures on a differentiable manifold in terms of exterior, Lie, and co-derivatives.

Section 1 is introductory and in Section 2, we define certain structures and prove their inclusion relations, corresponding to the inclusion relations in Gray [1].

In the last section we define a conformal diffeomorphism between two differentiable manifolds and obtain some interesting results relating their structures.

1. An n-dimensional differentiable manifold V is said *to possess an f-structure* [5] if a non-null (1, 1) tensor field f of constant rank r is defined on it which satisfies $f^3+f = 0$. If the rank of f is such that $n-r \geqslant 1$, then there exist two complementary distributions L and M corresponding to the projection operators l and m respectively, defined as [5];

$$l = -f^2 \quad \text{and} \quad m = f^2+I, \tag{1.1}$$

where I denotes the identity operator. These projection operators satisfy the following relations:

$$\begin{aligned} lf &= fl = f, \quad mf = fm = 0,\\ f^2 l &= -l \quad \text{and} \quad f^2 m = 0. \end{aligned} \tag{1.2}$$

The above relations show that f acts as an almost complex structure on L and as a null operator on M. If the rank of f is r, then the dimensions of L and M are r and $(n-r)$ respectively [5].

Let $F(V)$ denote the ring of real-valued differentiable functions on V and $\mathfrak{X}(V)$ the module of derivations of $F(V)$. $\mathfrak{X}(V)$ is then a Lie algebra over real numbers and elements of $\mathfrak{X}(V)$ are called *vector fields*. The (1, 1) tensor field f is then a linear map over $\mathfrak{X}(V)$;

$$f\colon \mathfrak{X}(V) \to \mathfrak{X}(V).$$

Yano [5] has defined a positive definite Riemannian metric $\langle , \rangle$ in V, with respect to which the distributions L and M are orthogonal. Such a Riemannian metric satisfies the following relations [5]

$$(1.3) \qquad \langle X, Y\rangle = \langle fX, fY\rangle + \langle mX, Y\rangle \quad \text{for all } X, Y \in \mathfrak{X}(V).$$

Since L and M are orthogonal, (1.2) yields

$$(1.4) \qquad \langle fX, Y\rangle = \langle f^2X, fY\rangle, \quad \langle X, fY\rangle = \langle fX, f^2Y\rangle.$$

A 2-form F has been defined as [5]

$$(1.5) \qquad F(X, Y) = -F(Y, X) = \langle fX, Y\rangle,$$

and it is easy to verify that

$$(1.6) \qquad F(mX, Y) = 0 = F(X, mY).$$

The Nijenhins tensor N of type (1, 2) is defined as [4]

$$(1.7) \qquad N(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y] \quad \text{for all } X, Y \in \mathfrak{X}(V).$$

2. Using the definitions of the Riemannian connexion ∇_X and the Lie derivative $\mathscr{L}_X$, we have the following relations:

$$(2.1) \quad \nabla_X(f)(Y) = \nabla_X(fY) - f\nabla_X Y, \quad (\mathscr{L}_X f)Y = [X, fY] - f[X, Y].$$

In view of (1.2) and the above relations, we have

$$m\nabla_X(f)(mY) = 0 \quad \text{and} \quad m(\mathscr{L}_X f)(mY) = 0.$$

Since f^2 is also a (1, 1) tensor, we have

$$(2.2) \qquad \nabla_X(f^2)(Y) = \nabla_X(f^2Y) - f^2\nabla_X Y.$$

We can easily check that the covariant derivative $\nabla_X(F)$ and the exterior derivative dF of F are given by the following:

$$(2.3) \qquad \nabla_X(F)(Y, Z) = \langle \nabla_X(f)Y, Z\rangle$$

and

$$(2.4) \qquad dF(X, Y, Z) = \underset{X,Y,Z}{\mathscr{C}} \nabla_X(F)(Y, Z),$$

where $\mathscr{C}$ denotes the cyclic sum over X, Y, Z.

THEOREM 2.1. *By using above formulae we get the following results:*

$$N(X, Y) = \nabla_{fX}(f)Y - \nabla_{fY}(f)X + f\nabla_Y(f)X - f\nabla_X(f)Y, \tag{2.5}$$
$$= (\mathcal{L}_{fX}f)Y - f(\mathcal{L}_X f)Y,$$

$$\nabla_{fX}(F)(fY, fZ) \tag{2.6}$$
$$= dF(fX, fY, fZ) - dF(fX, f^2Y, f^2Z) + \langle fX, N(fY, f^2Z)\rangle,$$

$$2\nabla_{fX}(F)(fY, fZ) + 2\nabla_{f^2X}(F)(f^2Y, fZ) \tag{2.7}$$
$$= dF(fX, fY, fZ) - dF(fX, f^2Y, f^2Z) + dF(fY, f^2Z, f^2X),$$

$$2\nabla_{f^2X}(F)(f^2Y, fZ) - 2\nabla_{fX}(F)(fY, fZ) \tag{2.8}$$
$$= \langle N(fX, f^2Y), fZ\rangle - \langle N(fX, fZ), f^2Y\rangle - \langle N(f^2Y, fZ), fX\rangle.$$

Proof. The proof of (2.5) follows from (2.1) and

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

while (2.6), (2.7) and (2.8) are consequences of (2.5) and the formula

$$\nabla_X(F)(f^2Y, fZ) = \nabla_X(F)(fY, f^2Z). \tag{2.9}$$

We shall call an f-structure manifold *fK-manifold* iff

$$\nabla_{fX}(f) = 0,$$

fAK-manifold iff

$$dF(fX, fY, fZ) = 0,$$

fNK-manifold iff

$$\nabla_{fX}(f)(fY) + \nabla_{fY}(f)(fX) = 0,$$

fQK-manifold iff

$$\nabla_{fX}(f)(fY) + \nabla_{f^2X}(f)(f^2Y) = 0,$$

and *fH-manifold* iff

$$N(fX, fY) = 0$$

for all $X, Y, Z \in \mathfrak{X}(V)$.

As a consequence of theorem (2.1) and the definitions of fH and fQK-manifold we get the following

THEOREM 2.2. $(\mathcal{L}_{f^2X}f)(fY) = f(\mathcal{L}_{fX}f)(fY)$ *for all* $X, Y \in \mathfrak{X}(V)$ *if and only if the manifold* V *is* fH, *while*

$$\nabla_{fX}(F)(fY, fZ) = -\nabla_{f^2X}(F)(f^2Y, fZ)$$

for all $X, Y, Z \in \mathfrak{X}(V)$ *if and only if the manifold* V *is fQK-manifold.*

We next study the inclusion relations between the special f-structure manifolds defined above and prove

THEOREM 2.3.

$$fK \begin{Bmatrix} \subseteq fAK \\ \subseteq fNK \end{Bmatrix} \subseteq fQK \quad \textit{and} \quad fK \subseteq fH.$$

Furthermore,

$$fK \subseteq fH \cap fQK \subseteq fAK \cap fNK.$$

Proof. That $fK \subseteq fAK$ follows from (2.3) and (2.4); $fAK \subseteq fQK$ follows from (2.3) and (2.7); while $fK \subseteq fH$ follows from (2.5). It is obvious that $fK \subseteq fNK$, while $fNK \subseteq fQK$ is a consequence of (2.9).

Furthermore, $fK \subseteq fH \cap fQK$ is obvious.

If the (1, 1) tensor field f satisfies

$$\nabla_{fX}(f)Y = f\nabla_X(f)Y, \tag{2.10}$$

then from (2.5)

$$N(X, Y) = 0,$$

and we get

THEOREM 2.4. *An f-structure manifold V is fH-manifold if the* (1, 1) *tensor field f satisfies*

$$\nabla_{fX}(f)Y = f\nabla_X(f)Y.$$

Also, if the f-structure satisfies (2.10), then

$$\nabla_{fX}(f)(fY) + \nabla_{f^2X}(f)(f^2Y) = f\nabla_X(f)(fY) + f^2\nabla_X(f)(f^2Y).$$

In view of (2.1) and the above result, we get

$$\nabla_{fX}(f)(fY) + \nabla_{f^2X}(f)(f^2Y) = 2f\nabla_X(f)(fY),$$

which provides the proof of the following

THEOREM 2.5. *An f-structure manifold V which satisfies* (2.10), *is fQK-manifold iff*

$$f\nabla_X(f)(fY) = 0.$$

3. Conformal diffeomorphism of f-structure manifolds. Let $(V, \langle , \rangle)$ and $(V^0, \langle , \rangle^0)$ be two Riemannian manifolds and $\Phi: V \to V^0$ be a diffeomorphism. If $X \in \mathfrak{X}(V)$, we denote by $X^0 \in \mathfrak{X}(V^0)$ the vector field corresponding to X induced by Φ. Then Φ is called a *conformal diffeomorphism* provided there exists $\sigma \in F(V)$ such that

$$\langle X^0, Y^0\rangle^0 \cdot \Phi = e^{2\sigma}\langle X, Y\rangle \tag{3.1}$$

for all $X, Y \in \mathfrak{X}(V)$. For $g \in F(V)$ we define $\operatorname{grad} g$ by

$$\langle \operatorname{grad} g, X\rangle = X(g) \tag{3.2}$$

for all $X \in \mathfrak{X}(V)$. The Riemannian connections ∇^0 and ∇ of V^0 and V satisfy the following relation [1]

$$(3.3)\qquad \nabla^0_{X^0} Y^0 = \{\nabla_X Y + X(\sigma) Y + Y(\sigma) X - \langle X, Y\rangle \operatorname{grad}\sigma\}^0.$$

Let V and V^0 be f-structure manifolds respectively. Suppose that Φ: $V \to V^0$ in addition to being a conformal diffeomorphism also preserves the f-structure, i.e. there exists a $(1, 1)$ tensor field f^0: $\mathfrak{X}(V^0) \to \mathfrak{X}(V^0)$ in V^0 such that

$$(3.4)\qquad f^0 X^0 = (fX)^0.$$

If $\langle , \rangle^0$ is the Riemannian metric in V^0, then this metric satisfies following relations:

$$\langle f^0 X^0, Y^0\rangle^0 = \langle (f^0)^2 X^0, f^0 Y^0\rangle^0$$

and

$$\langle X^0, f^0 Y^0\rangle^0 = \langle f^0 X^0, (f^0)^2 Y^0\rangle^0.$$

If Φ^* is the map induced by Φ which takes differential forms on V^0 back to the differential forms on V, then we have the following

THEOREM 3.1. *The structures of the spaces V and V^0 are related by the following*:

$$(3.5)\qquad F^0(X^0, Y^0)\cdot\Phi = e^{2\sigma} F(X, Y),$$

$$(3.6)\qquad \Phi^* F^0 = e^{2\sigma} F,$$

$$(3.7)\qquad \Phi^*(dF^0) = e^{2\sigma}\{2d\sigma \wedge F + dF\},$$

$$(3.8)\qquad \nabla^0_{X^0}(f^0) Y^0 = \{\nabla_X(f) Y + fY(\sigma)_X - Y(\sigma)(fX) + \langle fX, Y\rangle \operatorname{grad}\sigma + \\ + \langle X, Y\rangle f \operatorname{grad}\sigma\}^0,$$

$$(3.9)\qquad \nabla^0_{X^0}(F^0)(Y^0, Z^0)\cdot\Phi = e^{2\sigma}\{\nabla_X(F)(Y, Z) + fY(\sigma)\langle X, Z\rangle - \\ - Y(\sigma)F(X, Z) + F(X, Y)Z(\sigma) - \langle X, Y\rangle fZ(\sigma)\},$$

$$(3.10)\qquad N^0(X^0, Y^0) = \{N(X, Y)\}^0$$

for all $X, Y, Z \in \mathfrak{X}(V)$, where N^0 is the Nijenhuis tensor and F^0 is a 2-form in V^0 defined by

$$(3.11)\qquad F^0(X^0, Y^0) = \langle f^0 X^0, Y^0\rangle^0.$$

Proof. The proof of (3.5) follows from (3.1) and (3.4); (3.6) and (3.7) follow from the definition of Φ^* and (3.4); (3.8) follows from (2.1) and (3.3); (3.9) is a direct consequence of (2.3) and (3.8); while (3.10) follows from (2.5) and (3.8).

THEOREM 3.2. *Let Φ: $V \to V^0$ be a conformal diffeomorphism between f-structure manifolds. If $V \in fH$, then $V^0 \in fH$. On the other hand, suppose*

$\dim V \geq 3$ *and* Φ *is not homothetic; then if* V *is in one of the classes* fK, fAK, fNK *or* fQK, *then* V^0 *is not in any of the classes* fK, fAK, fNK *or* fQK.

Proof. If $V \in fH$, then from (3.10) it follows that $V^0 \in fH$. Next, if V is in one of the classes fK, fAK, fNK, fQK, then in view of theorem (2.3) V is necessarily fQK, and consequently theorem (3.1) shows that V^0 is not fQK and therefore cannot be in any of the classes fK, fAK, fNK or fQK.

Since V^0 is also an f-structure manifold, we define the complementary projection operators l^0 and m^0 in V^0 corresponding to the projection operators l and m in V, as follows:

$$l^0 = -(f^0)^2 \quad \text{and} \quad m^0 = (f^0)^2 + I^0, \tag{3.12}$$

where I^0 is the identity operator in V^0. From (3.4) we get

$$l^0 X^0 = (lX)^0 \quad \text{and} \quad m^0 X^0 = (mX)^0. \tag{3.13}$$

Let L^0 and M^0 be the distributions corresponding to operators l^0 and m^0 in V^0 respectively. Then from (3.11) and (3.13) we have the following

THEOREM 3.3.

$$N^0(m^0 X^0, m^0 Y^0) = \{N(mX, mY)\}^0, \tag{3.14}$$

$$N^0(l^0 X^0, l^0 Y^0) = \{N(lX, lY)\}^0, \tag{3.15}$$

$$N^0(l^0 X^0, m^0 Y^0) = \{N(lX, mY)\}^0. \tag{3.16}$$

The above theorem together with relation (3.10) provides the proof of the following

THEOREM 3.4. *The distribution* L *is integrable in* V *if and only if the distribution* L^0 *is integrable in* V^0.

THEOREM 3.5. *The distribution* M *is integrable in* V *if and only if the distribution* M^0 *is integrable in* V^0.

THEOREM 3.6. *The distributions* L *and* M *are both integrable in* V *if and only if the distributions* L^0 *and* M^0 *are both integrable in* V^0.

If the distribution L is integrable and, moreover, if the almost complex structure f' induced from f on each integral manifold of L is integrable, then we say that the f-structure is partially integrable [3]. A necessary and sufficient condition for an f-structure to be partially integrable is [3]

$$N(lX, lY) = 0;$$

using equation (3.15), we have the following

THEOREM 3.7. *The* f*-structure in* V *is partially integrable if and only if the* f*-structure is partially integrable in* V^0.

Also the f-structure is integrable in V [3] iff

$$N(X, Y) = 0$$

and consequently in view of (3.10), we have

THEOREM 3.8. *The f-structure is integrable in V if and only if the f-structure is integrable in V^0.*

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Reçu par la Rédaction le 21. 5. 1971