

## LIFTING VECTOR-VALUED MAPS

BY

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The aim of this paper is to study liftings of continuous and holomorphic functions with values in complex Banach spaces. In Section 1 we consider the case of continuous while in Section 2 the case of holomorphic functions.

**1. Lifting vector-valued continuous functions.** Given two complex Banach spaces  $B$  and  $\tilde{B}$ , let  $\text{Hom}(B, \tilde{B})$  denote the Banach space of continuous linear maps from  $B$  into  $\tilde{B}$  endowed with the uniform topology. Put

$$S(B, \tilde{B}) = \{T \in \text{Hom}(B, \tilde{B}) : \text{Im } T = \tilde{B}\}.$$

Then  $S(B, \tilde{B})$  is open ([5]).

Let  $S$  be a topological space. By  $C(S, B)$  we denote the locally convex space of all continuous maps from  $S$  into  $B$  endowed with the open-compact topology. Let  $g: S \times B \rightarrow \tilde{B}$  be a continuous map such that  $g_s \in \text{Hom}(B, \tilde{B})$  for all  $s \in S$ , where  $g_s(u) = g(s, u)$  for all  $u \in B$ . Then  $g$  induces a continuous linear map  $\hat{g}: C(S, B) \rightarrow C(S, \tilde{B})$  by  $[\hat{g}\sigma]s = g(s, \sigma(s))$ .

Now let  $g_s \in S(B, \tilde{B})$  for all  $s \in S$ . In [1] Bartle and Graves proved that the map  $\hat{g}$  is surjective if  $S$  is paracompact and the map  $\tilde{g}: S \rightarrow \text{Hom}(B, \tilde{B})$  associated with  $g$  is continuous.

In this section we shall find necessary and sufficient conditions for  $\hat{g}$  to be surjective.

**Definition 1.1** ([5]). Let  $g: S \times B \rightarrow \tilde{B}$  be a continuous map such that  $g_s \in S(B, \tilde{B})$  for all  $s \in S$ . Then the map  $g$  is called *locally uniformly open* if for every  $s_0 \in S$  there exists a neighbourhood  $G$  of  $s_0$  such that

$$\bigcap_{s \in G} \{g_s u : u \in B, \|u\| \leq 1\}$$

is a neighbourhood of zero in  $\tilde{B}$ .

In [5] Kurato and Kas proved that if  $\tilde{g}$  is continuous, then  $g$  is locally uniformly open. Let us consider the following examples.

**Example 1.1.** Let  $S$  be a compact space and let  $Ev: S \times C(S) \rightarrow C$  be

the evaluation map. Since  $Ev_s \{ \sigma \in C(S) : \|\sigma\| < \varepsilon \} \supset \{ z \in \mathbb{C} : \|z\| < \varepsilon \}$  for every  $s \in S$ , the map  $Ev$  is uniformly open. Obviously  $\tilde{E}v$  is continuous if and only if  $S$  is finite.

**Example 1.2.** Let  $H$  be a separable Hilbert space with  $\dim H = \infty$ . Put

$$S_2(H) = \{ T \in \mathcal{S}(H, H) : \|T\| \leq 2 \}$$

and

$$\varrho(T, P) = \sum_{k=1}^{\infty} \|Tu_k - Pu_k\| / 2^k (1 + \|Tu_k - Pu_k\|)$$

for all  $T, P \in S_2(H)$ , where  $\overline{\{u_k\}} = H$ . Consider the evaluation map  $Ev: S_2(H) \times H \rightarrow H$ . Let  $\varrho(T_n, T) \rightarrow 0$  and  $x_n \rightarrow x$  in  $H$ . Then

$$\begin{aligned} \|T_n x_n - Tx\| &\leq \|T_n x_n - T_n x\| + \|T_n x - Tx\| \leq \|T_n\| \|x_n - x\| + \|T_n x - Tx\| \\ &\leq 2 \|x_n - x\| + \|T_n x - Tx\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence  $Ev$  is continuous. Obviously  $\text{Im } Ev_T = H$  for all  $T \in S_2(H)$ . We show that  $Ev$  is not locally uniformly open. Contrarily, suppose that there exist  $\varepsilon > 0$ ,  $\delta > 0$  and  $z_1, \dots, z_p \in H$  such that for every  $T \in W = \{ T \in S_2(H) : \|Tz_j - z_j\| < \delta \}$  we have  $TU_1 \supseteq U_\varepsilon$ , where  $U_r = \{ u \in H : \|u\| < r \}$ . For every  $n$  put  $T_n(u' + u'') = u'/n + u''$ , where  $u' \in \text{span}\{z_1, \dots, z_p\}^\perp$  and  $u'' \in \text{span}\{z_1, \dots, z_p\}$ . Then  $\|T_n\| \leq 1$ ,  $\text{Im } T_n = H$  and  $\|T_n z_j - z_j\| = 0$  for all  $j = 1, \dots, p$ . Hence  $T_n \in W$  for all  $n$ . Thus  $T_n U_1 \supseteq U_\varepsilon$  for all  $n$ , whence

$$\sup_{\|u'\| \leq 1} \|T_n(u', 0)\| = \sup_{\|u'\| \leq 1} \|u'/n\| = 1/n \geq \varepsilon \quad \text{for all } n.$$

This contradiction shows that  $Ev$  is not locally uniformly open. For every topological space  $S$  and for every Banach space  $B$  we denote by  $C_b(S, B)$  the Banach space of all bounded continuous maps from  $S$  into  $B$  with the sup-norm.

**Definition 1.2.** Let  $g: S \times B \rightarrow \tilde{B}$  be a continuous map such that  $g_s \in \mathcal{S}(B, \tilde{B})$  for all  $s \in S$ . We say that  $g$  has the local  $b$ -lifting property if for every  $s_0 \in S$  there exists a neighbourhood  $G$  of  $s_0$  such that  $\sup \{\|g_s\| : s \in G\} < \infty$  and the map  $\hat{g}_G = \widehat{g|_G}: C_b(G, B) \rightarrow C_b(G, \tilde{B})$  is surjective.

**THEOREM 1.1.** Let  $S$  be a locally paracompact space and  $g: S \times B \rightarrow \tilde{B}$  be a continuous map such that  $g_s \in \mathcal{S}(B, \tilde{B})$  for all  $s \in S$ . Then the following conditions are equivalent:

- (i)  $g$  has the local  $b$ -lifting property;
- (ii)  $g$  is locally uniformly open;
- (iii) the map  $(\text{id}, g): S \times B \rightarrow S \times \tilde{B}$  is open.

**Proof.** (i)  $\Rightarrow$  (ii) Let  $s_0 \in S$  and  $G$  be a neighbourhood of  $s_0$  such that

$\sup \{\|g_s\|: s \in G\} < \infty$  and  $\hat{g}_G$  is surjective. By the open mapping theorem for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\hat{g}_G \{\sigma \in C_b(G, B): \|\sigma\| \leq \varepsilon\} \supseteq \{\sigma'' \in C_b(G, \tilde{B}): \|\sigma''\| < \delta\}.$$

From this relation we infer that

$$\bigcap_{s \in G} g_s U_\varepsilon \supseteq \{u'' \in B: \|u''\| < \delta\}.$$

Hence  $g$  is locally uniformly open.

(ii)  $\Rightarrow$  (iii) Let  $G \times U$  be an open set in  $S \times B$  and let  $(s_0, u'_0) \in (\text{id}, g)(G \times U)$ . Then  $u'_0 = g_{s_0} u_0$  for  $u_0 \in U$ . Take a  $\varepsilon > 0$  such that  $u_0 + U_\varepsilon \subseteq U$  and a neighbourhood  $G_0$  of  $s_0$  such that  $W'' = \bigcap \{g_s U_\varepsilon: s \in G_0\}$  is a neighbourhood of zero in  $\tilde{B}$ . By continuity of  $g$  there exists a neighbourhood  $G_1 \subseteq G_0$  of  $s_0$  such that  $g_{s_0} u_0 - g_s u_0 \in W_1''$  for all  $s \in G_1$ , where  $W_1''$  is a neighbourhood of zero in  $\tilde{B}$  such that  $W_1'' + W_1'' \subseteq W''$ . Let  $(s, u'_0 + u'') \in G_1 \times (u'_0 + W_1'')$ . Then

$$g_s u_0 - u'' = (g_s u_0 - g_{s_0} u_0) + g_{s_0} u_0 - u'' \in u'_0 - W''.$$

Therefore

$$u'_0 + u'' = g_s(u_0 + u), \quad u \in U_\varepsilon.$$

Hence

$$(s, u'_0) \in G_1 \times (u'_0 + W_1'') \subseteq (\text{id}, g)(G \times U).$$

Consequently  $(\text{id}, g)$  is open.

(iii)  $\Rightarrow$  (i) Let  $s_0 \in S$ . By hypothesis there exist a paracompact neighbourhood  $G$  of  $s_0$  and a  $\delta > 0$  such that  $\sup \{\|g_s\|: s \in G\} < \infty$  and  $g_s U_1 \supseteq U_\delta''$  for all  $s \in G$ . It follows that there exists a constant  $C > 0$  such that for every  $(s, u'') \in G \times \tilde{B}$  there is a  $u_s \in B$  such that  $g_s u_s = u''$  and  $\|u_s\| \leq C \|u''\|$ .

Let  $\sigma'' \in C_b(G, B)$  and  $\varepsilon > 0$ . By paracompactness of  $\tilde{B}$  it follows that there is  $\sigma_\varepsilon'' \in C_b(G, \tilde{B})$  such that

$$\|\sigma_\varepsilon'' - \sigma''\| \leq \varepsilon, \quad \|\sigma_\varepsilon''\| \leq \|\sigma''\|$$

and

$$\sigma_\varepsilon''(s) = \sum_{i \in I} \varphi_i(\sigma'' s) u_i'',$$

where  $\{\varphi_i\}$  is a locally finite partition of unity for an open cover of  $\tilde{B}$  and  $u_i'' \in \sigma''(G)$  for all  $i \in I$ .

Fix  $i \in I$ . For every  $s \in G$  we choose  $u_{s,i} \in B$  such that

$$g_s u_{s,i} = u_i'' \quad \text{and} \quad \|u_{s,i}\| \leq C \|u_i''\|.$$

Since  $g$  is continuous, there is a neighbourhood  $G_{s,i}$  of  $s$  in  $G$  such that  $\|g_t u_{s,i} - u'_i\| < \varepsilon$  for all  $t \in G_{s,i}$ . Suppose  $\{\varphi_{s,i}\}$  is a locally finite partition of unity inscribed into cover  $\{G_{s,i}\}$ . Put

$$\sigma_i(t) = \sum_{s \in G} \varphi_{s,i}(t) u_{s,i}.$$

Obviously  $\sigma_i \in C_b(G, B)$  and

$$\|\hat{g}\sigma_i - u'_i\| \leq \sup_{t \in G} \sum_{s \in G} \varphi_{s,i}(t) \|g_t u_{s,i} - u'_i\| \leq \varepsilon$$

as well as

$$\begin{aligned} \|\sigma_i\| &\leq \sup_{t \in G} \sum_{s \in G} \varphi_{s,i}(t) \|u_{s,i}\| \\ &\leq \sup_{s \in G} \|u_{s,i}\| \leq C \|\sigma''\|. \end{aligned}$$

Setting

$$\sigma(t) = \sum_{i \in I} \varphi_i(\sigma''(t)) \sigma_i(t)$$

we obtain an element  $\sigma$  belonging to  $C_b(G, B)$  such that

$$\begin{aligned} \|\hat{g}\sigma - \sigma''\| &\leq \|\hat{g}\sigma - \sigma''_i\| + \|\sigma''_i - \sigma''\| \\ &\leq \varepsilon + \sup_{t \in G} \sum_{i \in I} \varphi_i(\sigma''(t)) \|g_t \sigma_i(t) - u'_i\| \\ &\leq 2\varepsilon \end{aligned}$$

and

$$\|\sigma\| \leq \sup_{t \in G} \sum_{i \in I} \varphi_i(\sigma''(t)) \|\sigma_i(t)\| \leq C \|\sigma''\|.$$

Hence

$$\hat{g} \{ \sigma \in C_b(G, B) : \|\sigma\| \leq C \} \supseteq \{ \sigma'' \in C_b(G, \tilde{B}) : \|\sigma''\| \leq 1 \}.$$

Therefore, by the open mapping theorem we infer that  $\hat{g}_G$  is surjective.

**THEOREM 1.2.** *Let  $S$  be an arbitrary topological space and  $\tilde{g}: S \rightarrow S(B, \tilde{B})$  be a continuous map. Then the map  $(\text{id}, g): S \times B \rightarrow S \times \tilde{B}$  is locally trivial.*

**Proof.** a) First we assume that  $S$  is a contractible metric space and  $\tilde{g}$  is bounded. By Theorem 1.1 (or by [1]) the map  $\hat{g}: C_b(S, B) \rightarrow C_b(S, \tilde{B})$  is surjective. Hence there exists a continuous map  $q: C_b(S, \tilde{B}) \rightarrow C_b(S, B)$  such that  $gq = \text{id}$ . Given a commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{f_0 = (f_0^1, f_0^2)} & S \times B \\
 \downarrow k & & \downarrow (\text{id}, g) \\
 X \times I & \xrightarrow{h = (h^1, h^2)} & S \times \tilde{B}
 \end{array}$$

in which  $I = [0, 1]$ ,  $k(x) = (x, 0)$ ,  $f_0$  and  $h$  are continuous and  $X$  is an arbitrary topological space. For every  $(x, t) \in X \times I$  put

$$f(x, t) = (h^1(x, t), q(\overline{h^2(x, t)})[h^1(x, t)] - q(\overline{h^2(x, 0)})[h^1(x, t)] + f_0^2(x))$$

where  $\overline{u''}(s) = u''$ ,  $u'' \in \tilde{B}$ , for all  $s \in S$ . Then

$$fk(x) = f(x, 0) = (h^1(x, 0), f_0^2(x)) = f_0(x), \quad (\text{id}, g)f = h.$$

Hence  $(S \times B, (\text{id}, g), S \times \tilde{B})$  is a fiber space in sense of Hurewicz. By contractibility of  $S \times \tilde{B}$  and by [6] we infer that  $(S \times B, (\text{id}, g), S \times \tilde{B})$  is trivial.

b) Now we assume that  $S$  is a metric space. Without loss of generality we can assume that  $S$  is a closed subset in a normed space  $C$ . Since  $S(B, \tilde{B})$  is open,  $\tilde{g}$  can be extended to a continuous map  $\tilde{f}$  from an open neighbourhood  $W$  of  $S$  in  $C$  into  $S(B, \tilde{B})$ . By a) and by the local contractibility of  $W$ , the map  $(\text{id}, \tilde{f}): W \times B \rightarrow W \times \tilde{B}$  is locally trivial. Hence  $(\text{id}, g)$  is locally trivial.

c) Finally we assume that  $S$  is an arbitrary topological space. By  $S|_\rho$  we denote the metric space of equivalent classes:  $s \sim t \Leftrightarrow g_s = g_t$ . This space is endowed with the metric induced by  $\rho(\bar{s}, \bar{t}) = \|g_s - g_t\|$ . Since the map  $\tilde{g}_\rho: S|_\rho \rightarrow S(B, \tilde{B})$  is continuous, by b) the map  $(\text{id}, g_\rho): S|_\rho \times B \rightarrow S|_\rho \times \tilde{B}$  is locally trivial. Thus it is easy to see that the map  $(\text{id}, g)$  is locally trivial.  $\square$

**2. Lifting vector-valued holomorphic functions.** Let  $X$  be a complex space,  $B$  and  $\tilde{B}$  be complex Banach spaces. A map  $g: X \times B \rightarrow \tilde{B}$  is said to be a holomorphic family of continuous linear maps from  $B$  into  $\tilde{B}$  if  $g_z \in \text{Hom}(B, \tilde{B})$  for all  $z \in X$  and  $g(\cdot, u)$  is holomorphic for all  $u \in B$ .

By Lemma 2.1, the map  $\tilde{g}: X \rightarrow \text{Hom}(B, \tilde{B})$  associated with  $g$  is holomorphic. Therefore  $g$  induces a continuous linear map  $\hat{g}: \mathcal{O}(X, B) \rightarrow \mathcal{O}(X, \tilde{B})$  defined by

$$[\hat{g}\sigma]z = g(z, \sigma(z)).$$

Here  $\mathcal{O}(X, B)$  denotes the locally convex space of holomorphic maps from  $X$  into  $B$  endowed with the compact-open topology.

In this section we shall prove the following

**THEOREM 2.1.** *Let  $X$  be a complex space having a countable topology and let  $\mathcal{O}(X)$  determine the topology of  $X$ . Then the following conditions are equivalent.*

- (i)  $X$  is a Stein space.
- (ii)  $H^1(X, \mathcal{O}_\xi) = 0$  for every sheaf  $\mathcal{O}_\xi$  of germs of holomorphic sections on  $X$  of a holomorphic Banach bundle  $\xi$  over  $X$ .
- (iii) The map  $\hat{g}: \mathcal{O}(X, B) \rightarrow \mathcal{O}(X, \tilde{B})$  is surjective for every holomorphic family  $g$  on  $X$  with values in  $S(B, \tilde{B})$  such that  $\text{Ker}(\text{id}, g)$  has a structure of a holomorphic Banach bundle over  $X$  for which the canonical map  $i: \text{Ker}(\text{id}, g) \rightarrow \underline{B}$  is holomorphic, where  $\underline{B}$  denotes the trivial bundle over  $X$  with the fiber  $B$ .
- (iv) The map  $\hat{g}: \mathcal{O}(X, B) \rightarrow \mathcal{O}(X)$  is surjective for every holomorphic family  $g: X \times B \rightarrow C$  of continuous linear functionals on  $B$  such that  $g_z \neq 0$  for all  $z \in X$ .

Proof. (i) implies (ii) by a theorem of Bungart ([2]).

(ii)  $\Rightarrow$  (iii) By exactness of the cohomology sequence, it is enough to prove that the sequence

$$0 \rightarrow \mathcal{O}_{\text{Ker}g} \rightarrow \mathcal{O}_{\underline{B}} \rightarrow \mathcal{O}_{\tilde{B}} \rightarrow 0$$

is exact,  $\bar{g} = (\text{id}, g)$ .

This statement is an immediate consequence of the following

LEMMA 2.1. Let  $g: X \times B \rightarrow \tilde{B}$  be a holomorphic family of continuous linear maps from a Banach space  $B$  into a Banach space  $\tilde{B}$ . Then the map  $\tilde{g}: X \rightarrow \text{Hom}(B, \tilde{B})$  associated with  $g$  is holomorphic.

Proof. We assume that  $X$  is an analytic set in an open subset of  $C^n$  for some  $n$ . Obviously the map  $\bar{g}: B \rightarrow \mathcal{O}(X, \tilde{B})$  associated with  $g$  is holomorphic. First we show that the map  $\tilde{g}_c: X \rightarrow \text{Hom}_c(B, \tilde{B})$  is holomorphic, where  $\text{Hom}_c(B, \tilde{B})$  denotes the space  $\text{Hom}(B, \tilde{B})$  endowed with the open-compact topology. Suppose  $\theta: \mathcal{O}^*(X) \rightarrow \text{Hom}_c(B, \tilde{B}^{**})$  is a map defined by  $[\theta\omega(u)](u''^*) = \omega(u''^*\bar{g}u)$ ,  $\tilde{B}^*$  and  $\mathcal{O}^*(X)$  are strong dual spaces of  $\tilde{B}$  and  $\mathcal{O}(X)$  respectively. Since  $\omega$  is continuous,  $\theta\omega(u) \in (\tilde{B}_c^*)^* = \tilde{B}$ . Hence, by continuity of  $\bar{g}$ , it follows that  $\theta$  is a continuous linear map from  $\mathcal{O}^*(X)$  into  $\text{Hom}_c(B, \tilde{B})$ . Hence, by a result of Bungart ([3]) we infer that  $\tilde{g}_c$  is holomorphic. We check that  $\tilde{g}$  is holomorphic. Let  $z_0 \in X$ . Since  $\tilde{g}_c$  is holomorphic, there exist a polydisc  $\Delta(z_0, r)$  with centre at  $z_0$  and polyradius  $r > 0$  and  $\tilde{f} \in \mathcal{O}(\Delta(z_0, r), \text{Hom}_c(B, \tilde{B}))$  extending of  $\tilde{g}_c|_{\Delta(z_0, r) \cap X}$  ([3]). Since  $\{\sup\|\tilde{f}z\|: z \in K\} < \infty$  for every compact set  $K$  in  $\Delta(z_0, r)$  by the Cauchy integral formula we infer that  $\tilde{f} \in \mathcal{O}(\Delta(z_0, r), \text{Hom}(B, \tilde{B}))$ . Consequently  $\tilde{g}$  is holomorphic.  $\square$

LEMMA 2.2. Let  $g: X \times B \rightarrow \tilde{B}$  be as in Lemma 2.1. Let  $g_{z_0}$  be surjective. Then  $\hat{g}_{z_0}: \mathcal{O}_{\underline{B}, z_0} \rightarrow \mathcal{O}_{\tilde{B}, z_0}$  is surjective.

Proof. We can assume  $X$  is an analytic set in an open set in  $C^n$  for some  $n$  and  $z_0 = 0 \in X$ . By Lemma 2.1 the map  $\tilde{g}: X \rightarrow \text{Hom}(B, \tilde{B})$  is

holomorphic and therefore  $\tilde{g}$  can be extended to a holomorphic map  $\tilde{f}$  from  $\Delta_{2r} = \Delta(0, 2r)$  into  $\text{Hom}(B, \tilde{B})$  with  $r > 0$ .

Consider the Taylor expansion of  $f$  at zero:

$$f(z) = A_0 + \sum_{|\alpha| \geq 1} A_\alpha z^\alpha$$

where  $z = (z_1, \dots, z_n)$ ,  $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . By hypothesis,  $A_0$  is surjective. Hence

$$\frac{1}{2}C = \supinf_{\|u''\|=1, A_0 u''=u''} \|u\| < \infty.$$

Let

$$M = \sup \{ \|\tilde{f}_\xi\| : \xi \in \Delta_r \}, \quad B_0 = A_0,$$

$$B_\alpha = r^{|\alpha|} (4MC_1 n)^{-2kn|\alpha|} A_\alpha, \quad |\alpha| \geq 1$$

and

$$Z = (4MC_1 n)^{2kn} z/r,$$

where  $C_1$  and  $k$  are positive numbers such that  $C_1 \geq C$ ,  $(MC_1) > 1$  and  $(MC_1)^k > C^2$ . We can assume that  $C > 1$ . Obviously  $\|B_\alpha\| \leq (4MC_1 n)^{-kn|\alpha|}$  for all  $|\alpha| \geq 1$ .

a) Let  $u'' \in \tilde{B}$ . We find  $\sigma \in \mathcal{O}(\Delta_1, B)$  such that

$$F(Z)\sigma(Z) = u'' \quad \text{for all } Z \in \Delta_1, \quad \|\sigma\|_{\Delta_{1/2}} \leq \tilde{C} \|u''\|,$$

where

$$F(Z) = B_0 + \sum_{|\alpha| \geq 1} B_\alpha Z^\alpha$$

and  $\tilde{C}$  is a constant independent of  $u''$ . We can assume that  $\|u''\| = 1$ . We find  $\sigma$  from the formula:

$$\sigma(Z) = \sum_{\alpha} u_\alpha Z^\alpha.$$

From the condition

$$F(Z)\sigma(Z) = u'' \quad \text{for all } Z \in \Delta_1$$

we get the relations

$$B_0 u_0 = u''$$

and

$$B_0 u_\beta = - \sum_{\substack{\alpha+\gamma=\beta \\ 0 < \alpha \leq \beta, 0 \leq \gamma < \beta}} B_\alpha u_\gamma.$$

We choose  $u_0 \in B$  such that  $B_0 u_0 = u''_0$  and  $\|u_0\| \leq C \|u''\| = C$ . Let  $u_\beta \in B$  for  $0 < \beta < \beta_0$  be such that

$$B_0 u_\beta = \sum_{\substack{\alpha+\gamma=\beta \\ 0 < \alpha \leq \beta, 0 \leq \gamma < \beta}} B_\alpha u_\gamma$$

and

$$\|u_\beta\| \leq 1.$$

We choose  $u_{\beta_0} \in B$  such that

$$B_0 u_{\beta_0} = - \sum_{\substack{\alpha+\gamma=\beta_0 \\ 0 < \alpha \leq \beta_0, 0 \leq \gamma < \beta_0}} B_\alpha u_\gamma$$

and

$$\|u_{\beta_0}\| \leq C \left\| \sum_{\substack{\alpha+\gamma=\beta_0 \\ 0 < \alpha \leq \beta_0, 0 \leq \gamma < \beta_0}} B_\alpha u_\gamma \right\|.$$

Then

$$\begin{aligned} \|u_{\beta_0}\| &\leq C \sum_{0 < \alpha \leq \beta_0} C(4C_1 M n)^{-kn|\alpha|} \leq \sum_{|\alpha|=1}^{\infty} (4b)^{-kn|\alpha|} \\ &\leq 2^n (4n)^{-kn} + 3^n (4n)^{-2kn} + \dots \leq 1. \end{aligned}$$

Since  $\|u_\alpha\| \leq C$  for all  $\alpha$ , the series  $\sum_{\alpha} u_\alpha Z^\alpha$  converges to an element  $\sigma \in \mathcal{O}(\Delta_1, B)$ . Obviously  $FZ\sigma(Z) = u''$  for all  $Z \in \Delta_1$  and  $\|\sigma\|_{\Delta_{1/2}} \leq \tilde{C}$ , where  $\tilde{C}$  is a constant independent of  $u''$ . Setting

$$\tilde{\sigma}(z) = \sigma((4MC_1 n)^{2kn} z/r)$$

we obtain an element  $\tilde{\sigma} \in \mathcal{O}(\Delta_\delta, B)$  with  $\delta = r/(4MC_1 n)^{2kn}$  such that  $\tilde{f}_Z \tilde{\sigma}(z) = u''$  for all  $z \in \Delta_\delta$  and  $\|\tilde{\sigma}\|_{\Delta_{\delta/2}} \leq \tilde{C}$ . Now let  $\sigma'' \in \mathcal{O}_{\tilde{B},0}$  and let

$$\tilde{\sigma}''(z) = \sum_{\alpha} u''_{\alpha} z^{\alpha}$$

be a representative of  $\sigma''$  on a polydisc  $\Delta_\delta$ ,  $\delta > 0$ . For every  $u''_{\alpha}$ , by a) there exists  $\sigma_{\alpha} \in \mathcal{O}(\Delta_{\delta_0}, B)$  such that

$$\tilde{f}_Z \sigma_{\alpha}(z) = u''_{\alpha} \quad \text{for all } z \in \Delta_{\delta_0}$$

and

$$\|\sigma_{\alpha}\|_{\Delta_{\delta_0/2}} \leq \tilde{C} \|u''_{\alpha}\| \quad \text{for all } \alpha$$

where  $\delta_0, \tilde{C}$  are constants independent of  $u''_{\alpha}$ . Hence, setting

$$\sigma(z) = \sum_{\alpha} \sigma_{\alpha} z^{\alpha}$$

we get an element  $\sigma \in \mathcal{O}_{B,0}$  such that  $\hat{g}_0 \sigma = \sigma''$ .  $\square$



LEMMA 2.3. Let  $g: X \times B' \rightarrow B$  be a holomorphic family of isomorphisms  $g_z$  from a Banach space  $B'$  into a Banach space  $B$  and let  $\sigma: X \rightarrow B'$  be a map such that  $\hat{g}\sigma$  is holomorphic. Then  $\sigma$  is holomorphic.

Proof. First we show that  $(\text{id}, g): X \times B' \rightarrow X \times B$  is an embedding. Suppose  $(z_n, g(z_n, u_n)) \rightarrow (z, u)$ . Then  $z_n \rightarrow z$  and therefore by Lemma 2.1,  $g_{z_n} \rightarrow g_z$ . We show that  $\{g_z u_n\}$  is a Cauchy sequence in  $B$  and therefore  $u_n \rightarrow u$ . Since  $g_{z_n}$  is surjective and the map  $\tilde{g}^*: X \rightarrow \text{Hom}(B^*, B'^*)$  is continuous, it follows that  $\tilde{g}_{z_n}^*(U^*) \supseteq \varepsilon U'^*$  for some  $\varepsilon > 0$  and for all  $n > n_0$  ([5]), where  $U^*$  and  $U'^*$  are unit balls in  $B^*$  and  $B'^*$  respectively. Therefore for  $n > n_0$  we have

$$\begin{aligned} \varepsilon \|u_n\| &= \sup_{\|u^*\| \leq 1} |u^*(u_n)| \leq \sup_{\|u^*\| \leq 1} |\tilde{g}_{z_n}^* u^*(u_n)| \\ &= \sup_{\|u^*\| \leq 1} |u^* g_{z_n} u_n| \leq \sup_n \|g_{z_n} u_n\| < \infty. \end{aligned}$$

Hence

$$\|g_z u_n - g_z u_m\| \leq \|g_{z_n} - g_z\| \|u_n\| + \|g_{z_n} u_n - g_{z_m} u_m\| + \|g_{z_m} - g_z\| \|u_m\| \rightarrow 0.$$

Thus  $\sigma$  is continuous. Hence, in order to prove that  $\sigma$  is holomorphic it is enough to prove that  $u^* \sigma$  is holomorphic for all  $u^* \in B'^*$  ([3]).

Let  $z_0 \in X$  and  $u^* \in B'^*$ . By Lemma 2.2 there exists  $\gamma \in \mathcal{O}(U, B^*)$ , where  $U$  is a neighbourhood of  $z_0$ , such that  $g_z^* \gamma(z) = u^*$  for all  $z \in U$ . Therefore we have

$$u^* \sigma(z) = [g_z^* \gamma(z)] \sigma(z) = \gamma(z) (g_z \sigma(z)).$$

Since  $\hat{g}\sigma$  is holomorphic, we infer that  $u^* \sigma$  is holomorphic.  $\square$

(iii)  $\Rightarrow$  (iv) Proof is trivial, because  $\text{Ker } g_z$  is complemented in  $B$  for all  $z \in X$  and therefore  $\text{Ker } \bar{g}$  is a holomorphic Banach bundle.

(iv)  $\Rightarrow$  (i) By [4] it is enough to prove that  $X = \mathfrak{M}(\mathcal{O}(X))$ , where  $\mathfrak{M}(\mathcal{O}(X))$  denotes the spectrum of  $\mathcal{O}(X)$ . Since  $\mathcal{O}(X)$  determines the topology of  $X$  it suffices to show that  $V_\omega = \{z \in X: \sigma(z) = 0, \sigma \in \text{Ker } \omega\} \neq \emptyset$  for all  $\omega \in \mathfrak{M}(\mathcal{O}(X))$ . For a contradiction, let  $V_{\omega_0} = \emptyset$  for some  $\omega_0$ . Since  $\mathcal{O}(X)$  is a separable Fréchet space there exists  $\{\sigma_n\} \subset \text{Ker } \omega_0$  such that

$$\bigcap_{n=1}^{\infty} \{z \in X: \sigma_n(z) = 0\} = \emptyset \quad \text{and} \quad \sigma_n \rightarrow 0.$$

Consider the holomorphic family  $g: X \times l^1 \rightarrow C$  of continuous linear functionals on  $l^1$  defined by

$$g(z, \{\xi_n\}) = \sum_{n=1}^{\infty} \xi_n \sigma_n(z).$$

Since  $\bigcap_{n=1}^{\infty} \{z \in X: \sigma_n(z) = 0\} = \emptyset$  we infer that  $g_z \neq 0$  for all  $z \in X$ . By hy-

pothesis, it follows that there exists a sequence  $\{\beta_n\} \subset \mathcal{O}(X)$  such that

$$\sum_{n=1}^{\infty} \beta_n(z) \sigma_n(z) = 1 \quad \text{for all } z \in X.$$

This relation shows that  $\text{Ker } \omega_0 = \mathcal{O}(X)$ . But this is impossible, because  $\omega_0 \neq 0$ .  $\square$

#### REFERENCES

- [1] R. Bartle and L. Graves, *Mappings between function spaces*, Transactions of the American Mathematical Society 72 (1952), p. 400–413.
- [2] L. Bungart, *On analytic fiber bundles, I. Holomorphic fiber bundles with infinite-dimensional fibers*, Topology 7 (1968), p. 55–68.
- [3] —, *Holomorphic functions with values in locally convex spaces and applications to integral formulas*, Transactions of the American Mathematical Society 111 (1964), p. 317–343.
- [4] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, NJ, 1965.
- [5] A. Kurato and M. P. Kas, *Theorems on lifting of vector-valued functions*, Izvestija Akademii Nauk SSSR. Serija Matematičeskaja 4 (1975), p. 911–925.
- [6] S. Teleman, *Topologie und differenzierbare Mannigfaltigkeiten*, Berlin 1968.

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