COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 1

A CLASSIFICATION SCHEME AND CHARACTERIZATION OF CERTAIN CURVES

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1. Introduction. The theory of curves, i.e., 1-dimensional continua, was developed in large part in the 1920's and early 1930's with much of this development due to Karl Menger, to G. T. Whyburn, and to C. Kuratowski, and contained in their books [4, 5, 3].

This paper is devoted to characterizing some known types of curves, developing additional ones, and refining and adding to a classification of curves as described by Whyburn and Kuratowski. Several of the theorems proved here have easy corollaries giving characterizations of simple closed curves and dendrites.

Let $\mathscr{C}$ be the class of all continua that contain no continua of condensation, $\mathscr{R}_e$ the class of all regular curves, $\mathscr{H}$ the class of all hereditarily locally connected (hlc) continua, $\mathscr{R}_a$ the class of all rational curves and $\mathscr{I}$ the class of all curves. Then it is known ([3], p. 210, and [5], p. 99) that the following implications hold but that none can be reversed: $\mathscr{C} \subset \mathscr{R}_e \subset \mathscr{H} \subset \mathscr{R}_a \subset \mathscr{I}$. It is the purpose of the next section to develop some additional classes of curves and insert them in this classification scheme.

Many of the terms used in this paper are defined as they occur. All those used but not defined may be found in [3] and [5]. Always a *continuum* will be a compact, metric, closed and connected point set.

2. Some additional types of curves. A non-degenerate subcontinuum C of the continuum M is a *strong continuum of condensation* if C is contained in the closure of one of the components of $M-C$.

THEOREM 1. *Let M be a continuum that contains no strong continuum of condensation. Then M is hereditarily locally connected* (hlc).

Proof. An hereditarily locally connected continuum is characterized by containing no non-degenerate continuum of convergence; so let us assume that M contains a non-degenerate continuum of convergence K, i.e., $K = \lim K_i$, where $K_i \cdot K_j = \emptyset$ for $i \neq j$ and $K_i \cdot K = \emptyset$ for all i. Let P and Q be disjoint non-degenerate subcontinua of K. Now by [3],

p. 89, an infinite number of the K_i's lie in a component of $M-P$ or $M-Q$; so either P or Q is a strong continuum of condensation, contrary to hypothesis.

If $\mathscr{K}$ denotes the class of all continua that contain no strong continuum of condensation, then $\mathscr{C} \subset \mathscr{K} \subset \mathscr{H}$. Kuratowski [3], p. 210, gives an

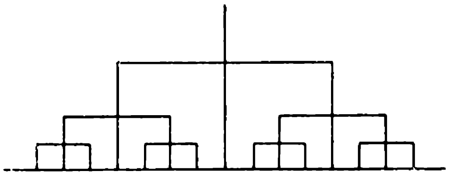

Fig. 1

example of a continuum that is hereditarily locally connected and is not a regular curve. This continuum also has no strong continuum of condensation which shows that $\mathscr{K} \not\subset \mathscr{R}_e$. The next example shows that $\mathscr{R}_e \not\subset \mathscr{K}$ so although $\mathscr{R}_e$ and $\mathscr{K}$ both lie between $\mathscr{C}$ and $\mathscr{H}$ neither is a subset of the other.

Example 1. In the plane let M consist of: (1) the interval $[0,1]$ of the x-axis, (2) the vertical interval of length $1/2$ with base $(1/2,0)$, two vertical intervals of length $1/4$ with bases at $(1/4,0)$ and $(3/4,0)$, four vertical intervals of length $1/8$ with bases at $(1/8,0)$, $(3/8,0)$, $(5/8,0)$, and $(7/8,0)$, etc., and (3) the horizontal interval with end points $(1/4,1/4)$, $(3/4,1/4)$, two horizontal intervals with end points at $(1/8,1/8)$, $(3/8,1/8)$, and $(5/8,1/8)$, $(7/8,1/8)$, etc.

This continuum (Fig. 1) is a regular curve but contains a strong continuum of condensation, e.g., the interval $[0,1]$.

A continuum that has no strong continuum of condensation may have a continuum of condensation, i.e., $\mathscr{C} \neq \mathscr{K}$, as the next example shows.

Example 2. Let M be the continuum that is the sum of the intervals in steps (1) and (2) of Example 1 (the continuum of Fig. 1 without the horizontal intervals except for the base). The interval $[0,1]$ is a continuum of condensation although M contains no strong continuum of condensation.

There is another way of describing a continuum that contains no strong continuum of condensation that is useful. A continuum M contains no strong continuum of condensation if and only if for every subcontinuum C of M and component K of $M-C$, the closure of K in C is totally disconnected, i.e., $\bar{K}\cdot C$ contains no non-degenerate component. With this in mind Theorem 1 has some immediate corollaries.

COROLLARY. *If for every non-degenerate subcontinuum C in the continuum M and for every component K of $M-C$, $\bar{K}\cdot C$ consists of exactly two points, then M is a simple closed curve.*

COROLLARY. *If for every subcontinuum C in the continuum M and every component K of $M-C$, $\bar{K}\cdot C$ consists of one point, then M is a dendrite and conversely.*

COROLLARY. *If for every subcontinuum C in the continuum M and every component K of $M-C$, $\bar{K}\cdot C$ consists of one point and in addition the complement of every continuum in M has at most two components, then M is an arc.*

The continuum M *is aposyndetic at x with respect to y* if there is an open set U and continuum H such that $x\in U\subset H\subset M-y$. If for every point $y\in M-x$ there is an open set U and continuum H so that $x\in U \subset H\subset M-y$, then M is *aposyndetic at x*. If $x\in M$ and for every pair of points y and z in $M-x$, there is an open set U and continuum H such that $x\in U\subset H\subset M-(y+z)$, then M is *2-aposyndetic at x*.

A more general result using these concepts can be obtained by weakening the hypothesis of Theorem 1.

THEOREM 2. *Let M be a continuum and suppose that for every continuum $C\subset M$ and every point $x\notin C$, there is a continuum C' containing C but not x such that the component K_x of $M-C'$ containing x has a totally disconnected closure in C', i.e., the components of $\bar{K}_x\cdot C'$ are degenerate. Then M is aposyndetic at every point of a dense subset of M.*

Proof. Suppose U is an open set in M and M is aposyndetic at no point in U. Let Q be a countable dense subset of M. According to a theorem of Grace [1], p. 102, there exist $x\in U$ and $y\in M$ such that M is not aposyndetic at x with respect to y, and such that y cuts x from each point of $Q-y$. Let V be an open set such that $x\in V\subset\bar{V}\subset U-y$. Let $x_1, x_2, \ldots$ be a sequence of points in $Q\cdot V$ converging to x such that each x_i lies in a component K_i of $\bar{V}$ where $K_i\neq K_j$ if $i\neq j$. Let C be the component of $\bar{V}$ containing x. Then $C\cdot K_i=\emptyset$ for all i. Let C' be a continuum containing C but not y so that the component K_y of $M-C'$ containing y has a totally disconnected closure in C'. Because y cuts x from each x_i, every K_i lies in the same component of $M-C'$ as y, that component being K_y. But $\limsup K_i$ will be a non-degenerate subcontinuum C'' of C. Then $\bar{K}_y\cdot C'$ contains C'', which is a contradiction.

In the conclusion of Theorem 2 nothing stronger than aposyndetic at every point of a dense subset can be claimed as the following examples show.

Example 3. Let M consist of concentric squares centered at (1/2,0) on the x-axis and passing through each point of the Cantor discontinuum on the interval $[0, 1]$ of the x-axis, together with alternating spanning arcs which are considered to be points. This continuum, illustrated in Fig. 2, satisfies the hypothesis of Theorem 2, is aposyndetic at all of its points but is 2-aposyndetic at none of its points.

To show that M need not even be aposyndetic at all of its points under the hypothesis of Theorem 2, consider the next example.

Example 4. Let M (Fig. 3) consist of (1) the vertical arcs in the plane from $(-1/n, 0)$ to $(-1/n, 1)$ and from $(1/n, 0)$ to $(1/n, 1)$ for

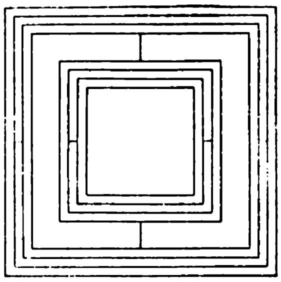

Fig. 2

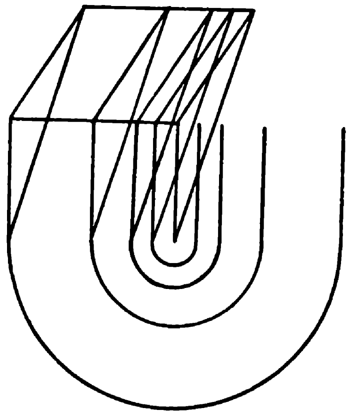

Fig. 3

$n = 1, 2, \ldots$; (2) arcs perpendicular to the plane from $(-1/n, 1, 0)$ to $(-1/n, 1, 1)$ for $n = 1, 2, \ldots$; (3) arcs joining $(-1/n, 1, 1)$ to $(-1/n, 0, 0)$ for $n = 1, 2, \ldots$; (4) the semi-circles in the xy-plane, $x^2+y^2 = 1/n^2$, $y \leqslant 0$ for $n = 1, 2, \ldots$; (5) the arcs from $(0, 0, 0)$ to $(0, 1, 0)$, from $(0, 0, 0)$ to $(0, 1, 1)$, from $(0, 1, 0)$ to $(0, 1, 1)$, from $(-1, 1, 0)$ to $(0, 1, 0)$ and from $(-1, 1, 1)$ to $(0, 1, 1)$. This continuum is not aposyndetic at any point of the half open interval $(0, 1)$ on the y-axis in the xy-plane.

If in Example 4 the subset of the continuum perpendicular to the plane is altered as in Fig. 4, then $\overline{K}_x \cdot C'$ in Theorem 2 is finite for all points x instead of just totally disconnected.

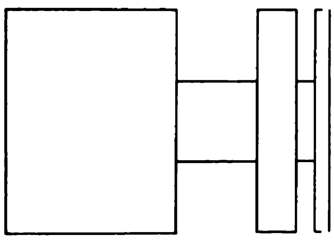

Fig. 4

Remark. As seen from Example 3 and the altered Example 4, for each continuum C and each $x \notin C$, there is a continuum C' containing C but not x such that if K_x is the component of $M - C'$ containing x, then $\overline{K}_x \cdot C'$ is finite. In fact, for most of the points in M, $\overline{K}_x \cdot C'$ consists of one or two points. So requiring $\overline{K}_x \cdot C'$ to be finite rather than totally disconnected does not result in a stronger conclusion. However, if $\overline{K}_x \cdot C'$

is required to be one point only, then M must be locally connected; in fact this characterizes a dendrite.

Curves with the property of Theorem 2 as we have seen can be rather general. In the following theorem this property is strengthened in a way that forces M to be a curve that is aposyndetic at *all* points.

THEOREM 3. *Suppose that for every subcontinuum C of the continuum M and every point $x \notin C$, there is a continuum C' containing C but not x so that the boundary of C',* $\mathrm{Fr}(C')$, *is totally disconnected. Then M is a curve that is aposyndetic at each of its points.*

Proof. Suppose that M is not aposyndetic at x with respect to y. Then by [2], p. 405, y plus the set of all points p in M such that M is not aposyndetic at p with respect to y is a continuum, N. Let C be a nondegenerate subcontinuum of N not containing y and let C' be a subcontinuum of M containing C but not y such that $\mathrm{Fr}(C')$ is totally disconnected. Now C must intersect the interior of C' and since $C \subset N$, there must be points of N in the interior of C'. This makes M aposyndetic at these points of N with respect to y which is a contradiction. It is immediate that M is a 1-dimensional continuum and the theorem is established.

Example 3 gives a continuum satisfying the hypothesis of the theorem that is 2-aposyndetic at none of its points. So it is impossible to claim anything stronger than aposyndetic for a continuum with this property.

Suppose we let $(*)$ denote the class of all continua satisfying the hypothesis of Theorem 3 except that $\mathrm{Fr}(C')$ is to be countable. Of course, every continuum M in $(*)$ is connected im kleinen on a dense subset of M as well as being aposyndetic at all of its points since M is now a rational curve. With this class we can refine the classification of curves between $\mathscr{H}$ (the hlc continua) and $\mathscr{R}_a$ (the rational curves). To aid in this refinement let $\mathscr{R}_{\mathrm{lc}}$ be the class of all rational, locally connected curves and $\mathscr{R}_{\mathrm{apo}}$ the class of all rational aposyndetic curves. Except for obvious arguments we have established:

$$\mathscr{H} \subset \mathscr{R}_{\mathrm{lc}} \subset (*) \subset \mathscr{R}_{\mathrm{apo}} \subset \mathscr{R}_a .$$

These inclusions cannot be reversed as shown by the next examples.

Example 5. Let M consist of the arcs in the plane from $(1/n, 0)$ to $(1/n, 1)$ for $n = 1, 2, \ldots$ together with the arcs from $(0, 0)$ to $(0, 1)$, from $(0, 1)$ to $(1, 1)$, and $(0, 0)$ to $(1, 0)$. Then $M \in (*)$, but $M \notin \mathscr{R}_{\mathrm{lc}}$.

Example 6. In this example, $M \in \mathscr{R}_{\mathrm{apo}}$ but $M \notin (*)$. Let M be comprised of a sequence of rectangles converging to a point p where each rectangle has in its interior a sequence of broken line segments converging to the perimeter (Fig. 5).

Another way of describing a continuum belonging to $(*)$ is provided by the next theorem. The proof is easy and is omitted.

THEOREM 4. *A continuum M belongs to $(*)$ if and only if for every pair of points $x, y \in M$, there is a continuum H such that $x \in H^0 \subset H \subset M-y$ and* $\mathrm{Fr}(H)$ *is countable.*

Remark. In Theorem 2 the conclusion was unchanged under broad changes in the cardinality of the totally disconnected set $\bar{K}_x \cdot C'$. An analogous statement cannot be made for the cardinality of $\mathrm{Fr}(C')$ in Theorem 3. As we have seen, making $\mathrm{Fr}(C')$ countable has made the

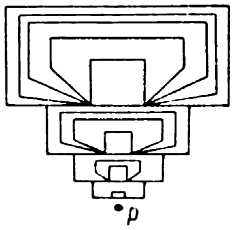

Fig. 5

curve rational and it is not difficult to see that if $\mathrm{Fr}(C')$ is finite, then this will characterize a regular curve. Furthermore requiring $\mathrm{Fr}(C')$ to be exactly one point characterizes a dendrite but making it precisely two points does not make M a simple closed curve. However, if the boundary of *every* subcontinuum of M consists of exactly two points, then M is a simple closed curve and conversely.

3. Characterizations of the class $\mathscr{H}$. In this section two characterizations of hereditarily locally connected continua are established.

THEOREM 5. *A continuum M is hereditarily locally connected if and only if M is hereditarily aposyndetic, i.e., M is aposyndetic at all of its points and so is each subcontinuum of M.*

Proof. Since an hlc continuum is characterized by containing no non-degenerate continuum of convergence, it suffices to show that if M is hereditarily aposyndetic, then M contains no non-degenerate continuum of convergence. So let us assume that M is hereditarily aposyndetic and K is a non-degenerate continuum of convergence, i.e., $K = \lim K_i$, where each K_i is a continuum, $K_i \cdot K_j = \emptyset$ for $i \neq j$ and $K \cdot K_i = \emptyset$ for all i. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be sequences where $x_i + y_i \subset K_i$, $x_i \neq y_i$, and x and y are their respective limit points in K with $x \neq y$. Let K_i' be an irreducible subcontinuum of K_i from x_i to y_i. Since K_i' is aposyndetic at all of its points, it is an arc. By taking subsequences if necessary there is a continuum $K' \subset K$ such that $K' = \lim K_i'$ and $x + y \subset K'$. (Here $K_1', K_2', \ldots$ may denote a subsequence of the original sequence of K_i''s.) Since M is aposyndetic at x with respect to y, there exist an open set U and a continuum H such that $x \in U \subset H \subset M-y$. The set H must intersect all but a finite number of the K_i''s; so no generality is lost by assuming that it intersects them all. Hence $P = K' + H + \sum K_i'$ is a continuum. Since y and all but a finite number of the y_i's lie in $M-H$, we can assume that H contains no y_i. Let k_i be the last point of K_i' (ordered from x_i to y_i) that lies in H. Let k be a limit point of $k_1, k_2, \ldots$ in K'. It is easy to see that P is not aposyndetic at y with respect to k. This contradiction establishes the theorem.

A non-degenerate continuum $K \subset M$ is called a *strong continuum of convergence* if $K = \lim K_i$ where the K_i's are mutually disjoint subcontinua of M, $K \cdot K_i = \emptyset$ for all i and all but a finite number of the K_i's lie in one component of $M-K$.

The condition that M contain no continuum of condensation is stronger than the condition that M contain no strong continuum of condensation as was previously noted but the following theorem shows that such is not the case for continua of convergence.

THEOREM 6. *A necessary and sufficient condition that a continuum M be hereditarily locally connected is that M contain no strong continuum of convergence.*

Proof. The necessity is clear, let us prove the sufficiency. Since each subcontinuum of M inherits the property of not containing a strong continuum of convergence if M does not, it suffices to prove that M is locally connected. First of all, if C is a subcontinuum of M and U is an open set such that $C \subset U$, then only a finite number of components of $M-C$ intersect $M-U$. For let $K_1, K_2, \ldots$ be an infinite sequence of components of $M-C$ such that $K_i \cdot (M-U) \neq \emptyset$ for all i. Let $C \subset V \subset \overline{V} \subset U$, where V is an open set. There is a non-degenerate continuum of convergence Q lying in $M-V$ such that $Q = \lim Q_i$ where $Q_i \subset K_i - V$ for suitable i, and $Q_i \cdot (M-U) \neq \emptyset \neq Q_i \cdot \mathrm{Fr}(V)$. Clearly $\sum Q_i \subset \sum K_i \subset \sum \overline{K}_i + C$ and the latter is a connected set lying in $M-Q$ (if Q happens to be in one of the K_i's, do not include that K_i in the sum). This makes Q a strong continuum of convergence which is a contradiction.

Now suppose that M is not connected im kleinen at x. Then there is an open set U containing x and a non-degenerate continuum C lying in U and containing x such that $C = \lim C_i$, where the C_i's are continua each lying in a different component Q_i of U and $Q_i \cdot Q = \emptyset$ for all i where Q is the component of U containing C. Now $\overline{Q}_i \cdot (M-U) \neq \emptyset$ and since only a finite number of components of $M-C$ intersect $M-U$, an infinite number of the $\overline{Q}_i$'s must lie in one component of $M-C$. Then an infinite number of the C_i's lie in one component of $M-C$ and C is a strong continuum of convergence which is impossible.

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Reçu par la Rédaction le 22. 8. 1967