Some structures on an \( f \)-structure manifold

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The idea of \( f \)-structure on a differentiable manifold was initiated and developed by Yano [3, 5]. Koto [4] defined and studied certain structures on almost Hermitian manifold, some of which were reformulated by Gray [1] in terms of exterior and co-derivatives. In the present paper we define and study some structures on a differentiable manifold in terms of exterior, Lie, and co-derivatives.

Section 1 is introductory and in Section 2, we define certain structures and prove their inclusion relations, corresponding to the inclusion relations in Gray [1].

In the last section we define a conformal diffeomorphism between two differentiable manifolds and obtain some interesting results relating their structures.

1. An \( n \)-dimensional differentiable manifold \( V \) is said to possess an \( f \)-structure [5] if a non-null \((1, 1)\) tensor field \( f \) of constant rank \( r \) is defined on it which satisfies \( f^2 + f = 0 \). If the rank of \( f \) is such that \( n - r \geq 1 \), then there exist two complementary distributions \( L \) and \( M \) corresponding to the projection operators \( l \) and \( m \) respectively, defined as [5];

\[
(1.1) \quad l = -f^2 \quad \text{and} \quad m = f^2 + I,
\]

where \( I \) denotes the identity operator. These projection operators satisfy the following relations:

\[
(1.2) \quad lf = fl = f, \quad mf = fm = 0, \\
\quad f^2 l = -l \quad \text{and} \quad f^2 m = 0.
\]

The above relations show that \( f \) acts as an almost complex structure on \( L \) and as a null operator on \( M \). If the rank of \( f \) is \( r \), then the dimensions of \( L \) and \( M \) are \( r \) and \( (n-r) \) respectively [5].

Let \( F(V) \) denote the ring of real-valued differentiable functions on \( V \) and \( \mathfrak{X}(V) \) the module of derivations of \( F(V) \). \( \mathfrak{X}(V) \) is then a Lie algebra
over real numbers and elements of $\mathfrak{X}(V)$ are called vector fields. The $(1, 1)$ tensor field $f$ is then a linear map over $\mathfrak{X}(V)$;

$$f: \mathfrak{X}(V) \rightarrow \mathfrak{X}(V).$$

Yano [5] has defined a positive definite Riemannian metric $\langle \cdot, \cdot \rangle$ in $V$, with respect to which the distributions $L$ and $M$ are orthogonal. Such a Riemannian metric satisfies the following relations [5]

$$\langle X, Y \rangle = \langle fX, fY \rangle + \langle mX, Y \rangle \quad \text{for all } X, Y \in \mathfrak{X}(V).$$

Since $L$ and $M$ are orthogonal, (1.2) yields

$$\langle fX, Y \rangle = \langle f^2X, fY \rangle, \quad \langle X, fY \rangle = \langle fX, f^2Y \rangle.$$

A 2-form $F$ has been defined as [5]

$$F(X, Y) = -F(Y, X) = \langle fX, Y \rangle,$$

and it is easy to verify that

$$F(mX, Y) = 0 = F(X, mY).$$

The Nijenhins tensor $N$ of type $(1, 2)$ is defined as [4]

$$N(X, Y) = \{fX, fY\} - f\{fX, Y\} - f\{X, fY\} + f^2\{X, Y\} \quad \text{for all } X, Y \in \mathfrak{X}(V).$$

2. Using the definitions of the Riemannian connexion $V_X$ and the Lie derivative $\mathcal{L}_X$, we have the following relations:

$$V_X(f)(Y) = V_X(fY) - fV_XY, \quad (\mathcal{L}_X f)Y = [X, fY] - f[X, Y].$$

In view of (1.2) and the above relations, we have

$$mV_X(f)(mY) = 0 \quad \text{and} \quad m(\mathcal{L}_X f)(mY) = 0.$$

Since $f^2$ is also a $(1, 1)$ tensor, we have

$$V_X(f^2)(Y) = V_X(f^2 Y) - f^2V_XY.$$

We can easily check that the covariant derivative $V_X(F)$ and the exterior derivative $dF$ of $F$ are given by the following:

$$V_X(F)(Y, Z) = \langle V_X(f)Y, Z \rangle$$

and

$$dF(X, Y, Z) = \sum_{x,y,z} V_X(F)(Y, Z),$$

where $\sum$ denotes the cyclic sum over $X, Y, Z$. 
THEOREM 2.1. By using above formulae we get the following results:

(2.5) \[ N(X, Y) = V_{fX}(f) Y - V_{fY}(f) X + fV_X(f) X - fV_Y(f) Y, \]
\[ = (\mathcal{L}_{fX}) Y - f(\mathcal{L}_X f) Y, \]

(2.6) \[ V_{fX}(F)(fY, fZ) = dF(fX, fY, fZ) - dF(fX, f^2 Y, f^2 Z) + \langle fX, N(fY, f^2 Z) \rangle, \]

(2.7) \[ 2V_{fX}(F)(fY, fZ) + 2V_{f^2 X}(F)(f^2 Y, fZ) \]
\[ = dF(fX, fY, fZ) - dF(fX, f^2 Y, f^2 Z) + dF(fY, f^2 Z, f^2 X), \]

(2.8) \[ 2V_{f^2 X}(F)(f^2 Y, fZ) - 2V_{fX}(F)(fY, fZ) \]
\[ = \langle N(fX, f^2 Y), fZ \rangle - \langle N(fX, fZ), f^2 Y \rangle - \langle N(fY, fZ), fX \rangle. \]

Proof. The proof of (2.5) follows from (2.1) and
\[ V_X Y - V_Y X = [X, Y], \]
while (2.6), (2.7) and (2.8) are consequences of (2.5) and the formula

(2.9) \[ V_X(F)(f^2 Y, fZ) = V_X(F)(fY, f^2 Z). \]

We shall call an $f$-structure manifold $fK$-manifold iff
\[ V_{fX}(f) = 0, \]
$fAK$-manifold iff
\[ dF(fX, fY, fZ) = 0, \]
$fNK$-manifold iff
\[ V_{fX}(fY) + V_{fY}(f) = 0, \]
$fQK$-manifold iff
\[ V_{fX}(fY) + V_{f^2 X}(f) = 0, \]
and $fH$-manifold iff
\[ N(fX, fY) = 0 \]
for all $X, Y, Z \in \mathfrak{X}(V)$.

As a consequence of theorem (2.1) and the definitions of $fH$ and $fQK$-manifold we get the following

THEOREM 2.2. \[ (\mathcal{L}_{f^2 X} f)(fY) = f(\mathcal{L}_{fX} f)(fY) \] for all $X, Y \in \mathfrak{X}(V)$ if and only if the manifold $V$ is $fH$, while
\[ V_{fX}(F)(fY, fZ) = -V_{f^2 X}(F)(f^2 Y, fZ) \]
for all $X, Y, Z \in \mathfrak{X}(V)$ if and only if the manifold $V$ is $fQK$-manifold.

We next study the inclusion relations between the special $f$-structure manifolds defined above and prove
THEOREM 2.3.

\[ fK \begin{cases} \equiv fAK \\ \equiv fNK \end{cases} \subseteq fQK \quad \text{and} \quad fK \subseteq fH. \]

Furthermore,

\[ fK \subseteq fH \cap fQK \subseteq fAK \cap fNK. \]

Proof. That \( fK \subseteq fAK \) follows from (2.3) and (2.4); \( fAK \subseteq fQK \) follows from (2.3) and (2.7); while \( fK \subseteq fH \) follows from (2.5). It is obvious that \( fK \subseteq fNK \), while \( fNK \subseteq fQK \) is a consequence of (2.9).

Furthermore, \( fK \subseteq fH \cap fQK \) is obvious.

If the \((1,1)\) tensor field \( f \) satisfies

\[ \nabla_{fx}(f)Y = f\nabla_x(f)Y, \]

then from (2.5)

\[ N(X, Y) = 0, \]

and we get

THEOREM 2.4. An \( f \)-structure manifold \( V \) is \( fH \)-manifold if the \((1,1)\) tensor field \( f \) satisfies

\[ \nabla_{fx}(f)Y = f\nabla_x(f)Y. \]

Also, if the \( f \)-structure satisfies (2.10), then

\[ \nabla_{fx}(fY) + \nabla_{fx}(f^2)Y = f\nabla_x(fY) + f^2\nabla_x(fY). \]

In view of (2.1) and the above result, we get

\[ \nabla_{fx}(fY) + \nabla_{fx}(f^2)Y = 2f\nabla_x(fY), \]

which provides the proof of the following

THEOREM 2.5. An \( f \)-structure manifold \( V \) which satisfies (2.10), is \( fQK \)-manifold iff

\[ f\nabla_x(fY) = 0. \]

3. Conformal diffeomorphism of \( f \)-structure manifolds. Let \((V, \langle, \rangle)\) and \((V^0, \langle, \rangle^0)\) be two Riemannian manifolds and \( \Phi: V \to V^0 \) be a diffeomorphism. If \( X \in \mathfrak{X}(V) \), we denote by \( X^0 \in \mathfrak{X}(V^0) \) the vector field corresponding to \( X \) induced by \( \Phi \). Then \( \Phi \) is called a conformal diffeomorphism provided there exists \( \sigma \in F(V) \) such that

\[ \langle X^0, Y^0 \rangle^0 = e^{2\sigma} \langle X, Y \rangle \]

for all \( X, Y \in \mathfrak{X}(V) \). For \( g \in F(V) \) we define \( \text{grad} g \) by

\[ \langle \text{grad} g, X \rangle = X(g) \]
for all $X \in \mathfrak{X}(V)$. The Riemannian connections $V^0$ and $V$ of $V^0$ and $V$ satisfy the following relation [1]

$$V^0_{X^0} Y^0 = \{V_X Y + X(\sigma) Y + Y(\sigma) X - \langle X, Y \rangle \text{ grad } \sigma \rangle^0.$$  

(3.3)

Let $V$ and $V^0$ be $f$-structure manifolds respectively. Suppose that $\Phi: V \rightarrow V^0$ in addition to being a conformal diffeomorphism also preserves the $f$-structure, i.e. there exists a $(1, 1)$ tensor field $f^0: \mathfrak{X}(V^0) \rightarrow \mathfrak{X}(V^0)$ in $V^0$ such that

$$f^0 X^0 = (fX)^0.$$  

(3.4)

If $\langle , \rangle^0$ is the Riemannian metric in $V^0$, then this metric satisfies following relations:

$$\langle f^0 X^0, Y^0 \rangle^0 = \langle (f^0)^2 X^0, f^0 Y^0 \rangle^0$$

and

$$\langle X^0, f^0 Y^0 \rangle^0 = \langle f^0 X^0, (f^0)^2 Y^0 \rangle^0.$$  

If $\Phi^*$ is the map induced by $\Phi$ which takes differential forms on $V^0$ back to the differential forms on $V$, then we have the following

**Theorem 3.1.** The structures of the spaces $V$ and $V^0$ are related by the following:

$$F^0(X^0, Y^0) \cdot \Phi = e^{2\sigma} F(X, Y),$$

(3.5)

$$\Phi^* F^0 = e^{2\sigma} F,$$

(3.6)

$$\Phi^* (dF^0) = e^{2\sigma} [2d\sigma \wedge F + dF],$$

(3.7)

$$\mathcal{P}^0_\mathcal{X}^0 (f^0) Y^0 = \{V_X (f) Y + fY(\sigma) X - Y(\sigma) (fX) + \langle fX, Y \rangle \text{ grad } \sigma + \langle X, Y \rangle f \text{ grad } \sigma \},$$

(3.8)

$$\mathcal{P}^0_\mathcal{X}^0 (F^0)(Y^0, Z^0) \cdot \Phi = e^{2\sigma} \{V_X (F)(Y, Z) + fY(\sigma) \langle X, Z \rangle - Y(\sigma) F(X, Z) + F(X, Y) Z(\sigma) - X, Y \rangle f \text{ grad } \sigma \},$$

(3.9)

$$\mathcal{N}^0(X^0, Y^0) = [N(X, Y)]^0$$

for all $X, Y, Z \in \mathfrak{X}(V)$, where $N^0$ is the Nijenhuis tensor and $F^0$ is a 2-form in $V^0$ defined by

$$F^0(X^0, Y^0) = \langle f^0 X^0, Y^0 \rangle^0.$$  

(3.10)

**Proof.** The proof of (3.5) follows from (3.1) and (3.4); (3.6) and (3.7) follow from the definition of $\Phi^*$ and (3.4); (3.8) follows from (2.1) and (3.3); (3.9) is a direct consequence of (2.3) and (3.8); while (3.10) follows from (2.5) and (3.8).

**Theorem 3.2.** Let $\Phi: V \rightarrow V^0$ be a conformal diffeomorphism between $f$-structure manifolds. If $V \in fH$, then $V^0 \in fH$. On the other hand, suppose
\[ \dim V \geq 3 \text{ and } \Phi \text{ is not homothetic; then if } V \text{ is in one of the classes } fK, fAK, fNK \text{ or } fQK, \text{ then } V^0 \text{ is not in any of the classes } fK, fAK, fNK \text{ or } fQK. \]

Proof. If \( V \in fH \), then from (3.10) it follows that \( V^0 \in fH \). Next, if \( V \) is in one of the classes \( fK, fAK, fNK, fQK \), then in view of theorem (2.3) \( V \) is necessarily \( fQK \), and consequently theorem (3.1) shows that \( V^0 \) is not \( fQK \) and therefore cannot be in any of the classes \( fK, fAK, fNK \) or \( fQK \).

Since \( V^0 \) is also an \( f \)-structure manifold, we define the complementary projection operators \( l^0 \) and \( m^0 \) in \( V^0 \) corresponding to the projection operators \( l \) and \( m \) in \( V \), as follows:

\[ l^0 = -(f^0)^2 \quad \text{and} \quad m^0 = (f^0)^2 + I^0, \]

where \( I^0 \) is the identity operator in \( V^0 \). From (3.4) we get

\[ l^0 X^0 = (lX)^0 \quad \text{and} \quad m^0 X^0 = (mX)^0. \]

Let \( L^0 \) and \( M^0 \) be the distributions corresponding to operators \( l^0 \) and \( m^0 \) in \( V^0 \) respectively. Then from (3.11) and (3.13) we have the following

**Theorem 3.3.**

\[ N^0(m^0 X^0, m^0 Y^0) = \{N(mX, mY)\}^0, \]

\[ N^0(l^0 X^0, l^0 Y^0) = \{N(lX, lY)\}^0, \]

\[ N^0(l^0 X^0, m^0 Y^0) = \{N(lX, mY)\}^0. \]

The above theorem together with relation (3.10) provides the proof of the following

**Theorem 3.4.** The distribution \( L \) is integrable in \( V \) if and only if the distribution \( L^0 \) is integrable in \( V^0 \).

**Theorem 3.5.** The distribution \( M \) is integrable in \( V \) if and only if the distribution \( M^0 \) is integrable in \( V^0 \).

**Theorem 3.6.** The distributions \( L \) and \( M \) are both integrable in \( V \) if and only if the distributions \( L^0 \) and \( M^0 \) are both integrable in \( V^0 \).

If the distribution \( L \) is integrable and, moreover, if the almost complex structure \( f' \) induced from \( f \) on each integral manifold of \( L \) is integrable, then we say that the \( f \)-structure is partially integrable [3]. A necessary and sufficient condition for an \( f \)-structure to be partially integrable is [3]

\[ N(lX, lY) = 0; \]

using equation (3.15), we have the following

**Theorem 3.7.** The \( f \)-structure in \( V \) is partially integrable if and only if the \( f \)-structure is partially integrable in \( V^0 \).
Also the $f$-structure is integrable in $V$ [3] iff
\[ N(X, Y) = 0 \]
and consequently in view of (3.10), we have

**Theorem 3.8.** The $f$-structure is integrable in $V$ if and only if the $f$-structure is integrable in $V^0$.

**References**


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