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LIFTING VECTOR-VALUED MAPS

BY

NGUYEN VAN KHUE (WARSZAWA)

The aim of this paper is to study liftings of continuous and holomorphic functions with values in complex Banach spaces. In Section 1 we consider the case of continuous while in Section 2 the case of holomorphic functions.

1. Lifting vector-valued continuous functions. Given two complex Banach spaces B and $\tilde{B}$, let $\mathrm{Hom}(B, \tilde{B})$ denote the Banach space of continuous linear maps from B into $\tilde{B}$ endowed with the uniform topology. Put

$$S(B, \tilde{B}) = \{T \in \mathrm{Hom}(B, \tilde{B}):\ \mathrm{Im}\, T = \tilde{B}\}.$$

Then $S(B, \tilde{B})$ is open ([5]).

Let S be a topological space. By $C(S, B)$ we denote the locally convex space of all continuous maps from S into B endowed with the open-compact topology. Let $g\colon S \times B \to \tilde{B}$ be a continuous map such that $g_s \in \mathrm{Hom}(B, \tilde{B})$ for all $s \in S$, where $g_s(u) = g(s, u)$ for all $u \in B$. Then g induces a continuous linear map $\hat{g}\colon C(S, B) \to C(S, \tilde{B})$ by $[\hat{g}\sigma]s = g(s, \sigma(s))$.

Now let $g_s \in S(B, \tilde{B})$ for all $s \in S$. In [1] Bartle and Graves proved that the map $\hat{g}$ is surjective if S is paracompact and the map $\tilde{g}\colon S \to \mathrm{Hom}(B, \tilde{B})$ associated with g is continuous.

In this section we shall find necessary and sufficient conditions for $\hat{g}$ to be surjective.

Definition 1.1 ([5]). Let $g\colon S \times B \to \tilde{B}$ be a continuous map such that $g_s \in S(B, \tilde{B})$ for all $s \in S$. Then the map g is called *locally uniformly open* if for every $s_0 \in S$ there exists a neighbourhood G of s_0 such that

$$\bigcap_{s \in G} \{g_s u:\ u \in B,\ \|u\| \leqslant 1\}$$

is a neighbourhood of zero in $\tilde{B}$.

In [5] Kurato and Kas proved that if $\tilde{g}$ is continuous, then g is locally uniformly open. Let us consider the following examples.

Example 1.1. Let S be a compact space and let $Ev\colon S \times C(S) \to C$ be the evaluation map. Since $Ev_s\{\sigma \in C(S)\colon \|\sigma\| < \varepsilon\} \supset \{z \in C\colon \|z\| < \varepsilon\}$ for every $s \in S$, the map Ev is uniformly open. Obviously $\widetilde{Ev}$ is continuous if and only if S is finite.

Example 1.2. Let H be a separable Hilbert space with $\dim H = \infty$. Put

$$S_2(H) = \{T \in S(H, H)\colon \|T\| \leqslant 2\}$$

and

$$\varrho(T, P) = \sum_{k=1}^{\infty} \|Tu_k - Pu_k\|/2^k(1 + \|Tu_k - Pu_k\|)$$

for all $T, P \in S_2(H)$, where $\overline{\{u_k\}} = H$. Consider the evaluation map $Ev\colon S_2(H) \times H \to H$. Let $\varrho(T_n, T) \to 0$ and $x_n \to x$ in H. Then

$$\begin{aligned}\|T_n x_n - Tx\| &\leqslant \|T_n x_n - T_n x\| + \|T_n x - Tx\| \leqslant \|T_n\|\,\|x_n - x\| + \|T_n x - Tx\| \\ &\leqslant 2\|x_n - x\| + \|T_n x - Tx\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence Ev is continuous. Obviously $\operatorname{Im} Ev_T = H$ for all $T \in S_2(H)$. We show that Ev is not locally uniformly open. Contrarily, suppose that there exist $\varepsilon > 0$, $\delta > 0$ and $z_1, \ldots, z_p \in H$ such that for every $T \in W = \{T \in S_2(H)\colon \|Tz_j - z_j\| < \delta\}$ we have $TU_1 \geqslant U_\varepsilon$, where $U_r = \{u \in H\colon \|u\| < r\}$. For every n put $T_n(u' + u'') = u'/n + u''$, where $u' \in \operatorname{span}\{z_1, \ldots, z_p\}^\perp$ and $u'' \in \operatorname{span}\{z_1, \ldots, z_p\}$. Then $\|T_n\| \leqslant 1$, $\operatorname{Im} T_n = H$ and $\|T_n z_j - z_j\| = 0$ for all $j = 1, \ldots, p$. Hence $T_n \in W$ for all n. Thus $T_n U_1 \geqslant U_\varepsilon$ for all n, whence

$$\sup_{\|u'\| \leqslant 1} \|T_n(u', 0)\| = \sup_{\|u'\| \leqslant 1} \|u'/n\| = 1/n \geqslant \varepsilon \quad \text{for all } n.$$

This contradiction shows that Ev is not locally uniformly open. For every topological space S and for every Banach space B we denote by $C_b(S, B)$ the Banach space of all bounded continuous maps from S into B with the sup-norm.

Definition 1.2. Let $g\colon S \times B \to \tilde{B}$ be a continuous map such that $g_s \in S(B, \tilde{B})$ for all $s \in S$. We say that g *has the local b-lifting property* if for every $s_0 \in S$ there exists a neighbourhood G of s_0 such that $\sup\{\|g_s\|\colon s \in G\} < \infty$ and the map $\hat{g}_G = \widehat{g|G}\colon C_b(G, B) \to C_b(G, \tilde{B})$ is surjective.

THEOREM 1.1. *Let S be a locally paracompact space and $g\colon S \times B \to \tilde{B}$ be a continuous map such that $g_s \in S(B, \tilde{B})$ for all $s \in S$. Then the following conditions are equivalent*:

(i) *g has the local b-lifting property*;

(ii) *g is locally uniformly open*;

(iii) *the map* (id, g): $S \times B \to S \times \tilde{B}$ *is open.*

Proof. (i) $\Rightarrow$ (ii) Let $s_0 \in S$ and G be a neighbourhood of s_0 such that

$\sup\{\|g_s\|: s\in G\} < \infty$ and $\hat{g}_G$ is surjective. By the open mapping theorem for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\hat{g}_G\{\sigma\in C_b(G, B): \|\sigma\| \leqslant \varepsilon\} \supseteq \{\sigma''\in C_b(G, \tilde{B}): \|\sigma''\| < \delta\}.$$

From this relation we infer that

$$\bigcap_{s\in G} g_s U_\varepsilon \supseteq \{u''\in B: \|u''\| < \delta\}.$$

Hence g is locally uniformly open.

(ii) $\Rightarrow$ (iii) Let $G\times U$ be an open set in $S\times B$ and let $(s_0, u_0'')\in$ $\in(\mathrm{id}, g)(G\times U)$. Then $u_0'' = g_{s_0}u_0$ for $u_0\in U$. Take a $\varepsilon > 0$ such that $u_0+U_\varepsilon \subseteq U$ and a neighbourhood G_0 of s_0 such that $W'' = \bigcap\{g_s U_\varepsilon: s\in G_0\}$ is a neighbourhood of zero in $\tilde{B}$. By continuity of g there exists a neighbourhood $G_1 \subseteq G_0$ of s_0 such that $g_{s_0}u_0 - g_s u_0 \in W_1''$ for all $s\in G_1$, where W_1'' is a neighbourhood of zero in $\tilde{B}$ such that $W_1''+W_1'' \subseteq W''$. Let $(s, u_0''+$ $+u'')\in G_1\times(u_0''+W_1'')$. Then

$$g_s u_0 - u'' = (g_s u_0 - g_{s_0}u_0) + g_{s_0}u_0 - u'' \in u_0'' - W''.$$

Therefore

$$u_0''+u'' = g_s(u_0+u), \qquad u\in U_\varepsilon.$$

Hence

$$(s, u_0'')\in G_1\times(u_0''+W_1'') \subseteq (\mathrm{id}, g)(G\times U).$$

Consequently (id, g) is open.

(iii) $\Rightarrow$ (i) Let $s_0\in S$. By hypothesis there exist a paracompact neighbourhood G of s_0 and a $\delta > 0$ such that $\sup\{\|g_s\|: s\in G\} < \infty$ and $g_s U_1 \supseteq U_\delta''$ for all $s\in G$. It follows that there exists a constant $C > 0$ such that for every $(s, u'')\in G\times\tilde{B}$ there is a $u_s\in B$ such that $g_s u_s = u''$ and $\|u_s\| \leqslant C\|u''\|$.

Let $\sigma''\in C_b(G, B)$ and $\varepsilon > 0$. By paracompactness of $\tilde{B}$ it follows that there is $\sigma_\varepsilon''\in C_b(G, \tilde{B})$ such that

$$\|\sigma_\varepsilon'' - \sigma\| \leqslant \varepsilon, \qquad \|\sigma_\varepsilon''\| \leqslant \|\sigma''\|$$

and

$$\sigma_\varepsilon''(s) = \sum_{i\in I} \varphi_i(\sigma'' s)u_i'',$$

where $\{\varphi_i\}$ is a locally finite partition of unity for an open cover of $\tilde{B}$ and $u_i''\in\sigma''(G)$ for all $i\in I$.

Fix $i\in I$. For every $s\in G$ we choose $u_{s,i}\in B$ such that

$$g_s u_{s,i} = u_i'' \quad \text{ and } \quad \|u_{s,i}\| \leqslant C\|u_i''\|.$$

Since g is continuous, there is a neighbourhood $G_{s,i}$ of s in G such that $\|g_t u_{s,i} - u_i''\| < \varepsilon$ for all $t \in G_s$. Suppose $\{\varphi_{s,i}\}$ is a locally finite partition of unity inscribed into cover $\{G_{s,i}\}$. Put

$$\sigma_i(t) = \sum_{s \in G} \varphi_{s,i}(t) u_{s,i}.$$

Obviously $\sigma_i \in C_b(G, B)$ and

$$\|\hat{g}\sigma_i - u_i''\| \leqslant \sup_{t \in G} \sum_{s \in G} \varphi_{s,i}(t) \|g_t u_{s,i} - u_i''\| \leqslant \varepsilon$$

as well as

$$\|\sigma_i\| \leqslant \sup_{t \in G} \sum_{s \in G} \varphi_{s,i}(t) \|u_{s,i}\|$$

$$\leqslant \sup_{s \in G} \|u_{s,i}\| \leqslant C\|\sigma''\|.$$

Setting

$$\sigma(t) = \sum_{i \in I} \varphi_i(\sigma_t'') \sigma_i(t)$$

we obtain an element σ belonging to $C_b(G, B)$ such that

$$\|\hat{g}\sigma - \sigma''\| \leqslant \|\hat{g}\sigma - \sigma_\varepsilon''\| + \|\sigma_\varepsilon'' - \sigma''\|$$

$$\leqslant \varepsilon + \sup_{t \in G} \sum_{i \in I} \varphi_i(\sigma'' t) \|g_t \sigma_i(t) - u_i''\|$$

$$\leqslant 2\varepsilon$$

and

$$\|\sigma\| \leqslant \sup_{t \in G} \sum_{i \in I} \varphi_i(\sigma'' t) \|\sigma_i(t)\| \leqslant C\|\sigma''\|.$$

Hence

$$\hat{g}\{\sigma \in C_b(G, B): \|\sigma\| \leqslant C\} \supseteq \{\sigma'' \in C_b(G, \tilde{B}): \|\sigma''\| \leqslant 1\}.$$

Therefore, by the open mapping theorem we infer that $\hat{g}_G$ is surjective.

THEOREM 1.2. *Let S be an arbitrary topological space and $\tilde{g}$: $S \to S(B, \tilde{B})$ be a continuous map. Then the map* (id, g): $S \times B \to S \times \tilde{B}$ *is locally trivial.*

Proof. a) First we assume that S is a contractible metric space and $\tilde{g}$ is bounded. By Theorem 1.1 (or by [1]) the map $\hat{g}$: $C_b(S, B) \to C_b(S, \tilde{B})$ is surjective. Hence there exists a continuous map q: $C_b(S, \tilde{B}) \to C_b(S, B)$ such that $gq = \text{id}$. Given a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f_0=(f_0^1, f_0^2)"] \arrow[d, "k"'] & S\times B \arrow[d, "(\mathrm{id}, g)"] \\ X\times I \arrow[r, "h=(h^1, h^2)"] & S\times\tilde{B} \end{tikzcd}$$

in which $I=[0,1]$, $k(x)=(x,0)$, f_0 and h are continuous and X is an arbitrary topological space. For every $(x,t)\in X\times I$ put

$$f(x,t)=\big(h^1(x,t),\, q\big(\overline{h^2(x,t)}\big)[h^1(x,t)]-q\big(\overline{h^2(x,0)}\big)[h^1(x,t)]+f_0^2(x)\big)$$

where $\overline{u''}(s)=u''$, $u''\in\tilde{B}$, for all $s\in S$. Then

$$fk(x)=f(x,0)=\big(h^1(x,0), f_0^2(x)\big)=f_0(x), \qquad (\mathrm{id}, g)f=h.$$

Hence $\big(S\times B, (\mathrm{id}, g), S\times\tilde{B}\big)$ is a fiber space in sense of Hurewicz. By contractibility of $S\times\tilde{B}$ and by [6] we infer that $\big(S\times B, (\mathrm{id}, g), S\times\tilde{B}\big)$ is trivial.

b) Now we assume that S is a metric space. Without loss of generality we can assume that S is a closed subset in a normed space C. Since $S(B,\tilde{B})$ is open, $\tilde{g}$ can be extended to a continuous map $\tilde{f}$ from an open neighbourhood W of S in C into $S(B,\tilde{B})$. By a) and by the local contractibility of W, the map (id, f): $W\times B\to W\times\tilde{B}$ is locally trivial. Hence (id, g) is locally trivial.

c) Finally we assume that S is an arbitrary topological space. By $S|_\varrho$ we denote the metric space of equivalent classes: $s\sim t \;\Leftrightarrow\; g_s=g_t$. This space is endowed with the metric induced by $\varrho(\bar{s},\bar{t})=\|g_s-g_t\|$. Since the map $\tilde{g}_\varrho$: $S|_\varrho\to S(B,\tilde{B})$ is continuous, by b) the map (id, g_ϱ): $S|_\varrho\times B\to S|_\varrho\times\tilde{B}$ is locally trivial. Thus it is easy to see that the map (id, g) is locally trivial. □

2. Lifting vector-valued holomorphic functions. Let X be a complex space, B and $\tilde{B}$ be complex Banach spaces. A map g: $X\times B\to\tilde{B}$ is said to be a holomorphic family of continuous linear maps from B into $\tilde{B}$ if $g_z\in\mathrm{Hom}(B,\tilde{B})$ for all $z\in X$ and $g(\cdot, u)$ is holomorphic for all $u\in B$.

By Lemma 2.1, the map $\tilde{g}$: $X\to\mathrm{Hom}(B,\tilde{B})$ associated with g is holomorphic. Therefore g induces a continuous linear map $\hat{g}$: $\mathcal{O}(X,B)\to\mathcal{O}(X,\tilde{B})$ defined by

$$[\hat{g}\sigma]z=g\big(z,\sigma(z)\big).$$

Here $\mathcal{O}(X,B)$ denotes the locally convex space of holomorphic maps from X into B endowed with the compact-open topology.

In this section we shall prove the following

THEOREM 2.1. *Let X be a complex space having a countable topology and let $\mathcal{O}(X)$ determine the topology of X. Then the following conditions are equivalent.*

(i) *X is a Stein space.*

(ii) *$H^1(X, \mathcal{O}_\xi) = 0$ for every sheaf $\mathcal{O}_\xi$ of germs of holomorphic sections on X of a holomorphic Banach bundle ξ over X.*

(iii) *The map $\hat{g}$: $\mathcal{O}(X, B) \to \mathcal{O}(X, \tilde{B})$ is surjective for every holomorphic family g on X with values in $S(B, \tilde{B})$ such that* $\mathrm{Ker}(\mathrm{id}, g)$ *has a structure of a holomorphic Banach bundle over X for which the canonical map i:* $\mathrm{Ker}(\mathrm{id}, g) \to \underline{B}$ *is holomorphic, where $\underline{B}$ denotes the trivial bundle over X with the fiber B.*

(iv) *The map $\hat{g}$: $\mathcal{O}(X, B) \to \mathcal{O}(X)$ is surjective for every holomorphic family g: $X \times B \to C$ of continuous linear functionals on B such that $g_z \neq 0$ for all $z \in X$.*

Proof. (i) implies (ii) by a theorem of Bungart ([2]).

(ii) $\Rightarrow$ (iii) By exactness of the cohomology sequence, it is enough to prove that the sequence

$$0 \to \mathcal{O}_{\mathrm{Ker}\bar{g}} \to \mathcal{O}_{\underline{B}} \to \mathcal{O}_{\underline{\tilde{B}}} \to 0$$

is exact, $\bar{g} = (\mathrm{id}, g)$.

This statement is an immediate consequence of the following

LEMMA 2.1. *Let g: $X \times B \to \tilde{B}$ be a holomorphic family of continuous linear maps from a Banach space B into a Banach space $\tilde{B}$. Then the map $\tilde{g}$: $X \to \mathrm{Hom}(B, \tilde{B})$ associated with g is holomorphic.*

Proof. We assume that X is an analytic set in an open subset of C^n for some n. Obviously the map $\bar{g}$: $B \to \mathcal{O}(X, \tilde{B})$ associated with g is holomorphic. First we show that the map $\tilde{g}_c$: $X \to \mathrm{Hom}_c(B, \tilde{B})$ is holomorphic, where $\mathrm{Hom}_c(B, \tilde{B})$ denotes the space $\mathrm{Hom}(B, \tilde{B})$ endowed with the open-compact topology. Suppose θ: $\mathcal{O}^*(X) \to \mathrm{Hom}_c(B, \tilde{B}^{**})$ is a map defined by $[\theta\omega(u)](u''^*) = \omega(u''^*\bar{g}u)$, $\tilde{B}^*$ and $\mathcal{O}^*(X)$ are strong dual spaces of $\tilde{B}$ and $\mathcal{O}(X)$ respectively. Since ω is continuous, $\theta\omega(u) \in (\tilde{B}^*_c)^* = \tilde{B}$. Hence, by continuity of $\bar{g}$, it follows that θ is a continuous linear map from $\mathcal{O}^*(X)$ into $\mathrm{Hom}_c(B, \tilde{B})$. Hence, by a result of Bungart ([3]) we infer that $\tilde{g}_c$ is holomorphic. We check that $\tilde{g}$ is holomorphic. Let $z_0 \in X$. Since $\tilde{g}_c$ is holomorphic, there exist a polydisc $\Delta(z_0, r)$ with centre at z_0 and polyradius $r > 0$ and $\tilde{f} \in \mathcal{O}(\Delta(z_0, r), \mathrm{Hom}_c(B, \tilde{B}))$ extending of $\tilde{g}_c | \Delta(z_0, r) \cap X$ ([3]). Since $\{\sup \|\tilde{f}z\|$: $z \in K\} < \infty$ for every compact set K in $\Delta(z_0, r)$ by the Cauchy integral formula we infer that $\tilde{f} \in \mathcal{O}(\Delta(z_0, r), \mathrm{Hom}(B, \tilde{B}))$. Consequently $\tilde{g}$ is holomorphic. □

LEMMA 2.2. *Let g: $X \times B \to \tilde{B}$ be as in Lemma 2.1. Let g_{z_0} be surjective. Then $\hat{g}_{z_0}$: $\mathcal{O}_{\underline{B}, z_0} \to \mathcal{O}_{\underline{\tilde{B}}, z_0}$ is surjective.*

Proof. We can assume X is an analytic set in an open set in C^n for some n and $z_0 = 0 \in X$. By Lemma 2.1 the map $\tilde{g}$: $X \to \mathrm{Hom}(B, \tilde{B})$ is holomorphic and therefore $\tilde{g}$ can be extended to a holomorphic map $\tilde{f}$ from $\Delta_{2r} = \Delta(0, 2r)$ into $\mathrm{Hom}(B, \tilde{B})$ with $r > 0$.

Consider the Taylor expansion of f at zero:

$$f(z) = A_0 + \sum_{|\alpha| \geqslant 1} A_\alpha z^\alpha$$

where $z = (z_1, \ldots, z_n)$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. By hypothesis, A_0 is surjective. Hence

$$\frac{1}{2} C = \sup_{\|u''\| = 1} \inf_{A_0 u = u''} \|u\| < \infty.$$

Let

$$M = \sup \{\|\tilde{f}_\xi\|: \xi \in \Delta_r\}, \qquad B_0 = A_0,$$

$$B_\alpha = r^{|\alpha|} (4MC_1 n)^{-2kn|\alpha|} A_\alpha, \qquad |\alpha| \geqslant 1$$

and

$$Z = (4MC_1 n)^{2kn} z/r,$$

where C_1 and k are positive numbers such that $C_1 \geqslant C$, $(MC_1) > 1$ and $(MC_1)^k > C^2$. We can assume that $C > 1$. Obviously $\|B_\alpha\| \leqslant (4MC_1 n)^{-kn|\alpha|}$ for all $|\alpha| \geqslant 1$.

a) Let $u'' \in \tilde{B}$. We find $\sigma \in \mathcal{O}(\Delta_1, B)$ such that

$$F(Z)\sigma(Z) = u'' \quad \text{for all } Z \in \Delta_1, \qquad \|\sigma\|_{\Delta_{1/2}} \leqslant \tilde{C} \|u''\|,$$

where

$$F(Z) = B_0 + \sum_{|\alpha| \geqslant 1} B_\alpha Z^\alpha$$

and $\tilde{C}$ is a constant independent of u''. We can assume that $\|u''\| = 1$. We find σ from the formula:

$$\sigma(Z) = \sum_\alpha u_\alpha Z^\alpha.$$

From the condition

$$F(Z)\sigma(Z) = u'' \quad \text{for all } Z \in \Delta_1$$

we get the relations

$$B_0 u_0 = u''$$

and

$$B_0 u_\beta = - \sum_{\substack{\alpha + \gamma = \beta \\ 0 < \alpha \leqslant \beta,\ 0 \leqslant \gamma < \beta}} B_\alpha u_\gamma.$$

We choose $u_0 \in B$ such that $B_0 u_0 = u_0''$ and $\|u_0\| \leqslant C\,\|u''\| = C$. Let $u_\beta \in B$ for $0 < \beta < \beta_0$ be such that

$$B_0 u_\beta = \sum_{\substack{\alpha+\gamma=\beta \\ 0<\alpha\leqslant\beta,\, 0\leqslant\gamma<\beta}} B_\alpha u_\gamma$$

and

$$\|u_\beta\| \leqslant 1.$$

We choose $u_{\beta_0} \in B$ such that

$$B_0 u_{\beta_0} = - \sum_{\substack{\alpha+\gamma=\beta_0 \\ 0<\alpha\leqslant\beta_0,\, 0\leqslant\gamma<\beta_0}} B_\alpha u_\gamma$$

and

$$\|u_{\beta_0}\| \leqslant C \Big\| \sum_{\substack{\alpha+\gamma=\beta_0 \\ 0<\alpha\leqslant\beta_0,\, 0\leqslant\gamma<\beta_0}} B_\alpha u_\gamma \Big\|.$$

Then

$$\|u_{\beta_0}\| \leqslant C \sum_{0<\alpha\leqslant\beta_0} C(4C_1 Mn)^{-kn|\alpha|} \leqslant \sum_{|\alpha|=1}^{\infty} (4b)^{-kn|\alpha|}$$
$$\leqslant 2^n (4n)^{-kn} + 3^n (4n)^{-2kn} + \ldots \leqslant 1.$$

Since $\|u_\alpha\| \leqslant C$ for all α, the series $\sum_\alpha u_\alpha Z^\alpha$ converges to an element $\sigma \in \mathcal{O}(\Delta_1, B)$. Obviously $FZ\sigma(Z) = u''$ for all $Z \in \Delta_1$ and $\|\sigma\|_{\Delta_{1/2}} \leqslant \tilde{C}$, where $\tilde{C}$ is a constant independent of u''. Setting

$$\tilde{\sigma}(z) = \sigma\big((4MC_1 n)^{2kn} z/r\big)$$

we obtain an element $\tilde{\sigma} \in \mathcal{O}(\Delta_\delta, B)$ with $\delta = r/(4MC_1 n)^{2kn}$ such that $\tilde{f}z\tilde{\sigma}(z) = u''$ for all $z \in \Delta_\delta$ and $\|\tilde{\sigma}\|_{\Delta_{\delta/2}} \leqslant \tilde{C}$. Now let $\sigma'' \in \mathcal{O}_{\tilde{B},0}$ and let

$$\tilde{\sigma}''(z) = \sum_\alpha u_\alpha'' z^\alpha$$

be a representative of σ'' on a polydisc Δ_δ, $\delta > 0$. For every u_α'', by a) there exists $\sigma_\alpha \in \mathcal{O}(\Delta_{\delta_0}, B)$ such that

$$\tilde{f}z\sigma_\alpha(z) = u_\alpha'' \quad \text{for all } z \in \Delta_{\delta_0}$$

and

$$\|\sigma_\alpha\|_{\Delta_{\delta_0/2}} \leqslant \tilde{C}\,\|u_\alpha''\| \quad \text{for all } \alpha$$

where δ_0, $\tilde{C}$ are constants independent of u_α''. Hence, setting

$$\sigma(z) = \sum_\alpha \sigma_\alpha z^\alpha$$

we get an element $\sigma \in \mathcal{O}_{B,0}$ such that $\hat{g}_0 \sigma = \sigma''$. □

LEMMA 2.3. *Let g: $X \times B' \to B$ be a holomorphic family of isomorphisms g_z from a Banach space B' into a Banach space B and let σ: $X \to B'$ be a map such that $\hat{g}\sigma$ is holomorphic. Then σ is holomorphic.*

Proof. First we show that (id, g): $X \times B' \to X \times B$ is an embedding. Suppose $(z_n, g(z_n, u_n)) \to (z, u)$. Then $z_n \to z$ and therefore by Lemma 2.1, $g_{z_n} \to g_z$. We show that $\{g_z u_n\}$ is a Cauchy sequence in B and therefore $u_n \to u$. Since g_{z_n} is surjective and the map $\tilde{g}^*$: $X \to \mathrm{Hom}(B^*, B'^*)$ is continuous, it follows that $\tilde{g}^*_{z_n}(U^*) \supseteq \varepsilon U'^*$ for some $\varepsilon > 0$ and for all $n > n_0$ ([5]), where U^* and U'^* are unit balls in B^* and B'^* respectively. Therefore for $n > n_0$ we have

$$\varepsilon \|u_n\| = \sup |u^*(u_n)| \leqslant \sup_{\|u^*\| \leqslant 1} |\tilde{g}^*_{z_n} u^*(u_n)|$$
$$= \sup_{\|u^*\| \leqslant 1} |u^* g_{z_n} u_n| \leqslant \sup_n \|g_{z_n} u_n\| < \infty.$$

Hence

$$\|g_z u_n - g_z u_m\| \leqslant \|g_{z_n} - g_z\| \|u_n\| + \|g_{z_n} u_n - g_{z_m} u_m\| + \|g_{z_m} - g_z\| \|u_m\| \to 0.$$

Thus σ is continuous. Hence, in order to prove that σ is holomorphic it is enough to prove that $u^* \sigma$ is holomorphic for all $u^* \in B'^*$ ([3]).

Let $z_0 \in X$ and $u^* \in B'^*$. By Lemma 2.2 there exists $\gamma \in \mathcal{O}(U, B^*)$, where U is a neighbourhood of z_0, such that $g_z^* \gamma(z) = u^*$ for all $z \in U$. Therefore we have

$$u^* \sigma(z) = [g_z^* \gamma(z)] \sigma(z) = \gamma(z)(g_z \sigma(z)).$$

Since $\hat{g}\sigma$ is holomorphic, we infer that $u^* \sigma$ is holomorphic. □

(iii) $\Rightarrow$ (iv) Proof is trivial, because $\mathrm{Ker}\, g_z$ is complemented in B for all $z \in X$ and therefore $\mathrm{Ker}\, \tilde{g}$ is a holomorphic Banach bundle.

(iv) $\Rightarrow$ (i) By [4] it is enough to prove that $X = \mathfrak{M}(\mathcal{O}(X))$, where $\mathfrak{M}(\mathcal{O}(X))$ denotes the spectrum of $\mathcal{O}(X)$. Since $\mathcal{O}(X)$ determines the topology of X it suffices to show that $V_\omega = \{z \in X\colon \sigma(z) = 0,\ \sigma \in \mathrm{Ker}\, \omega\} \neq \emptyset$ for all $\omega \in \mathfrak{M}(\mathcal{O}(X))$. For a contradiction, let $V_{\omega_0} = \emptyset$ for some ω_0. Since $\mathcal{O}(X)$ is a separable Fréchet space there exists $\{\sigma_n\} \subset \mathrm{Ker}\, \omega_0$ such that

$$\bigcap_{n=1}^{\infty} \{z \in X\colon \sigma_n(z) = 0\} = \emptyset \quad \text{and} \quad \sigma_n \to 0.$$

Consider the holomorphic family g: $X \times l^1 \to C$ of continuous linear functionals on l^1 defined by

$$g(z, \{\xi_n\}) = \sum_{n=1}^{\infty} \xi_n \sigma_n(z).$$

Since $\bigcap_{n=1}^{\infty} \{z \in X\colon \sigma_n(z) = 0\} = \emptyset$ we infer that $g_z \neq 0$ for all $z \in X$. By hypothesis, it follows that there exists a sequence $\{\beta_n\} \subset \mathcal{O}(X)$ such that

$$\sum_{n=1}^{\infty} \beta_n(z)\sigma_n(z) = 1 \quad \text{for all } z \in X.$$

This relation shows that $\operatorname{Ker}\omega_0 = \mathcal{O}(X)$. But this is impossible, because $\omega_0 \neq 0$. □

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA, POLAND

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