Two-sided $L^1$-estimates for Szegő kernels on classical domains

by JOSEPHINE MITCHELL (Buffalo, N.Y.) and G. SAMPSON (Auburn, Al.)

This paper is dedicated to the memory of Loo-Keng Hua. The first author presented it at the "Symposium for Hua", Beijing, PRC, August 1-7, 1988.

Abstract. Let $D$ be a bounded symmetric domain in $C^N(N > 1)$ with Bergman–Shilov boundary $b$ and $S(z,t)$ $(z \in D, t \in b)$ the Szegő kernel of $D$. The order of the integral $\int_S S(z,t) \, ds_t$ $(t \in b, 0 \leq r < 1)$ is found for the classical domains $R_{nk}$ (the hyperbolic space of Lie spheres) and the matrix spaces $R_{s}(n,n)$, $R_{s}$ and $R_{s,n}$ (n even) (using Hua's notation in [6]). An upper bound is obtained for $R_{m,n}$ and $R_{s,n}$ (n odd). The results are applied to the family of operators $\int_b S(z,t) f(t) \, ds_t$ (1) where $S(z,t)$ is a function of the Szegő kernel $S(r,u)$ $(u \in b)$ and a related operator to obtain necessary and sufficient conditions on $\gamma, \delta$ for the operators to map $L^p(b)$ into $H^q(D)$. Using the Harish-Chandra realization of an irreducible bounded symmetric domain, Faraut and Korányi obtain the order of $\int_S S(z,t)^{1+\gamma} \, ds_t$ for $q > q_0 > 0$ [1]. This gives a mapping theorem for a related operator to map $L^p(b)$ into $H^q(D)$ for $p \geq 1$.

1. Introduction

1. We begin with the family of operators

\begin{equation}
(\mathcal{F}f)(z) = \int_b S(z,t) f(t) \, ds_t;
\end{equation}

Here $b$ is the Bergman–Shilov (B–S) boundary of a classical domain $D$ in $C^N(N > 1)$, $S(z,w)$ $(z \in D, w \in$ the closure $D^-$ of $D$) is the Szegő kernel of $D([6], p. 88)$. Note that $S(z,w)$ is called the Cauchy kernel in [6]. Additional properties of the domains $D$ are described in Part 2.

The idea for this paper comes from the fact that the operator (1) does not map $L^1(b)$ into $L^1(b)$ independently of $r$. However, there is a well-known necessary and sufficient condition for the operator $T_r$ given by

\begin{equation}
(T_rf)(z) = (\mathcal{F}f)(z) \cdot R(r)
\end{equation}

$(z \in D_r = \{rw: w \in D\})$, $R(r)$ a function of $r$) to be a bounded operator on
Here the necessary and sufficient condition is given in the
lemma. Our method of proof is to obtain two-sided estimates in terms of \( r \) for
the integral

\[
\int_b |S(rv, t)| \, ds, \quad (v \in b).
\]

The exact order of (3) is in terms of the function

\[
R_{a, \beta}(r) = (1 - r)^{\alpha} \log^{\beta} \frac{1}{1 - r}
\]

(\( \alpha \geq 0, \beta \geq 0 \)), where \( \alpha, \beta \) depend on the classical domain \( D \) and its dimension.
(See Theorems 1 and 2.) These precise estimates are used to prove mapping
theorems from \( L^1(b) \) to \( H^1(D) \) for the family of operators (2), which are
independent of \( r \). Necessary and sufficient conditions are obtained for the
family (2) to map \( L^1(b) \) into \( H^1(D) \) and for a related family to map \( L^1(b) \) into
\( L^2(b) \). (See Theorems 3 and 4.) A mapping theorem is also proved for \( p \geq 1 \). (See
Theorem 5.) The estimates and mapping theorems are generalizations to
\( C^N \) (\( N > 1 \)) of the corresponding results for the unit disk.

For \( N = 1 \) an example of (2) is given by the family

\[
F_r(\varphi) = \log^{-1} \left( \frac{1}{1 - r} \right) (H_r f)(\varphi),
\]

where \( (H_r f)(\varphi) = \int_{\theta \in |\theta| < 1} \theta^{-1} f(\theta - \varphi) \, d\theta \) is the cut-off Hilbert transform. (See
\cite{14}, Chapter V, or \cite{16}.) In order to obtain \( L^1 \) mapping properties of
the family \( \{H_r f: 0 \leq r < 1\} \) with bound independent of \( r \) we must use the
operators \( F_r(\varphi) \). This is so because

\[
\sup_{\|f\|_1 \neq 0} \frac{\|H_r f\|_1}{\|f\|_1} = O \left( \log \frac{1}{1 - r} \right) \equiv c(r),
\]

which is a well-known result. (See also lemma in Section 4.1.) Since \( f \in L^1 \),
lim \( (H_r f)(\varphi) \) exists for almost all \( \varphi \) (\cite{14}, p. 132, Theorem 100). Hence
\( r \rightarrow 1^- \lim F_r(\varphi) = 0 \) for almost all \( \varphi \). This last fact was pointed out by the reviewer
in \cite{9} for operators similar to (2).

2. The domains \( D \) considered are bounded symmetric domains in the
complex vector space \( C^N(N > 1) \) with \( 0 \in D \). They possess the following
properties, which are used either explicitly or implicitly in the analysis. The
domain \( D \) has a group of holomorphic automorphisms \( G \), which is transitive.
on \( D \) and extends continuously to the topological boundary of \( D \), \( D \) has a \( (B-S) \)
boundary \( b \), which is a compact real-analytic submanifold of \( C^N \). The domain
\( D \) is circular and star-shaped with respect to \( 0 \) and \( b \) is circular; also, \( b \) is
invariant under $G$, and the isotropy group $G_0 = \{g \in G: g(0) = 0\}$ is transitive on $b$ and can be represented by unitary matrices. The boundary $b$ has a unique normalized $G_0$-invariant measure $\mu$ given by $d\mu_t = (1/V)ds_t$, $V$ the Euclidean volume of $b$ and $ds_t$ the Euclidean volume element at $t \in b$. If $D$ is irreducible, it can be realized as either one of the classical domains $R_j (j = I, II, III, IV)$, which are generalizations of the unit disk in $C^1$, or one of the special domains with $N = 16$ or 27. The groups $G$ of the domains $R_j$ are classical semi-simple Lie groups. (See [4], [5], [6], [7].)

Any bounded symmetric domain has a Szegő kernel, $S(z, w)$, $z \in D$, $w \in D^-$, which is holomorphic in $(z, \bar{w})$ on $D \times D^-$, continuous on $D \times D^-$ and has a singularity on $b$ at $z = w$. Let $0 < r < 1$. Also the slice function $S$, defined by $S_r(z, w) = S(rz, w)$ is hermitian symmetric, that is $S_r(z, w) = S_r(w, z)$. If $t \in b$, then

$$S(rt, rt) = \frac{1}{V(1-r^2)^N}$$

[6]. If $z \in D$, then

$$\int_b |S(z, t)|^p ds_t = O((1-r)^{-N(p-1)}) \quad \text{for} \quad p \geq 2$$

[10]. For $R_I$ and $R_H$, the lower bound for $p$ is sharpened in Theorem 2 of [10]. Note that the function $R_{a,\theta}$ of (4) can be expressed in terms of the Szegő kernel $S(rv, rv)(v \in b)$. In [13] (pp. 17–19) the order of $\int_b |S(z, t)|^p ds_t$ is obtained for the complex unit ball. Also see [7]. Using the Harish-Chandra realization of irreducible bounded symmetric domains, Faraut and Korányi generalize these inequalities to bounded symmetric domains in Theorem 4.1 of [1]. For the Harish-Chandra realization and its connection with E. Cartan's classification of globally symmetric spaces, which includes the classical domains, see ([5], pp. 311–327, 281, 354 and [8]). Since the Szegő kernel is unique, the Szegő kernel of a bounded symmetric domain $D$, obtained from the Harish-Chandra realization is the same as the Szegő kernel of $D$ obtained from the E. Cartan classification.

In the remainder of the paper $A$, $B$, $C$, ... are constants, depending on certain parameters but independent of $r$ and $t$. The constants are not necessarily the same at each occurrence. Also any complex powers are taken in the principal value sense.

2. The hyperbolic space $R_{IV}$ of Lie spheres

1. Preliminaries. The hyperbolic space of Lie spheres is given by

$$R_{IV} = \{z: |zz'|^2 + 1 - 2z\bar{z}' > 0, |zz'| < 1\} \quad (N \geq 2)$$

($\bar{z}'$ = conjugate transpose of $z$) with B–S boundary

$$b_{IV} = \{t: t = e^{i\theta}x, \quad 0 \leq \theta \leq \pi, \quad x \text{ a real vector with } xx' = 1\}.$$
The complex dimension of \( R_{IV} \) is \( N \) and the real dimension of \( b_{IV} \) is \( N \). Its Szegö kernel at the point \( e = (1, 0, \ldots, 0) \in b_{IV} \) is
\[
S(re, t) = (1/V) \left[ 1 + r^2 e^{-2i\theta} - 2rx_1 e^{-i\theta} \right]^{-N/2}.
\]

We find upper and lower bounds for
\[
I(re) = \int_s |S(re, t)| \, ds_t.
\]

Set
\[
g(r, x_1) = (1/V) \int_0^\pi |1 + r^2 e^{-2i\theta} - 2rx_1 e^{-i\theta}|^{-N/2} \, d\theta.
\]

Using ([7], pp. 1040–1041),
\[
I(re) = 2\pi \int_{xx' = 1} g(r, x_1) \hat{x} \quad (\hat{x} \text{ the volume element of } xx' = 1)
\]
\[
= 2\pi \int_{x_1^2 < 1} g(r, x_1) \, dx_1 \int_{xx' < (1 - x_1^2)} \hat{x} \quad (\hat{x} = (x_2, \ldots, x_{N-2}))
\]
\[
= C \int_{x_1^2 < 1} g(r, x_1) (1 - x_1^2)^{3(N-3)/2} \, dx_1.
\]

Following ([3], p. 527) set \( x_1 = \cos \varphi \). Then
\[
1 + r^2 e^{-2i\theta} - 2rx_1 e^{-i\theta} = (1 - re^{-i(\theta + \varphi)})(1 - re^{-i(\theta - \varphi)}) \quad \text{and}
\]
\[
|1 - re^{-i(\theta \pm \varphi)}|^2 = (1 - r)^2 + 2r(1 - \cos(\theta \pm \varphi)) = [...].
\]

Thus
\[
I(re) = C \int_0^\pi \sin^{N-2} \varphi \, d\varphi \int_0^{\pi} [...]^ {-N/4} [...]^{-N/4} \, d\theta.
\]

The order of (1) is given in

**Theorem 1.** Let \( N \) be a positive integer \( \geq 2 \) and \( 0 \leq r < 1 \). For the domain
\( D = R_{IV}(N) \),
\[
B R_{x, \beta}^{-1}(r) \leq \int_b |S(re, t)| \, ds_t \leq A R_{x, \beta}^{-1}(r)
\]

for \( v \in b \) and \( r \) sufficiently close to 1, where \( \alpha = (N/2) - 1 \), \( \beta = 0 \) for \( N > 2 \), \( \alpha = 0 \), \( \beta = 2 \) for \( N = 2 \) and \( R_{x, \beta} \) is given by (1.4).

To prove the theorem we need the estimates
\[
C_1 t^2 \leq 1 - \cos t \leq C_2 t^2 \quad \text{for } 0 \leq t \leq (11/6)\pi,
\]
\[
C_3 (2\pi - t)^2 \leq 1 - \cos t \leq C_4 (2\pi - t)^2 \quad \text{for } (4\pi \leq t \leq 2\pi,}
\]
(3c) \[ 0 \leq \sin t \leq t \quad \text{if} \quad 0 \leq t \leq \pi, \]
(3d) \[ 0 \leq \sin t \leq (\pi-t) \quad \text{if} \quad {\frac}{1}{2}\pi \leq t \leq \pi, \]
and for \( \frac{1}{2} \leq r < 1 \), \( N > 2 \) the estimates

(4a) \[ \int_0^{2\pi} \int_0^{1-r} x^{\frac{N}{2}-2} dx \int_0^1 (1-r)^{-\frac{N}{2}} dt \leq C(1-r)^{-\frac{(N/2-1)}{2}}, \]
(4b) \[ \int_0^{\frac{\pi}{2}} \int_0^{1-r} x^\frac{N}{2}-2 dx \int_0^1 t^{-\frac{N}{2}} dt \leq C(1-r)^{-\frac{(N/2-1)}{2}}, \]

(5) \[ \int_0^{\frac{\pi}{2}} x^{\frac{N}{2}-2} [(1-r)^2 + x^2]^{-\frac{N}{4}} dx \int_0^{\frac{\pi}{2}} [(1-r)^2 + t^2]^{-\frac{N}{4}} dt \leq C(1-r)^{-\frac{(N/2-1)}{2}}. \]

If \( N = 2 \) (4a) is bounded by \( \log[1/(1-r)] \) and (4b) and (5) by \( \log^2[1/(1-r)] \). In (4b) note that

\[ \int_0^1 t^{-\frac{N}{2}} dt \leq C(1-r)^{-\frac{(N/2-1)}{2}} \quad \text{and} \quad \int_0^{2\pi} x^{-\frac{N}{2}-2} dx \leq C \]

if \( N > 2 \) so that (4b) holds. The estimate is clear for \( N = 2 \). The proofs of (4a) and (5) are similar.

The method of proof to obtain (2) for \( N > 2 \) used here is to subdivide the integral \( I(re) \) in (1) into convenient pieces and to use certain estimates to get the order. Another method to find the order of \( I(re) \), that has been suggested, is to use the residue theorem to estimate the cases \( N = 4M (M = 1, 2, 3, \ldots) \); then Hölder's inequality gives the intermediate cases \( 4M < N < 4(M+1) \). The cases \( N = 2 \) and \( 2 < N < 4 \) are evaluated separately.

2. Proof of the upper bounds in Theorem 1 for \( N > 2 \). Since \( I(re) \), given by (1), is bounded for \( 0 \leq r \leq 1/2 \), we need only consider \( 1/2 < r < 1 \). We break \( I(re) \) into:

\[ I = \left\{ \int_0^{1-r} \int_0^{\pi} \int_0^{\pi} \right\} \mathcal{A}_r(\phi, \theta) \equiv I_1 + I_2, \]

\[ I_2 = \int_0^{\pi} \int_0^{\pi} \mathcal{A}_r(\phi, \theta), \]

\[ I_3 = \int_0^{\pi} \int_0^{\pi} \mathcal{A}_r(\phi, \theta), \]

\[ I_4 = \left\{ \int_0^{1-r} \int_0^{\pi} \int_0^{\pi/4} \int_0^{\pi/4} \mathcal{A}_r(\phi, \theta) \right\} \equiv I_{V_1} + I_{V_2} + I_{V_3}, \]

where

\[ \mathcal{A}_r(\phi, \theta) = \frac{C(\sin \phi)^{\frac{N-2}{2}} d\phi d\theta}{[\ldots]^\frac{1}{N/4} [\ldots]^\frac{1}{N/4}}. \]
Estimates for I−IV. We show that I is bounded and that II−IV have the upper bound \( O((1-r)^{-N/2+1}) \).

(a) By (3a), (3b) and (3d)

\[
IV_3 \leq C \int_{n/4}^{\pi-(1-r)} (\pi-\varphi)^{N-2} d\varphi \int_{\varphi+(1-r)}^{\pi} \left[ (1-r)^2 + (\theta-\varphi)^2 \right]^{-N/4} \times \left[ (1-r)^2 + (2\pi-(\theta+\varphi))^2 \right]^{-N/4} d\theta.
\]

After making the change of variable \( u = \theta - \varphi, v = \pi - \varphi \) the estimate follows by (4b). By similar arguments estimates for \( IV_1 \) and \( IV_2 \) are obtained.

(b) By (3c) and (3d) we have

\[
III \leq C \left\{ \int_{1/2}^{n/2} \varphi^{N-2} d\varphi + \int_{n/2}^{\pi} (\pi-\varphi)^{N-2} d\varphi \right\} \times \int_{0}^{\pi} \left[ \ldots \right]^{-N/4} \left[ \ldots \right]^{-N/4} d\theta \equiv III_1 + III_2.
\]

(i) By (3a) and (3b)

\[
III_2 \leq C \int_{n/2}^{\pi} (\pi-\varphi)^{N-2} d\varphi \int_{0}^{\varphi-(1-r)} \left[ (1-r)^2 + (\theta-\varphi)^2 \right]^{-N/4} \times \left[ (1-r)^2 + (2\pi-(\theta+\varphi))^2 \right]^{-N/4} d\theta.
\]

Set \( u = \varphi - \theta \) and \( \psi = \pi - \varphi \) and the estimate follows from (4b) and (5).

(ii). By (3a)

\[
III_1 \leq C \int_{1/2}^{\pi/2} \varphi^{N-2} d\varphi \int_{0}^{\varphi-(1-r)} (\varphi-\theta)^{-N/2} (\varphi+\theta)^{-N/2} d\theta \leq C \int_{1/2}^{\pi/2} \varphi^{N/2-2} d\varphi \int_{1/2}^{\pi/2} u^{-N/2} du,
\]

were \( u = \varphi - \theta \) and the result follows by (4b).

(c) For II we have

\[
II \leq C(1-r)^{-N/2} \left\{ \int_{1-r}^{\pi-(1-r)} \right\} \left( \frac{7}{8} \right)^{n} \frac{\pi-(1-r)}{\varphi-(1-r)} \sin^{N-2} \varphi \, d\varphi \times \int_{\varphi+(1-r)}^{\varphi-(1-r)} \left[ (1-r)^2 + 2r(1-\cos(\theta+\varphi)) \right]^{-N/4} d\theta \\
\equiv II_1 + II_2.
\]

(i) By (3b) and (3d)

\[
II_2 \leq C(1-r)^{-N/2} \int_{(7/8)n}^{\pi-(1-r)} (\pi-\varphi)^{N-2} d\varphi \int_{\varphi-(1-r)}^{\varphi+(1-r)} \left[ 2\pi-(\theta+\varphi) \right]^{-N/2} d\theta.
\]
Setting \( u = 2\pi - (\theta + \varphi) \) and then \( s = 2(\pi - \varphi) - (1 - r) \) gives
\[
II_2 \leq C(1-r)^{-N/2}\int_{1-r}^{r} s^{N-2} ds \int_{s}^{u} u^{-N/2} du,
\]
from which the result follows by (4a).

(ii) By (3a) and (3c)
\[
II_1 \leq C(1-r)^{-N/2}\int_{1-r}^{r} \varphi^{N-2} d\varphi \int_{1-r}^{r} (\theta + \varphi)^{-N/2} d\theta
\]
\[
\leq C(1-r)^{1-N/2}\int_{1-r}^{r} \varphi^{N-2} d\varphi \leq C(1-r)^{1-N/2}.
\]

(d) In \( I_2 \) set \( \psi = \pi - \varphi, \tau = \pi - \theta \). Then \( I_2 = I_1 \) and
\[
I_1 \leq \int_{0}^{1-r} \varphi^{N-2} d\varphi \int_{0}^{1-r} [(1-r)^2 + 2r(1-\cos(\varphi + \theta))]^{-N/4} \left[(1-r)^2 + 2r(1-\cos(\varphi - \theta))\right]^{-N/4} d\theta
\]
\[
\leq C(1-r)^{-2}\int_{0}^{1-r} d\varphi \int_{0}^{2(1-r)} d\theta \leq C.
\]
This completes the proof of the upper bound in (2).

3. Proof of the lower bound in Theorem 1. The integrands of \( I, II, III_1, III_2, IV \) and intervals of integration are all non-negative for \( r \geq 1/2 \). Hence
\[
I(re) \geq III \geq III_1 = C \int_{1-r}^{r} \int_{0}^{\pi/2} \mathcal{A}_r(\varphi, \theta).
\]
We show that the lower bound of \( III_1 \) for \( N > 2 \) is \( C(1-r)^{-N/2+1} \) and for \( N = 2 \) it is \( C \log^2 [1/(1-r)] \).

For \( III_1 \) since \( 1-r \leq \varphi + \theta \), \( 1-\cos(\varphi + \theta) \leq (\varphi + \theta)^2 \), \( \sin \varphi \geq \frac{2}{\pi} \varphi \) in \( \left[0, \frac{\pi}{2}\right] \) and \( \varphi + \theta \leq 2\varphi \), we have
\[
I(re) \geq C \int_{1-r}^{r} \varphi^{N-2} d\varphi \int_{0}^{\varphi-(1-r)} (\varphi-\theta)^{-N/2} (\varphi+\theta)^{-N/2} d\theta
\]
\[
\geq C \int_{1-r}^{r} \varphi^{N/2-2} d\varphi \int_{0}^{\varphi-(1-r)} (\varphi-\theta)^{-N/2} d\theta.
\]
Set \( u = \varphi - \theta \) in the inner integral and integrate. We get if \( N > 2 \)
\[
I(re) \geq C \int_{1-r}^{r} \left(\frac{\varphi^{N/2-2}}{(1-r)^{N/2-1} - \varphi^{-1}}\right) d\varphi \geq C \left(\frac{1}{s} - 1 - \log \frac{1}{s}\right),
\]
where \( s = [2(1-r)/\pi]^{N/2-1} \). But \( 1/2s \gg 1 + \log(1/s) \) for \( 1-r \) sufficiently small. Thus \( I(re) \geq C/(1-r)^{N/2-1} \). If \( N = 2 \) the inner integral in (6) is 
\[
\int_{1-r}^{e} u^{-1} \, du = \log \phi + \log \frac{1}{1-r} \geq \log \phi, \quad \text{and} \quad I(re) \geq C \log^2 \frac{1}{1-r}.
\]
This completes the proof of Theorem 1.

Remark. Theorem 1 is likely true for any \( p, \ 0 < p < \infty \), but the details of the proof would be more complicated.

3. The matrix spaces \( R_j (j = I, II, III) \)

1. Preliminaries. The classical domains \( R_j (j = I, II, III) \) are defined by 
\[
D = \{ z : I - zz^* > 0 \},
\]
where \( z \) is a matrix of complex numbers, \( z^* \) its conjugate transpose and \( I \) an identity matrix. If \( D = R_I (m,n) \) \( (m \leq n) \), \( z \) is of order \( m \times n \) and \( I \) of order \( m \); \( R_I (1,n) \) is the complex unit ball in \( \mathbb{C}^n \); if \( D = R_{II} (n) \), \( z \) is a symmetric matrix of order \( n \); if \( D = R_{III} (n) \), \( z \) is a skew-symmetric matrix of order \( n \). The B-S boundary is given by 
\[
b = \{ z : zz^* = I \}
\]
where \( z \) is an \( m \times n \) matrix for \( R_I (m,n) \), a symmetric unitary matrix of order \( n \) for \( R_{II} (n) \), a skew-symmetric unitary matrix of order \( n \) for \( R_{III} (n) \), \( n \) even. For \( n \) odd 
\[
b = \{ UDU^* : U \text{ is unitary and } D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ldots 0 \}.
\]
The complex dimension of the domains \( R_j \) is \( mn \), \( (n/2)(n+1) \), \( (n/2)(n-1) \) respectively, while the real dimension of their B-S boundaries is \( m(2n-m) \), \( (n/2)(n+1) \), \( [n+(1+(-1)^{n-1})]/2 \) respectively [6].
The Szegö kernel is 
\[
S(z, w) = \frac{1}{V \det^*(I - zw^*)} \quad (z \in D, \ w \in D^-),
\]
where \( V \) is the volume of the domain \( b \), \( \alpha = n \) for \( R_I (m,n) \), \( (n+1)/2 \) for \( R_{II} (n) \), \( (n-1)/2 \) for \( R_{III} (n) \), \( n \) even, and \( \frac{1}{2}n \) for \( R_{III} (n) \), \( n \) odd. 

2. Upper and lower bounds for the integral 
\[
I(rv) = \int_{b} |S(rv, t)| \, ds, \quad (v \in b, \ 0 \leq r < 1)
\]
are given in

**Theorem 2.** For matrix spaces \( R_I (n,n) \), \( R_{II} (n) \), and \( R_{III} (n \text{ even}) \)
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(1) \[ B \mathcal{R}_{s,1}^{-1}(r) \leq \int_{b} |S(rv,t)| \, ds_i \leq A \mathcal{R}_{s,1}^{-1}(r), \]

for all $v \in b$ and $1 - r$ sufficiently small. If $n$ is even for $R_I$ and $R_{II}$, then $\beta = 0$ and $\alpha = n^2/4$ for $R_I$ and $n^2/8$ for $R_{II}$; if $n$ is odd $\beta = 1$ and $\alpha = 1/2(n^2 - 1)$ for $R_I$ and $1/2(n^2 - 1)$ for $R_{II}$. For $R_{III}$ if $n = 4u$, $u$ a positive integer, then $\beta = 0$ and $\alpha = n^2/8$, if $n = 4u - 2$ then $\beta = 1$ and $\alpha = (n - 4)^2/8$. Upper bounds for $R_I$ ($m < n$) and $R_{III}$ ($n$ odd) are given by (30) and (27).

Proof of Theorem 2. The upper bounds in (1) were obtained for the spaces $R_I(n,n)$ and $R_{II}(n)$ in [12].

3. Lower bound for the space $R_I(n) \equiv R_I(n,n)$. By formula (2.1) of [12]

\[ I(rv) = C \prod_{k=1}^{n} \left\{ \frac{2 \pi}{1 - re^{i\theta_k}} - n \int_{\alpha}^{\beta} |e^{i\theta_j} - e^{i\theta_k}|^2 \right\}, \]

where $\sigma$ is a positive integer to be chosen later. It can be shown that

\[ I(rv) \geq C \prod_{k=1}^{n} \left\{ \frac{2 \pi}{1 - re^{i\theta_k}} - n \int_{\alpha}^{\beta} |e^{i\theta_j} - e^{i\theta_k}|^2 \right\} \equiv C \mathbb{H} \]

and

(2) \[ \mathbb{H} \geq \prod_{k=1}^{n/2} \int_{1-r}^{\pi/2} |1 - re^{i\theta_k}|^{-n} d\theta_k \prod_{1 < j < k < \sigma} |e^{i\theta_j} - e^{i\theta_k}|^2. \]

From the inequality $|a - b| \geq |a| - |b|$ follows for $0 < r \leq 1$

(3) \[ |e^{i\theta_j} - e^{i\theta_k}|^2 \geq (|1 - re^{i\theta_j}| - |1 - re^{i\theta_k}|)^2, \]

also, there exists a constant $b$, $0 < b < 1$, independent of $r$, $\theta_j$ and $\theta_k$ such that if $\theta_j \geq 2\theta_k$ for $j < k$, $1 - r < \theta_j < \pi$ and $r \geq \frac{1}{2}$

(4) \[ |1 - re^{i\theta_k}| \leq b |1 - re^{i\theta_j}|. \]

Under these restrictions on $\theta_j$, $\theta_k$ and $r$ by (3) and (4)

(5) \[ |e^{i\theta_j} - e^{i\theta_k}|^2 \geq B |1 - re^{i\theta_j}|^a \] \quad ($B = 1 - b$)

for any $a > 0$. Inequality (5) implies that

(6) \[ \prod_{k=j+1}^{\sigma} |e^{i\theta_j} - e^{i\theta_k}|^a \geq B |1 - re^{i\theta_j}|^{a(\sigma - j)} \] \quad ($1 \leq j < \sigma$).
Also since \( \theta_k \geq 1 - r \) and \( r \geq \frac{1}{2} \)

\[
\theta_k^2 \geq \frac{1}{2}(\theta_k^2 + r\theta_k^2) \geq \frac{1}{2}[(1-r)^2 + r2^2] \geq \frac{1}{2}|1-re^{\theta_0}|^2.
\]

Use (6) with \( \alpha = 2 \) in (2). This gives

\[
 \mathcal{U} \geq C \int_{1-r}^{n/2} \int_{2\theta_0}^{n/2-1} \int_{2\theta_1}^{n/2} \cdots \int_{2\theta_{n-1}}^{n/2} \prod_{k=1}^{n/2} |1-re^{\theta_0}|^{2(\sigma - k)-n}.
\]

If \( n \) is odd, take \( \sigma = \frac{1}{2}(n+1) \). Then \( 2(\sigma - k) - n = 1 - 2k \) and by (7)

\[
 \mathcal{U} \geq C \int_{1-r}^{n/2} \int_{2\theta_0}^{n/2-1} \int_{2\theta_1}^{n/2} \cdots \int_{2\theta_{n-1}}^{n/2} \prod_{k=1}^{n/2} \theta_k^{1-2k}.
\]

In the following assume that \( 1 - r \) is so small that the lower bounds are all positive. Note that \( \theta_k < 1 \) for \( 2 \leq k \leq \sigma \) so that \( \theta_k < \theta_k^{1/2} \). Also, the set

\[
\{ \theta : 2\theta_k \leq \theta \leq A \} \ni \{ \theta : 2\theta_k \leq \theta \leq 2\theta_k^{1/2}, \theta_k \leq (A/2)^2 \},
\]

since \( 2\theta_k^{1/2} \leq A \). Here \( A \) takes on the values \( \pi/2^4 \). When \( A = \pi/2 \)

\[
\mathcal{U} \geq C \int_{1-r}^{n/2} \int_{2\theta_0}^{n/2-1} \int_{2\theta_1}^{n/2} \cdots \int_{2\theta_{n-1}}^{n/2} \prod_{k=2}^{n/2} \theta_k^{1-2k} \int_{\theta_1}^{1} d\theta_1.
\]

The inner integral equals \( \frac{1}{2}\log(1/\theta_2) \). Repeating with \( A_1 = (A/2)^2 \)

\[
\mathcal{U} \geq C \int_{1-r}^{n/2} \int_{2\theta_0}^{n/2-1} \int_{2\theta_1}^{n/2} \cdots \int_{2\theta_{n-1}}^{n/2} \prod_{k=3}^{n/2} \theta_k^{1-2k} \int_{\theta_1}^{1} d\theta_1 \int_{\theta_3}^{1} d\theta_3 \frac{1}{2}\log(1/\theta_2) d\theta_2.
\]

The inner integral \( \geq B(1/2^2\theta_3)\theta_3^{-2} \). Since \( \theta_2 \to \theta_2^{1/2} \) and the subscript \( k \) increases by 1. Thus we get

\[
\mathcal{U} \geq C \int_{1-r}^{n/2} \int_{2\theta_0}^{n/2-1} \int_{2\theta_1}^{n/2} \cdots \int_{2\theta_{n-1}}^{n/2} \prod_{k=3}^{n/2} \theta_k^{1-2k} \theta_3^{-2} \log(1/2^2\theta_3) d\theta_2.
\]

Notice that upon integrating the power of \( \theta_2 \) is decreased by 1 and the subscript in the denominator increased by 1, that is, \( \theta_2 \to \theta_2^{1/2} \). Repeating the above argument the exponent decreases by 1 for each \( \theta_k \), \( 2 \leq k \leq \sigma - 1 \), while the subscript \( k \) increases by 1. Thus we get

\[
\mathcal{U} \geq C \int_{1-r}^{(A-1)^2} \frac{1}{c\theta_\sigma} \log \frac{1}{c\theta_\sigma} \prod_{k=3}^{\sigma-1} \int_{1-r}^{1} \log \frac{1}{c\theta} \theta^{-(\sigma^2 - \sigma + 1)} d\theta,
\]

where \( c = 2^x, x > 1 \). Since \( (c\theta)^{-1} \geq (c^2(1-r))^{-1/2} \) and \( \log(1/c\theta) \geq B \log(1/(1-r)) \) for \( 1 - r \) sufficiently small

\[
I(r) \geq C \log \frac{1}{1-r} \int_{1-r}^{1} \theta^{-(\sigma^2 - \sigma + 1)} d\theta = C(1-r)^{-(\sigma^2 - \sigma + 1)/2} \log \frac{1}{1-r}.
\]

If \( n \) is even take \( \sigma = n/2 \). Repeating the above procedure, we get

\[
I(r) \geq C(1-r)^{-n/2}.
\]
The lower bounds (10) and (11) for \( I(rv) \) for the domain \( R_I(n) \) have the same order as the upper bounds in [12].

4. **Lower bound of** \( I(rv) \) **for the space** \( R_{II}(n) \). We proceed as for \( R_I(n) \). By formula (2.2) of [12]

\[
I(rv) = C \prod_{k=1}^{n} \int_{0}^{2\pi} d\theta_k |1-re^{i\theta_k}|^{-(n+1)/2} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|.
\]

Similarly as for (8), by (6) with \( \alpha = 1 \)

\[
I(rv) \geq C \prod_{k=1}^{n/2} \int_{1-r}^{1-r^{n/2}} d\theta_k \prod_{k=1}^{n/2} \int_{2\theta_0}^{2\theta_k} d\theta_{\sigma-1} \cdots \int_{2\theta_0}^{2\theta_1} d\theta_1 \prod_{k=1}^{n/2} |1-re^{i\theta_k}|^{\sigma-k-(n+1)/2}.
\]

If \( n \) is odd, take \( \sigma = (n+1)/2 \) which gives by (7)

\[
I(rv) \geq C \prod_{k=1}^{n/2} \int_{1-r}^{1-r^{n/2}} d\theta_k \prod_{k=1}^{n/2} \int_{2\theta_0}^{2\theta_k} d\theta_{\sigma-1} \cdots \int_{2\theta_0}^{2\theta_1} d\theta_1 \prod_{k=1}^{n/2} \theta_k^{-k} \geq C \log \frac{1}{1-r} \int_{1-r}^{1-r^{n/2}} d\theta \theta^{-(n^2-n)/2} \geq C(1-r)^{-(n^2-1)/8} \log \frac{1}{1-r}.
\]

If \( n \) is even take \( \sigma = (n+1)/2 \), which gives

\[
I(rv) \geq C(1-r)^{-n^2/8}.
\]

The lower bounds for \( I(rv) \) for \( R_{II}(n) \) given by (13) and (14) have the same order as the upper bounds in [12].

5. **Bounds for** \( I(rv) \) **for the space** \( R_{III}(n) \). For \( R_{III}(n) \)

\[
I(rk_0) = \frac{1}{V_b} |\text{det}(I+rk_0 \beta)|^{-\alpha} ds_n, \quad \alpha = \frac{n-1}{2} \quad \text{for } n \text{ even and } \alpha = \frac{n}{2} \quad \text{for } n \text{ odd}
\]

where \( k_0 \) is a skew-symmetric unitary matrix and \( 0 \leq r < 1 \). Let \( n \) be even. Without loss of generality take

\[
k_0 = D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (m = \frac{n}{2} \times n \text{ times}).
\]

By ([11], p. 385)

\[
|\text{det}(I+rd_1 \beta)| = |\text{det}(I-rD'_1)| = \prod_{k=1}^{m} |1-re^{i\theta_k}|^2
\]

since \( tD'_1 \) is unitary and hence unitarily equivalent to a diagonal matrix \( d = [e^{i\theta_1}, \ldots, e^{i\theta_m}] \), where \( \theta_{2k-1} = \theta_{2k} = \phi_k \) (\( 1 \leq k \leq m \)). Under the transformation \( v = tD'_1 b = b' = \{ v : v = tD'_1 U \} \). By a calculation following ([6], p. 56) and using ([15], p. 60) the volume element of \( b' \) is

\[
ds_v = C_n^{1/2} \prod_{1 \leq j < k \leq m} |e^{i\theta_j} - e^{i\theta_k}|^4 d\phi_1 \cdots d\phi_m U.
\]
Then (15) with \( k_0 = D_1 \) equals

\[
I(rD_1) = C \prod_{k=1}^{m} \int_0^{2\pi} |1 - re^{i\varphi_k}|^{-(n-1)} d\varphi_k \prod_{1 \leq j < k \leq m} |e^{i\varphi_j} - e^{i\varphi_k}|^4.
\]

Upper bounds for the integral (16). The lemma in ([12], p. 374) with \( a_j = |1 - re^{i\varphi_j}|, \alpha = 2, \beta = n-1, \) gives

\[
I(rD_1) \leq \prod_{k=1}^{m} \int_0^{2\pi} |1 - re^{i\varphi_k}|^{n-1-4k} d\varphi_k.
\]

It is well known that \( \int_0^{2\pi} |1 - re^{i\varphi}|^{-q} d\varphi_k = O((1-r)^{-q-1}) \) if \( q > 1, \)
\( = O(\log(1/(1-r))) \) if \( q = 1 \) and \( = O(1) \) if \( q < 1. \)

If \( n = 4u, u \) a positive integer, \( 4k - n - 1 < 1 \) for \( 1 \leq k \leq u \) and \( > 1 \) for \( u+1 \leq k \leq m = 2u. \) This gives

\[
I(rD_1) \leq C \prod_{u+1}^{2u} \int_0^{2\pi} |1 - re^{i\varphi_k}|^{n-1-4k} d\varphi_k \leq C(1-r)^{-n^2/8}.
\]

If \( n = 4u-2, 4k - n - 1 = 1 \) for \( k = u, < 1 \) for \( 1 \leq k \leq u - 1 \) and \( > 1 \) for \( u+1 \leq k \leq 2u-1 = m. \) This gives

\[
I(rD_1) \leq C(1-r)^{-(n^2-4)/8} \log \frac{1}{1-r}.
\]

Lower bounds for the integral (16). As for \( R_1(n) \) and \( R_{1/4}(n) \) by (6) with \( \alpha = 4 \) (see (8) and (12))

\[
I(rD_1) \geq C \int_1^{\pi/2} d\varphi \int_{2\varphi}^{\pi/2} d\varphi - 1 \int_{2\varphi_0}^{\pi/2} d\varphi_1 \prod_{k=1}^{\sigma} |1 - re^{i\varphi_k}|^{4(\sigma-k)-(n-1)}.
\]

Choose \( \sigma \) as the largest integer \(<(n+3)/4). \) If \( n/4 \) is an integer, \( \sigma = n/4. \) If \((n+2)/4 \) is an integer, \( \sigma = (n+2)/4. \) The integrands in (19) are

\[
\prod_{k=1}^{\sigma} |1 - re^{i\varphi_k}|^{-(4k-1)} \text{ if } \sigma = n/4 \text{ and } \prod_{k=1}^{\sigma} |1 - re^{i\varphi_k}|^{-(4k-3)} \text{ if } \sigma = (n+2)/4.
\]

By (7) these integrals are greater than or equal to

\[
C \prod_{k=1}^{\sigma} \varphi_k^{-(4k-1)} \quad (\sigma = n/4), \quad C \prod_{k=1}^{\sigma} \varphi_k^{-(4k-3)} \quad (\sigma = (n+2)/4)
\]

respectively. Integrate (20) with respect to \( \varphi_k (1 \leq k \leq \sigma - 1). \) By the same reasoning as in Part 3 we get

\[
I(rD_1) \geq C(1-r)^{-n^2/8} \quad \text{if } \quad u = n/4
\]

and

\[
I(rD_1) \geq C(1-r)^{-(n^2-4)/8} \log \frac{1}{1-r} \quad \text{if } \quad u = (n+2)/4.
\]
The order of the lower bounds given in (21) and (22) for \( R_{III} \) \((n \text{ even})\) agrees with the order of the upper bounds (17) and (18) respectively.

6a. The case \( R_{III} \) \((n \text{ odd})\). By ([7], p. 1073) the closure of \( R_{III}(n) \) can be embedded into that of \( R_{III}(n+1) \) and \( b_n = b_{III}(n) \subset b_{n+1} = b_{III}(n+1) \).

The Szegő kernel of \( R_{III}(n) \) is

\[ S(z,t) = \frac{1}{V_n} \det(I + z\bar{t})^{n/2} \quad (z \in R_{III}(n), \ t \in b_n \text{ and } V_n = V(b_n)). \]

By ([7], p. 1073) any \( t \in b_{n+1} \) can be written in the form \( t_1 = \begin{pmatrix} t & U' & h' \\ -h & 0 \end{pmatrix} \)

\[ t = U'DU, \ U \text{ an arbitrary unitary matrix and } h = (0, \ldots, 0, e^{i\theta}), \ 0 \leq \theta \leq 2\pi. \]

Thus \( t \in b_n \). Also

\[ V_n = (1/2\pi) V_{n+1}. \]

Let \( z_1 = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \). It is easy to check that \( z_1 \in R_{III}(n+1) \). Also

\[ \det(I^{n+1} + z_1 \bar{t}_1) = \det(I^n + z\bar{t}). \]

Hence, by (25), \( \det(I^{n+1} + z_1 \bar{t}_1) \) is independent of \( \bar{U}' \bar{R} \) and by (24) we get the formula

\[ \frac{1}{V_n} \int_{b_n} \frac{ds_t}{|\det(I^n + z\bar{t})|^{n/2}} = \frac{1}{V_{n+1}} \int_{b_{n+1}} \frac{ds_{t_1}}{|\det(I^{n+1} + z_1 \bar{t}_1)|^{n/2}} \]

Now set \( z = rv \), where \( v \in b_n \), \( 0 \leq r < 1 \) and \( v_1 = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} ; v_1 \notin b_{n+1} \), \( v_1 \in R_{III}(n+1) - R_{III}(n+1) - b(n+1) \). Let \( v_0 \) be an arbitrary point of \( b_{n+1} \). Then

\[ \sup_{v \in b_n} \frac{1}{V_n} \int_{b_n} \frac{ds_t}{|\det(I^n + rv_0 \bar{t})|^{n/2}} \leq \sup_{v_1 \in b_{n+1}} \frac{1}{V_{n+1}} \int_{b_{n+1}} \frac{ds_{t_1}}{|\det(I^{n+1} + rv_0 \bar{t}_1)|^{n/2}} \]

by the maximum principle.

The upper bound for \( R_{III} \) \((n \text{ odd})\) follows by (26) and (27) from the upper bound for \( R_{III}(n+1) \).

6b. The case \( R_{i} \) \((m < n)\). Here we wish to find the order of the integral

\[ \int_{b_{mn}} |S(rv,t)| \, ds_t, \]

where \( b_{mn} \) is the B–S boundary of \( R_{i}(m < n) \) and \( v \in b_{mn} \).
Upper bound for (28). We obtain a formula for (28) over the B-S boundary \( b_n \) of \( R_f(n) \). Then use the upper bound given in Theorem 2. Let \( z_1 = \begin{pmatrix} z \\ 0 \end{pmatrix} \), \( t_1 = \begin{pmatrix} t \\ u \end{pmatrix} \). The point \( z_1 \in R_f(n) \) since \( I^n - z_1 z_1^* = \begin{pmatrix} I^n - zz^* & 0 \\ 0 & I^{n-m} \end{pmatrix} \) is positive definite. However, note that the point \( v_1 = \begin{pmatrix} v \\ 0 \end{pmatrix} \) \((v \in b_{mn}) \notin b_n \) since \( v_1 v_1^* = \begin{pmatrix} I^n & 0 \\ 0 & 0 \end{pmatrix} \neq I^n \).

Now \( \det(I^n - z_1 t_1^*) = \det(I^n - z_1^* z) \) so that by ([6], p. 94)

\[
\int_{b_{mn}} |S(z,t)| \, ds_t = \frac{1}{V_{mn}} \int_{b_{mn}} \frac{ds_t}{|\det(I^n - z_1 t_1^*)|^n} = \frac{1}{V_{n,b_n}} \int |\det(I^n - z_1 t_1^*)|^n. \tag{29}
\]

Set \( z = rv \) \((v \in b_{mn})\), \( z_1 = rv_1 = r \begin{pmatrix} v \\ 0 \end{pmatrix} \). By (29), the maximum principle and Theorem 2

\[
\int_{b_{mn}} |S(rv,t)| \, ds_t \leq \sup_{v \in b_n} \int_{b_{mn}} |S(rv_1,t_1)| \, ds_{t_1}
\]

\[
\leq A(1-r)^{-n^2/4} \quad \text{if } n \text{ is even},
\]

\[
\leq A(1-r)^{-(n^2-1)/4} \log \frac{1}{1-r} \quad \text{if } n \text{ is odd.} \tag{30}
\]

(A independent of \( r \)).

Lower bound for (28). Since there is no minimum principle, it is not possible to get a lower bound by this method. A determinantal inequality due to Hua (Scientia Sinica, Notes 1, vol. XIV, No. 5 (1964)), viz, for \( I - w w^* > 0 \), \( I - z z^* > 0 \),

\[
\det(I - w w^*) \det(I - z z^*) \leq |\det(I - w z)|^2 \leq \det(I + w w^*) \det(I + z z^*),
\]
yields

\[
\frac{1}{|\det(I^n - rv_1 t_1^*)|^2} \geq \frac{1}{(1+r)^{2n}},
\]

so that by (29)

\[
\int_{b_{mn}} |S(rv,t)| \, ds_t \geq \frac{1}{V_n(1+r)^{2n}}. \tag{31}
\]

This completes the proof of Theorem 2.

4. Mapping theorems

1. Let \( \{U_r\} \) be a family of linear operators defined on a measurable set \( b \) and depending on a parameter \( r, 0 \leq r < 1 \). We say that the family \( \{U_r; 0 \leq r < 1\} \) maps \( L^1(b) \) into \( L^1(b) \) uniformly if

\[
\|U_r f\|_1 \leq C \|f\|_1,
\]

where \( C \) is independent of \( f \) and \( r \).
For example, the family \( \{ F_r(\varphi) : 0 \leq r < 1 \} \) of Section 1.1 maps \( L^1([-\pi, \pi]) \) into \( L^1([-\pi, \pi]) \) uniformly.

Let \( D \) be a bounded circular domain in \( C^N \) with \( 0 \in D \), which is star-shaped with respect to \( 0 \) and has a measurable B-S boundary \( b \). Let \( K(z, w) \) be a measurable function defined on \( D \times D^c \) with the properties:

(i) the slice function \( K_s \) is hermitian symmetric on \( b \times b \);

(ii) for each \( r \), \( 0 \leq r < 1 \), and \( \nu \in b \),

\[
\int B_r(\nu, t) d\ell_t \leq B_r < \infty, \quad \text{where } B_r \text{ is independent of } \nu.
\]

Let \( T_r \) be the operator defined as in Section 1 by

\[
(T_r f)(z) = \int B_r(\nu, t) f(t) d\ell_t, \quad (z \in D, f \in L^1(b)).
\]

We give a necessary and sufficient condition that the family of operators \( \{ T_r : 0 \leq r < 1 \} \) maps \( L^1(b) \) into \( L^1(b) \) uniformly.

**Lemma.** A necessary and sufficient condition that the family of operators \( \{ T_r : 0 \leq r < 1 \} \), given by (1), maps uniformly from \( L^1(b) \) to \( L^1(b) \) is that

\[
\sup_{0 \leq r < 1} \sup_{\nu \in b} \int B_r(\nu, t) d\ell_t \leq C < \infty.
\]

**Proof.** Condition (2) is sufficient. The inequality \( \| T_r f \|_1 \leq C \| f \|_1 \) follows immediately from Fubini's theorem, the hermitian symmetry of \( K_s(\nu, t) \) and (2).

Condition (2) is necessary. Assume for arbitrary \( f \in L^1(b) \) that

\[
\| T_r f \|_1 \leq C \| f \|_1,
\]

where \( C \) is independent of \( f \) and \( r \), and prove that (2) holds.

If \( g \in L^\infty(b) \), then by Fubini's theorem, the hermitian symmetry of \( K_s(\nu, t) \) and (3)

\[
\int B_r(\nu, t) f(t) d\ell_t \leq \| g \|_\infty \int B_r(\nu, t) f(t) d\ell_t \leq C \| g \|_\infty \| f \|_1,
\]

where \( C \) independent of \( f \), \( r \) and \( g \). The linear functional given by

\[
S_{\nu}(f) = \int B_r(\nu, t) f(t) d\ell_t, \quad (f \in L^1(b), \ g \in L^\infty(b))
\]

is bounded, since by (4)

\[
\| S_{\nu} \| = \sup_{\| f \|_1 \geq 0} \frac{|S_{\nu}(f)|}{\| f \|_1} \leq C \| g \|_\infty,
\]

where \( C \) is independent of \( g \) and \( r \). The rest of the proof of condition (2) is standard and the necessity of the lemma follows.
2. First mapping theorem. Apply the lemma to the domains $R_{\gamma,\delta}$ Let

$$K_{\gamma,\delta}(z, w) = S(z, w) R_{\gamma,\delta}(r) \quad (z \in D_{\gamma}, \, w \in D^{-}),$$

where $R_{\gamma,\delta}$ is defined by (1.4) with values $\gamma, \delta$ related to $\alpha, \beta$ in Theorems 1 and 2. $K_{\gamma,\delta}$ satisfies properties (i) and (ii) of Section 1; (ii) follows since $K_{\gamma,\delta}$ is continuous. By (1)

$$\left( T_{\gamma,\delta}^\gamma f \right)(z) = \int_b\int S(z, t) f(t) ds_r R_{\gamma,\delta}(r) \quad (z \in D_{\gamma}). \tag{5}$$

Then:

**Theorem 3.** Let $D$ be one of the classical domains $R_1(n, n), R_{II}, R_{III}$ ($n$ even) or $R_{IV}$ with B-S boundary $b$. The family of operators $\mathcal{F}_{\gamma,\delta} = \{T_{\gamma,\delta}^\gamma : 0 \leq r < 1\}$, $T_{\gamma,\delta}^\gamma f$ given by (5) maps uniformly from $L^1(b)$ to $H^1(D)$ if and only if either $\gamma > \alpha, \delta$ arbitrary or $\gamma = \alpha, \delta \geq \beta$. For $R_1(m < n)$ and $R_{III}$ ($n$ odd) the condition is sufficient for the family $\mathcal{F}_{\gamma,\delta}$ to map uniformly from $L^1(b)$ to $H^1(D)$.

**Proof.** Assume that either $\gamma > \alpha, \delta$ arbitrary or $\gamma = \alpha, \delta \geq \beta$ and prove that $T_{\gamma,\delta}^\gamma$ maps $L^1(b)$ uniformly into $H^1(D)$.

By Theorems 1 and 2 and the hermitian symmetry of $S_r(v, t)$

$$B \leq \int_b |S(rv, t)| ds_r R_{\gamma,\beta}(r) \leq A.$$

Multiply by $R_{\gamma - a, \delta - \beta}(r)$. Since $(1 - r)^a \log^b [1/(1 - r)]$ is bounded on $0 \leq r < 1$ if $a > 0, b$ arbitrary or $a = 0, b \geq 0$, we have

$$BR_{\gamma - a, \delta - \beta}(r) \leq \int_b |S(rv, t)| ds_r R_{\gamma,\delta}(r) \leq A R_{\gamma - a, \delta - \beta}(r) \leq A. \tag{6}$$

Since $T_{\gamma,\delta}^\gamma f$ is holomorphic in $z$ on $D$, we use the $H^1$ metric

$$||T_{\gamma,\delta}^\gamma f||_{H^1} = \sup_{0 \leq r < 1} \int_b |(T_{\gamma,\delta}^\gamma f)(grv)| ds_r.$$

By (5), Fubini's theorem and the monotonicity of the mean $\int_b |S(grv, t)| ds_r$ in $g$ ([4], p. 523)

$$||T_{\gamma,\delta}^\gamma f||_{H^1} \leq \int_b |f(t)| ds_t \int_b |S(rv, t)| ds_r R_{\gamma,\delta}(r) \leq A ||f||_1$$

by (6), where $A$ is independent of $f$ and $r$. Thus the family $\mathcal{F}_{\gamma,\delta}$ maps $L^1(b)$ uniformly into $H^1(D)$ if $\gamma > \alpha, \delta$ arbitrary or $\gamma = \alpha, \delta \geq \beta$.

Conversely assume that the family $\mathcal{F}_{\gamma,\delta}$ maps $L^1(b)$ uniformly into $H^1(D)$ for the domains $R_1(n, n), R_{II}, R_{III}$ ($n$ even) and $R_{IV}$ and prove that either $\gamma > \alpha, \delta$ arbitrary or $\gamma = \alpha, \delta \geq \beta$. We give a proof by contradiction. Suppose first that $\gamma < \alpha$ and $\delta$ is arbitrary. In (6) take sup with respect to $t$, giving
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$$BR_{\gamma-\alpha,\delta-\beta}(r) \leq \sup_{\text{tub } b} \int |S(rv, t)| \, ds_v \, R_{r,\delta}(r) \leq A \, R_{\gamma-\alpha,\delta-\beta}(r).$$

Hence

$$\sup_{0 < r < 1} \sup_{\text{tub } b} \int |S(rv, t)| \, ds_v \, R_{r,\delta}(r) = \infty$$

and by the Lemma the family $\mathcal{F}_{\gamma,\delta}$ does not map $L^1(b)$ uniformly into $H^1(D)$. Thus $\gamma > \alpha$. Similarly if $\alpha = \gamma$ and $\delta < \beta$ (7) holds and $\mathcal{F}_{\gamma,\delta}$ does not map $L^1(b)$ uniformly into $H^1(D)$. 

3. Second mapping theorem. Let $D = R_I$, $R_{II}$, $R_{III}$ or $R_{IV}$. Take $\varepsilon$, $0 < \varepsilon < 1$, and define the operator $L^{\varepsilon,\delta}_r$ by

$$(L^{\varepsilon,\delta}_r f)(v) = \inf_{1 - \varepsilon < r < 1} \int |S(rv, t)| \, |f(t)| \, ds_t \, R_{r,\delta}(r)$$

$(v \in b, f \in L^1(b))$. The function $L^{\varepsilon,\delta}_r f$ is non-decreasing in $\varepsilon$ as $\varepsilon \to 0$. Hence the limit exists as an extended real-valued function. Let

$$(8) \quad L^{\varepsilon,\delta} f = \lim_{\varepsilon \to 0} L^{\varepsilon,\delta}_r f.$$ 

Then:

**Theorem 4.** Let $D$ be one of the classical domains $R_I(n, n)$, $R_{II}$, $R_{III}$ $(n$ even $)$ or $R_{IV}$ with B–S boundary $b$ and $f \in L^1(b)$. Then $L^{\varepsilon,\delta} f$ given by (8) is a bounded operator from $L^1(b)$ to $L^1(b)$ if and only if either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta, \gamma > \beta$. The condition is sufficient for $R_I(m < n)$ and $R_{III}$ $(n$ odd $)$. 

**Proof.** Assume that either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha$, $\delta, \gamma > \beta$ and prove that $L^{\varepsilon,\delta} f$ maps $L^1(b)$ into. By a property of inf, Fubini’s theorem and Theorems 1 and 2 we have

$$\left\{ (L^{\varepsilon,\delta}_r f)(v) \, ds_v \right\} \leq A \|f\|_1 \, R_{\gamma-\alpha,\delta-\beta}(r) \leq A \|f\|_1$$

$(A$ independent of $\varepsilon)$, since $R_{\gamma-\alpha,\delta-\beta}(r)$ is bounded. Hence by the monotone convergence theorem

$$\|L^{\varepsilon,\delta} f\|_1 = \left\{ (L^{\varepsilon,\delta}_r f)(v) \, ds_v \right\} = \lim_{\varepsilon \to 0} \left\{ (L^{\varepsilon,\delta}_r f)(v) \, ds_v \right\} \leq A \|f\|_1$$

if $\gamma > \alpha, \delta$ arbitrary or $\gamma = \alpha, \delta, \gamma > \beta$ and $A$ independent of $r$ and $\varepsilon$.

For the necessity of Theorem 4 assume that $L^{\varepsilon,\delta} f$ maps $L^1(b)$ into $L^1(b)$ and prove that either $\gamma > \alpha$, $\delta$ arbitrary or $\gamma = \alpha, \delta, \gamma > \beta$. Assume that $\gamma < \alpha$, $\delta$ arbitrary. Take $f = 1$. Then $f \in L^1(b)$ and by Theorems 1 and 2

$$(L^{\varepsilon,\delta}_r 1)(v) = \inf_{1 - \varepsilon < r < 1} \int |S(rv, t)| \, ds_t \, R_{r,\delta}(r) \geq B \inf_{1 - \varepsilon < r < 1} R_{\gamma-\alpha,\delta-\beta}(r),$$

where $B$ is independent of $f, r$ and $\varepsilon$. Since $1/(1-r) \geq 1/\varepsilon$

$$(9) \quad (L^{\varepsilon,\delta}_r 1)(v) \geq \inf_{1 - \varepsilon < r < 1} \frac{\log^{\beta-\delta} \frac{1}{1-r}}{(1-r)^{\alpha-\gamma}} \quad (\gamma < \alpha, \delta \text{ arbitrary})$$

$$\geq (\log 1/\varepsilon)^{\beta-\delta} / \varepsilon^{\alpha-\gamma}$$
for \(1 - r\) sufficiently small. By the monotonicity of \(L^{\gamma, \delta}_e\) in \(e\) as \(e \to 0\)
\[
\int_b (L^{\gamma, \delta}_e)(v) ds_v \geq \int_b (L^{\gamma, \delta}_e)(v) ds_v \geq B \log^{\beta - \delta} \frac{1}{\epsilon} e^{\alpha - \gamma}
\]
by (9). Thus if \(\gamma < \alpha, \delta\) arbitrary the right side of (10) \(\to \infty\) as \(e \to 0^+\) so that
\[
\int_b (L^{\gamma, \delta}_e)(v) ds_v
\]
is not a bounded operator from \(L^1(b)\) to \(L^1(b)\) for all \(f \in L^1(b)\);
similarly if \(\gamma = \alpha, \delta < \beta\). Hence either \(\gamma > \alpha, \delta\) arbitrary or \(\gamma = \alpha, \delta \geq \beta\) and the necessity is proved.

4. Mapping theorem for \(p \geq 1\). Let \(f \in L^p(b), p \geq 1\). Define the operator \(T_\sigma\) by
\[
(T_\sigma f)(z) = \int_b K_\sigma(z, t) f(t) ds_t, \quad (z \in D),
\]
where \(\sigma > \sigma_0 > 0\) and
\[
K_\sigma(z, t) = \frac{S(z, t)^{1 + \sigma/N}}{S(z, z)^{\sigma/N}}.
\]
The constant \(\sigma_0\) depends on the complex dimension of the domain \(D\) and constants, which come from the underlying Lie group theory in the Harish-Chandra realization of an irreducible bounded symmetric domain.

Fix \(r\) in (0, 1) and set \(z = rv\) for \(v \in b\). Then \(T_\sigma\) depends on \(r\) and we call the operator \(T^{(r)}_\sigma\):

\[
(T^{(r)}_\sigma f)(v) = \int_b K_\sigma(rv, t) f(t) ds_t.
\]

Theorem 5. For the classical domains the operator \(T^{(r)}_\sigma (\sigma > \sigma_0)\) given by (12) is a bounded linear operator from \(L^p(b)\) to \(H^p(D)\) for \(p \geq 1\) and
\[
\|T^{(r)}_\sigma f\|_p \leq A \|f\|_p \quad (A \text{ independent of } r).
\]

Proof. Fix \(r\) in (0, 1) and note that
\[
\int_b |K_\sigma(rv, t)| ds_t = \int_b \frac{|S(rv, t)|^{1 + \sigma/N}}{|S(rv, rv)|^{\sigma/N}} ds_t = C(1 - r^2)^{\sigma} \int_b |S(rv, t)|^{1 + \sigma/N} ds_t,
\]
since as noted in Section 1 for the classical domains \(S(rv, rv) = 1/(1 - r^2)^N\) for all \(v \in b[6]\).

The bounds in Theorem 4.1 of [1] using our notation, are

\[
BS(z, z)^{\sigma/N} \leq \int_b |S(z, t)|^{1 + \sigma/N} ds_t \leq A S(z, z)^{\sigma/N}
\]
for \(\sigma > \sigma_0\). If \(z = rv (v \in b)\) (14) becomes
\[
B(1 - r^2)^{-\sigma} \leq \int_b |S(rv, t)|^{1 + \sigma/N} ds_t \leq A(1 - r^2)^{-\sigma}.
\]
Thus

\[(15a) \quad \int_b \left| K_\sigma(rv,t) \right| ds_t \leq A(1-r^2)^\sigma \cdot (1-r^2)^{-\sigma} \leq A,\]

where \( A \) is independent of \( r \). By the hermitian symmetry and homogeneity of \( K_\sigma \), also,

\[(15b) \quad \int_b \left| K_\sigma(rv,t) \right| ds_t \leq A.\]

By (15a) and (15b) and standard arguments we get our result.

The same theorem holds for the operator \( \tilde{T}^{(\sigma)}_{\varrho} \) given by

\[(\tilde{T}^{(\sigma)}_{\varrho} f)(z) = \int_b \frac{S(z,t)^{1+\sigma^N}}{S(\varrho t, \varrho t)^{\sigma^N}} f(t)ds_t \quad (0 \leq \varrho < 1),\]

where \( \|\tilde{T}^{(\sigma)}_{\varrho} f\|_p \leq C\|f\|_p \), and \( \lim_{\varrho \to 1} \tilde{T}^{(\sigma)}_{\varrho} f \) exists in \( H^p(b) \).

In this case \( (\tilde{T}^{(\sigma)}_{\varrho} f)(z) \) is a holomorphic function of \( z \) and the mapping is from \( H^p(b) \) into \( H^p(D) \).

References


DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK AT BUFFALO
and
DEPARTMENT OF MATHEMATICS
AUBURN UNIVERSITY, AUBURN, ALABAMA

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