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A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals

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Abstract. We prove a continuation result in locally convex topological vector spaces which contains (in the context of Banach spaces) the well-known Leray–Schauder continuation principle as well as (in the context of locally convex spaces) the famous Schauder–Tychonoff fixed point theorem. We give some applications to boundary value problems for ordinary differential equations in noncompact intervals.

Introduction. One of the most important topological tools to prove the existence of a solution for a system of n nonlinear equations in $\boldsymbol{R}^n$ is the following straightforward consequence of the Brouwer topological degree theory:

THEOREM A (continuation principle in $\boldsymbol{R}^n$). *Let U be an open subset of $\boldsymbol{R}^n$ and let $\Phi\colon \bar{U}\times[0,1]\to \boldsymbol{R}^n$ be a continuous mapping. Assume that*

(a) *the map $\Phi_0\colon \bar{U}\to \boldsymbol{R}^n$, given by $x\mapsto\Phi(x,0)$ is linear (or, more generally, affine);*

(b) *the equation $\Phi(x,0)=0$ has a solution in U;*

(c) *$\Phi^{-1}(0)$ is a compact subset of $U\times[0,1]$.*

Then the equation $\Phi(x,1)=0$ has a solution in U.

Observe that, because of the assumption (c), the linear (or affine) map Φ_0 of Theorem A is invertible. Therefore, the equation $\Phi(x,\lambda)=0$ can be equivalently put into the form

$$x = T(x,\lambda), \tag{0.1}$$

with $T\colon \bar{U}\times[0,1]\to \boldsymbol{R}^n$ continuous and such that $T(x,0)\equiv\Phi_0^{-1}(0)\in U$.

Now, as in $\boldsymbol{R}^n$, many nonlinear equations in Banach spaces can be transformed into fixed point problems of the type

$$x = \Psi(x), \tag{0.2}$$

where $\Psi\colon \bar{U} \to E$ is a compact mapping defined on the closure of some open subset of a Banach space E. This fact (and the difficulties in finding an appropriate bounded closed subset of $\bar{U}$ which is mapped by Ψ into itself) induced in 1934 [3] J. Leray and J. Schauder to introduce, in the Banach spaces context, a topological degree theory for compact perturbations of the identity. In particular, they obtained the following extension of Theorem A:

THEOREM B (Leray–Schauder continuation principle). *Let U be an open subset of a Banach space E and let $T\colon \bar{U} \times [0, 1] \to E$ be a continuous map, sending bounded subsets of $\bar{U} \times [0, 1]$ into relatively compact subsets of E.*

Assume that

(a) $T(x, 0) = x_0 \in U$ *for all* $x \in \bar{U}$;

(b) *the fixed point set*

$$F = \{x \in \bar{U}\colon\ x = T(x, \lambda) \text{ for some } \lambda \in [0, 1]\}$$

is bounded and does not meet the boundary ∂U of U (i.e., F is a compact subset of U).

Then the map $x \mapsto T(x, 1)$ has a fixed point in U.

In spite of the fact that the Leray–Schauder degree theory has been extended to the context of locally convex spaces (see [4], [8]) the following simple example shows that the Leray–Schauder continuation principle cannot be extended just replacing the Banach space E by an arbitrary locally convex (or even Fréchet) space:

Let $C[0, \infty)$ denote the Fréchet space of all the continuous real functions defined on the half line $[0, \infty)$ with the topology of the uniform convergence on compact subsets of $[0, \infty)$. Define the map Ψ from $C[0, \infty)$ into itself by

$$\Psi(x)(t) = \int_0^t \left(1 + x(s)^2\right) ds.$$

Clearly Ψ is continuous and by Ascoli's theorem sends bounded sets into relatively compact sets. Moreover, since the integral equation

$$x(t) = \lambda \int_0^t \left(1 + x(s)^2\right) ds$$

admits a solution in $C[0, \infty)$ only when $\lambda = 0$, the fixed point set $F = \{x \in C[0, \infty)\colon\ x = \lambda\Psi(x) \text{ for some } \lambda \in [0, 1]\}$ is a compact subset of $U = C[0, \infty)$. On the other hand, the equation $x = \Psi(x)$ does not admit a solution in the space $C[0, \infty)$.

The reason why Nagumo's degree does not apply to the above example is probably due to the fact that a Hausdorff locally convex space does not admit bounded open subsets, unless it is normable. On the other hand, in the Nagumo notion of degree for a map $x \mapsto x - \Psi(x)$ one needs to assume Ψ defined on an open subset of the space, and with image contained in a compact set. This condition is clearly not restrictive in normed spaces (since, in this case, compact maps are also locally compact) but very much so in locally convex spaces, even in the case of linear operators.

In [8] an alternative extension of the Leray–Schauder degree theory for map $x \mapsto x - \Psi(x)$, where Ψ needs not send the whole domain into a relatively compact set, has been given. In that extension, however, the domain U of Ψ is a finitely bounded open subset of the space E (i.e., the intersection of U with any finite dimensional subspace of E is bounded). Unfortunately, there are no such open sets in many important locally convex spaces ($C[0, \infty)$ included).

Our aim is to prove an extension of the Leray–Schauder continuation principle in locally convex spaces (Theorem 1.2 below) which contains, as a special case, the well-known Schauder–Tychonoff fixed point theorem. Roughly speaking, our result establishes the existence of fixed points for locally compact operators which are defined on a relatively open subset U of a closed convex subset Q of the space E and which take values in E (and not merely in Q). When $Q = E$, and E is Banach, our result coincides with the Leray–Schauder continuation principle, while, when $U = Q$ we get a generalization of the Schauder–Tychonoff fixed point theorem. As far as we know such an extension seems to be unknown even in the case $E = \boldsymbol{R}^n$.

The reason why we prefer to deal with maps defined on convex sets instead of on open subsets of the whole space is motivated by the applications to boundary value problems for ordinary differential equations in noncompact intervals. In Examples 2.1 and 2.2 below we shall use in fact a result of [1] on the continuity and compactness of operators associated to boundary value problems for ODE's in noncompact intervals. We point out that in [1], for any given differential equation

$$\dot{x}(t) = f(t, x(t)), \quad t \in J, \tag{0.3}$$

with f continuous and J a real interval, the space E considered is always the same, no matter what boundary condition is associated with (0.3); namely, $E = C(J, \boldsymbol{R}^n)$, the Fréchet space of all the continuous real functions $x\colon J \to \boldsymbol{R}^n$ with the topology of uniform convergence on compact subintervals of J. On the other hand, when J is noncompact, the operator associated with a boundary condition for (0.3) is usually not defined (or, if defined, not locally compact) in an open subset of $C(J, \boldsymbol{R}^n)$, but it does have nice properties when restricted to a suitable subset of the space (see [1], [7]). Many authors who deal with the asymptotic behavior of differential equations prefer to associate to any given boundary condition for (0.3) an appropriate Banach space in such a way that the integral operator corresponding to that condition turns out to be well defined on an open set of the considered space. However, when using this method, the continuity and compactness properties of the operator turn out to be, sometimes, really hard to check. On the other hand, as it was pointed out in [1], given a boundary value problem for an ODE, the continuity and the compactness in $C(J, \boldsymbol{R}^n)$ of the related operator are straightforward consequences of the existence of suitable a priori bounds for the solutions of the given boundary value problem. We believe that the effort in proving the existence of such a priori bounds cannot be avoided with any other method.

1. Abstract continuation principles. Let E be a Hausdorff locally convex topological vector space and let $\mathscr{P}$ be a family of seminorms $p\colon E \to \boldsymbol{R}_+$ generating the topology of E. In the sequel we will denote by $\mathscr{A}$ the partially ordered and direct set

$$\mathscr{A} = \{\alpha = (p, \varepsilon)\colon\ p \in \mathscr{P},\ \varepsilon > 0\}$$

where for $\alpha = (p, \varepsilon)$, $\beta = (q, \eta) \in \mathscr{A}$ we have $\alpha \prec \beta$ if and only if $p(x) \leqslant q(x)$ for all $x \in \boldsymbol{E}$ and $\varepsilon > \eta$.

Theorem 1.1 below prepares the proof of a more general continuation principle which will be stated in Theorem 1.2. Anyhow, we point out that Theorem 1.1 is sufficient for all the applications to differential equations we are interested in (see Section 2).

THEOREM 1.1. *Let Q be a convex closed subset of $\boldsymbol{E}$ and let $T\colon Q \times [0, 1] \to \boldsymbol{E}$ be a continuous map with relatively compact image.*

Assume that

(i) *$T(x, 0) \in Q$ for any $x \in Q$;*

(ii) *for any $(x, \lambda) \in \partial Q \times [0, 1)$ with $T(x, \lambda) = x$ there exist open neighborhoods U_x of x in $\boldsymbol{E}$ and I_λ of λ in $[0, 1)$ such that $T\big((U_x \cap \partial Q) \times I_\lambda\big) \subset Q$.*

Then the equation

$$x = T(x, 1)$$

has a solution.

Remark 1.1. If E is metrizable, assumption (ii) can be expressed equivalently in the following form

(ii') *if $\{(x_j, \lambda_j)\}_{j \in \boldsymbol{N}}$ is a sequence in $\partial Q \times [0, 1]$ converging to (x, λ) with $T(x, \lambda) = x$ and $0 \leqslant \lambda < 1$, then $T(x_j, \lambda_j) \in Q$ for j sufficiently large.*

Remark 1.2. We point out that Q always coincides with ∂Q when E is not normable and Q is bounded (as it happens, for instance, in the examples given in Section 2).

In proving Theorem 1.1 we need the following

LEMMA 1.1. *Let Q be as in Theorem* 1.1 *and let K be a compact subset of E such that $K \cap Q \neq \emptyset$. Then, for any $\alpha = (p, \varepsilon) \in \mathscr{A}$, there exists a continuous map γ_α: $K \to E$ whose image is contained in a finite dimensional space and such that*

(a) $\gamma_\alpha(K \cap Q) \subset Q$,

(b) $p(\gamma_\alpha(x) - x) < \varepsilon$ *for all* $x \in K$.

Proof. Let $\alpha = (p, \varepsilon)$ be given. For any $x \in E$, denote

$$U_\alpha(x) = \{y \in E:\ p(y-x) < \varepsilon\}.$$

Since $K \cap Q$ is compact, there exist $x_1, x_2, \ldots, x_r \in K \cap Q$ such that $K \cap Q \subset \bigcup_{i=1}^{r} U_\alpha(x_i)$. Consequently, $K \setminus \bigcup_{i=1}^{r} U_\alpha(x_i)$ does not intersect $K \cap Q$ and so, since it is itself a compact set, there exist $\varepsilon_0 < \varepsilon$ and $x_{r+1}, \ldots, x_s \in K \setminus \bigcup_{i=1}^{r} U_\alpha(x_i)$ such that $K \setminus \bigcup_{i=1}^{r} U_\alpha(x_i) \subset \bigcup_{i=r+1}^{s} U_{\alpha_0}(x_i)$ and $(K \cap Q) \cap \big(\bigcup_{i=r+1}^{s} U_{\alpha_0}(x_i)\big) = \emptyset$, where $\alpha_0 = (p, \varepsilon_0)$.

Let ϱ_i: $K \to \boldsymbol{R}_+$ be given by

$$\varrho_i(x) = \begin{cases} \varepsilon - p(x-x_i), & x \in U_\alpha(x_i),\ i = 1, 2, \ldots, r, \\ 0, & \text{elsewhere}, \end{cases}$$

$$\varrho_i(x) = \begin{cases} \varepsilon_0 - p(x-x_i), & x \in U_{\alpha_0}(x_i),\ i = r+1, \ldots, s, \\ 0, & \text{elsewhere}, \end{cases}$$

and define γ_α: $K \to E$ by $\gamma_\alpha(x) = \sum_{i=1}^{s} \sigma_i(x) x_i$, where $\sigma_i(x) = \varrho_i(x)\big(\sum_{j=1}^{s} \varrho_j(x)\big)^{-1}$.

Clearly, γ_α is continuous, its image is contained in a finite dimensional space and $p(\gamma_\alpha(x) - x) \leqslant \sum_{i=1}^{s} \sigma_i(x) p(x - x_i) < \varepsilon$. Moreover, if $x \in K \cap Q$ then $\varrho_i(x) = 0$ for $i = r+1, \ldots, s$, so that, for $x \in K \cap Q$, $\gamma_\alpha(x)$ turns out to be in fact a convex combination of the elements $x_1, \ldots, x_r$ of Q. Thus, since Q is convex, $\gamma_\alpha(K \cap Q)$ is contained in Q. Q.E.D.

Proof of Theorem 1.1. Let us suppose first $\dim E < \infty$. Without loss of generality we may assume that E is endowed with an euclidean norm. So, the map r: $E \to Q$ which associates to any x the closest point of Q is well-defined.

Set

$$F = \{x \in E:\ T(r(x), \lambda) = x \text{ for some } \lambda \in [0, 1]\},$$
$$F_\lambda = \{x \in E:\ T(r(x), \lambda) = x\}.$$

Since for each $\lambda \in [0, 1]$ the map $x \to T(x, \lambda)$ has relatively compact image, from the Brouwer fixed point theorem, it follows that F_λ is nonempty for every λ. Moreover, F and F_λ are compact sets and, by (i), F_0 is contained in Q.

We have to show that $F_1 \cap Q \neq \emptyset$. By contradiction let us suppose the emptiness of $F_1 \cap Q$. Let V be any open neighborhood of Q such that $F_1 \cap \bar{V} = \emptyset$. We will prove that there exists $(y, \lambda) \in \partial V \times [0, 1)$ such that $T(r(y), \lambda) = y$. Suppose not and define σ: $E \to [0, 1]$ by

$$\sigma(x) = \max\left\{1 - \frac{\operatorname{dist}(x, F \cap \bar{V})}{\operatorname{dist}(\partial V, F \cap \bar{V})}, 0\right\}.$$

Clearly σ is continuous and $\sigma(x) = 1$ in $F \cap \bar{V}$, $\sigma(x) = 0$ in $E \setminus V$. Hence, by using again Brouwer's theorem, we obtain that there exists $y \in E$ such that $T(r(y), \sigma(y)) = y$. Since, by (i), $F_0 \subset Q \subset V$, we deduce immediately that y is in fact belonging to V. Thus, $\sigma(y) = 1$ so that $y \in F_1 \cap V$, contradicting the definition of V.

Let now $\varepsilon > 0$ be such that $\operatorname{dist}(F_1, Q) > \varepsilon$ and define

$$V_j = \{x \in E\colon \operatorname{dist}(x, Q) < \varepsilon/j\}, \quad j \in N.$$

By the above considerations, for each $j \in N$ there exists $(y_j, \lambda_j) \in \partial V_j \times [0, 1)$ such that $T(r(y_j), \lambda_j) = y_j$. Without loss of generality we may assume $\lambda_j \to \bar{\lambda}$ and $y_j \to \bar{x} \in \partial Q$, so that $T(\bar{x}, \bar{\lambda}) = \bar{x}$. Observe that it also results $\bar{\lambda} < 1$ since we are supposing $F_1 \cap Q = \emptyset$. Hence, by (ii), there exist open neighborhoods $U_{\bar{x}}$ and $I_{\bar{\lambda}}$ such that $T((U_{\bar{x}} \cap \partial Q) \times I_{\bar{\lambda}}) \subset Q$. On the other hand, $(r(y_j), \lambda_j)$ belongs to $(U_{\bar{x}} \cap \partial Q) \times I_{\bar{\lambda}}$ for $j > \bar{j}$ but $T(r(y_j), \lambda_j) \notin Q$ contradicting (ii). Thus $F_1 \cap Q \neq \emptyset$ and the theorem is proved in the case $\dim E < \infty$.

Let us now consider the general case. Observe that the assertion of the theorem is equivalent to the following:

$$\inf\{p(x - T(x, 1))\colon x \in Q\} = 0 \quad \text{for all } p \in \mathscr{P}. \tag{1.1}$$

In fact, if the above infimum is equal to zero for all $p \in \mathscr{P}$, then for any $\alpha = (p, \varepsilon) \in \mathscr{A}$ there exists $x_\alpha \in Q$ such that $p(x_\alpha - T(x_\alpha, 1)) < \varepsilon$. Thus the net $\{x_\alpha - T(x_\alpha, 1)\colon \alpha \in \mathscr{A}\}$ converges to zero and since the set $\{T(x_\alpha, 1)\colon \alpha \in A\}$ is relatively compact, without loss of generality we may also assume the convergence of $T(x_\alpha, 1)$ to some $\bar{x}$. So x_α converges to $\bar{x}$ as well and $T(\bar{x}, 1) = \bar{x}$.

We therefore will establish (1.1). By contradiction, suppose there exist $\bar{p} \in \mathscr{P}$ and $\bar{\varepsilon} > 0$ such that

$$\inf\{\bar{p}(x - T(x, 1))\colon x \in Q\} = \bar{\varepsilon}. \tag{1.2}$$

Then, in particular, $x \neq T(x, 1)$ on ∂Q so that, by assumption (ii), there exists an open subset Ω of $\partial Q \times [0, 1]$ containing the compact set $\{(x, \lambda) \in \partial Q \times [0, 1]\colon T(x, \lambda) = x\}$ and such that $T(\Omega) \subset Q$. By an argument similar to the one used above, it is not hard to see that the pair $(\bar{p}, \bar{\varepsilon})$ can also be chosen in such a way that

$$\{(x, \lambda)\in \partial Q \times[0, 1]:\ \bar{p}(x-T(x, \lambda)) < \bar{\varepsilon}\} \subset \Omega.$$

Let $K = \overline{T(Q\times[0, 1])}$. K is a compact set and $K\cap Q \neq \emptyset$ because of (i). Take γ_α: $K \to E$, $\alpha = (\bar{p}, \bar{\varepsilon})$, as in Lemma 1.1 and denote by E_1 the finite dimensional space spanned by the image of γ_α. Let T_α: $(E_1 \cap Q)\times[0, 1] \to E_1$ be defined by $T_\alpha = \gamma_\alpha \circ T$. Clearly, T_α is continuous and $T_\alpha(x, 0)\in Q$ for all $x\in Q$. Moreover, if x belongs to the (relative to E_1) boundary of $E_1\cap Q$ and $T_\alpha(x, \lambda) = x$ for some $\lambda\in[0, 1)$, then, in particular, $x\in\partial Q$ and $\bar{p}(T(x, \lambda)-x) < \bar{\varepsilon}$. Consequently $(x, \lambda)\in\Omega$, which implies that $T(x, \lambda)$ and, thus, $T_\alpha(x, \lambda)$ belong to Q. Therefore, the first part of our proof applies to the map T_α yielding the existence of $\bar{x}\in Q$ such that $T_\alpha(\bar{x}, 1) = \bar{x}$. Thus, $\bar{p}(\bar{x}-T(\bar{x}, 1)) < \bar{\varepsilon}$ contradicting (1.2). Q.E.D.

Sometimes, condition (ii) of Theorem 1.1 can be easily checked if one knows suitable a priori bounds on the image set $T(Q\times[0, 1])$, that is, in other words, $T(Q\times[0, 1])$ lies in an appropriate subset Σ of E. We have in fact the following consequence of Theorem 1.1.

COROLLARY 1.1. *Let T: $Q\times[0, 1]\to\Sigma\subset E$ be continuous with relatively compact image. Assume that*

(i) *$T(x, 0)\in Q$ for all $x\in Q$;*

(ii) *for any $\lambda\in[0, 1]$, the map $x\mapsto T(x, \lambda)$ does not admit fixed points on the boundary of Q relative to Σ $(\overline{\Sigma\setminus Q}\cap Q)$.*

Then the equation $x = T(x, 1)$ has a solution.

Let us now establish the following generalization of Theorem 1.1.

THEOREM 1.2. *Let Q be a convex closed subset of E and U a (relative to Q) open subset of Q. Let T: $U\times[0, 1]\to E$ be a locally compact continuous map and let T_0: $U\to E$ denote the map $T_0(x) = T(x, 0)$, $x\in U$.*

Assume that

(i) *$T_0(U)$ is a subset of Q;*

(ii) *the map T_0 admits a continuous extension $\hat{T}_0$: $Q\to Q$ with relatively compact image and without fixed points in $Q\setminus U$;*

(iii) *the set $\{x\in U:\ T(x, \lambda) = x$ for some $\lambda\in[0, 1]\}$ is compact;*

(iv) *for any $(x, \lambda)\in(U\cap\partial Q)\times[0, 1)$ with $T(x, \lambda) = x$, there exist open neighborhoods U_x of x in U and I_λ of λ in $[0, 1)$ such that $T((U_x\cap\partial Q)\times I_\lambda)\subset Q$.*

Then the equation

$$x = T(x, 1)$$

has a solution in U.

Proof. Since T is locally compact and since, by assumption (iii), the set

$$F = \{x \in U:\ T(x, \lambda) = x \text{ for some } \lambda \in [0, 1]\}$$

is compact, there exists an open neighborhood V of F in U such that $\bar{V} \subset U$ and $T(V \times [0, 1])$ is relatively compact. Without loss of generality we may assume that the family of seminorms $\mathscr{P}$ which generates the topology of $\boldsymbol{E}$ has the property that if $p \in \mathscr{P}$ and $\lambda \geqslant 0$, then $\lambda p \in \mathscr{P}$. So, for any $x \in F$, let $p_x \in \mathscr{P}$ be such that

$$\inf\{p_x(y-x),\ y \in Q \setminus V\} > 2$$

and denote $U_{(p_x, 1)}(x) = \{y \in \boldsymbol{E}:\ p_x(y-x) < 1\}$. There exist $x_1, \ldots, x_s \in F$ such that $\bigcup_{i=1}^{s} U_{(p_{x_i}, 1)}(x_i) \supset F$. Moreover,

$$\bigcup_{i=1}^{s} U_{(p_{x_i}, 1)}(x_i) \cap (Q \setminus V) = \emptyset .$$

Define δ: $\boldsymbol{R} \to [0, 1]$ by

$$\delta(t) = \begin{cases} 1, & t \leqslant 1, \\ 2-t, & 1 \leqslant t \leqslant 2, \\ 0, & t \geqslant 2, \end{cases}$$

and let σ: $Q \to [0, 1]$ be given by $\sigma(x) = \delta(\min\{p_{x_i}(x-x_i),\ i = 1, \ldots, s\})$. Clearly we have $\sigma(x) = 1$ for $x \in F$ and $\sigma(x) = 0$ for $x \in Q \setminus V$.

Let us now define $\hat{T}$: $Q \times [0, 1] \to \boldsymbol{E}$ as follows

$$\hat{T}(x, \lambda) = \begin{cases} T(x, \sigma(x)\lambda), & (x, \lambda) \in U \times [0, 1], \\ \hat{T}_0(x), & (x, \lambda) \in (Q \setminus U) \times [0, 1]. \end{cases}$$

By (i) and (ii), and since $\sigma(x) = 0$ in $U \setminus V$, it follows that the map $\hat{T}$ is continuous, has relatively compact image and $\hat{T}(x, 0) \in Q$ for all $x \in Q$. Observe also that $x \in Q$ satisfies $\hat{T}(x, \lambda) = x$ for some $\lambda \in [0, 1]$ if and only if $x \in F$ and $T(x, \lambda) = x$. So, if $(x, \lambda) \in \partial Q \times [0, 1)$ is such that $\hat{T}(x, \lambda) = x$, then $\sigma(x) = 1$ and, by assumption (iv), one can find neighborhoods $\hat{U}_x \subset U_x$ and $\hat{I}_\lambda \subset I_\lambda$ such that $\hat{T}((\hat{U}_x \cap \partial Q) \times \hat{I}_\lambda) \subset Q$. Therefore, the map $\hat{T}$ satisfies all the assumptions of Theorem 1.1 and, thus, there exists $\bar{x} \in Q$ such that $\hat{T}(\bar{x}, 1) = \bar{x}$. As observed above, this implies that $\bar{x}$ is a fixed point also of $T(\cdot, 1)$. Q.E.D.

Remark 1.3. Assumption (i) and (ii) of Theorem 1.2 are clearly satisfied in the case when $T(x, 0) = x_0 \in U$ for all $x \in U$.

Easy consequences of the above theorems are the following classical results:

Corollary 1.2. [2] (Schauder–Tychonoff fixed point theorem). *Let Q be* *a convex closed subset of a Hausdorff locally convex topological vector space* $\mathbf{E}$ *and let* $T: Q \to Q$ *be a continuous map with relatively compact image. Then* T *has a fixed point.*

COROLLARY 1.3 [6] (Rothe's type theorem). *Let* Q *be a convex closed subset of a Banach space* $\mathbf{E}$ *and let* $T: Q \to \mathbf{E}$ *be a continuous map with relatively compact image and such that* $T(\partial Q) \subset Q$. *Then* T *has a fixed point.*

Proof. Take $x_0 \in Q$ and apply Theorem 1.1 to the map $(x, \lambda) \in Q \times [0, 1] \mapsto \lambda T(x) + (1-\lambda) x_0$. Q.E.D.

The next result is an extension of the Leray–Schauder Continuation Principle and reduces to it when the convex set Q is the whole space. Moreover, it also turns out to be a generalization of Schauder's fixed point theorem.

COROLLARY 1.4. *Let* Q *be a convex closed subset of a Banach space* $\mathbf{E}$ *and* U *a (relative to* Q*) open subset of* Q. *Let* $T: U \times [0, 1] \to Q$ *be a compact map (i.e.,* T *sends bounded sets into relatively compact sets) such that* $T(x, 0) = x_0 \in U$ *for all* $x \in U$.

Assume that the set

$$\{x \in U: \ T(x, \lambda) = x \text{ for some } \lambda \in [0, 1]\}$$

is a compact subset of U.

Then the equation

$$T(x, 1) = x$$

has a solution in U.

2. Applications to differential systems. Let J be a (possibly noncompact) real interval and let $C^0(J, \mathbf{R}^n)$ denote the Fréchet space of all continuous functions $x: J \to \mathbf{R}^n$ with the topology of the uniform convergence on compact subintervals of J. As a generating family of seminorms for this topology one may consider $\{p_I: I \text{ compact subinterval of } J\}$, where $p_I(x) = \sup\{|x(t)|, t \in I\}$. We recall that a subset A of $C^0(J, \mathbf{R}^n)$ is bounded if and only if there exists a positive continuous function $\varphi: J \to \mathbf{R}$ such that $|x(t)| \leqslant \varphi(t)$ for all $t \in J$ and $x \in A$.

Consider the boundary value problem

$$\begin{aligned} &\dot{x}(t) = f(t, x(t)), \quad t \in J, \\ &x \in S, \end{aligned} \tag{2.1}$$

where $f: J \times \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function and S is a nonempty subset of $C^0(J, \mathbf{R}^n)$. Observe that, since S is contained into $C^0(J, \mathbf{R}^n)$, any solution of (2.1) is defined on the whole interval J.

Our aim is to show how the abstract continuation principles obtained in Section 1 can be used to deduce existence results for problem (2.1). To this end, it turns out to be convenient to apply a sort of Schauder's linearization technique to the system

$$\text{(2.2)}\qquad \begin{aligned} &\dot{x}(t) = g\big(t, x(t), x(t), \lambda\big), \qquad t \in J,\ \lambda \in [0, 1],\\ &x \in S, \end{aligned}$$

where $g\colon J \times \boldsymbol{R}^n \times \boldsymbol{R}^n \times [0, 1] \to \boldsymbol{R}^n$ is a continuous homotopy which, for $\lambda = 1$, coincides with f on the diagonal of $\boldsymbol{R}^n \times \boldsymbol{R}^n$. More precisely, we can state the following existence result:

THEOREM 2.1. *Let $g\colon J \times \boldsymbol{R}^n \times \boldsymbol{R}^n \times [0, 1] \to \boldsymbol{R}^n$ be continuous and such that $g(t, s, s, 1) = f(t, s)$ for all $(t, s) \in J \times \boldsymbol{R}^n$.*

Assume that

(c_1) *there exist a convex closed subset Q_1 of $C^0(J, \boldsymbol{R}^n)$ and a bounded closed subset S_1 of S such that the problem*

$$\text{(2.3)}\qquad \begin{aligned} &\dot{x}(t) = g\big(t, x(t), q(t), \lambda\big), \qquad t \in J,\ \lambda \in [0, 1],\\ &x \in S_1, \end{aligned}$$

is uniquely solvable for each $(q, \lambda) \in Q_1 \times [0, 1]$;

(c_2) *the solution of* (2.3) *corresponding to any $(q, 0)$ belongs to Q_1;*

(c_3) *if $\{(x_j, \lambda_j)\}_{j \in \boldsymbol{N}}$ is a sequence in $S_1 \times [0, 1]$, with $\lambda_j \to \lambda \in [0, 1)$ and x_j converging to a solution $x \in Q_1$ of* (2.2) *(corresponding to λ), then x_j belongs to Q_1 for j sufficiently large.*

Then system (2.1) *has a solution (in $Q_1 \cap S_1$).*

The proof of Theorem 2.1 requires the following continuity and compactness result obtained by Cecchi–Furi–Marini in [1].

LEMMA 2.1. *Let $h\colon J \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$ be a continuous function and let S be a nonempty subset of $C^0(J, \boldsymbol{R}^n)$.*

Assume that

(a_1) *there exists a bounded subset B of $C^0(J, \boldsymbol{R}^m)$ such that for any $b \in B$ the boundary value problem*

$$\begin{aligned} &\dot{x}(t) = h\big(t, x(t), b(t)\big), \qquad t \in J,\\ &x \in S, \end{aligned}$$

admits a unique solution $x = H(b)$;

(a_2) *$H(B)$ is bounded.*

Then $H(B)$ is a relatively compact subset of $C^0(J, \boldsymbol{R}^n)$.

Moreover, if S is closed, the operator $H\colon B \to S$ is continuous provided (a_1), (a_2) *are satisfied.*

Proof of Theorem 2.1. Let $T\colon Q_1 \times [0, 1] \to S_1$ be the operator which associates to any (q, λ) the unique solution $x = T(q, \lambda)$ of (2.3). Clearly, any fixed point of $T(\cdot, 1)$ is a solution of (2.1) belonging to $Q_1 \cap S_1$.

Observe first that the bounded set $Q_1 \cap S_1$ is nonempty since, by (c_2), $T(q, 0) \in Q_1$ for each q. Hence, $\overline{\mathrm{co}}(Q_1 \cap S_1)$ – the closed convex hull of $Q_1 \cap S_1$ – is bounded as well and so, by applying Lemma 2.1 with

$$B = \{b \in C^0(J, \mathbf{R}^{n+1}): b(t) = (q(t), \lambda),\ q \in \overline{\mathrm{co}}(Q_1 \cap S_1),\ \lambda \in [0, 1]\},$$

we obtain that the operator $T: \overline{\mathrm{co}}(Q_1 \cap S_1) \times [0, 1] \to S_1$ is continuous and has relatively compact image. We are, therefore, led to the situation considered in Theorem 1.1 with now T as above and $Q = \overline{\mathrm{co}}(Q_1 \cap S_1)$. Assumption (i) of Theorem 1.1 obviously follows from condition (c_1). In order to verify (ii), it suffices to show that if $\{(q_j, \lambda_j)\}_{j \in N}$ is a sequence in $(\partial \overline{\mathrm{co}}(Q_1 \cap S_1)) \times [0, 1]$ with $\lambda_j \to \lambda \in [0, 1)$ and q_j converging to a solution $x \in Q_1 \cap S_1$ of (2.2) corresponding to λ, then $T(q_j, \lambda_j) \in \overline{\mathrm{co}}(Q_1 \cap S_1)$ for j large enough (recall Remark 1.1). In fact, the sequence $\{T(q_j, \lambda_j)\}_{j \in N} \subset S_1$ converges to x, so that, by assumption (c_3), $T(q_j, \lambda_j)$ belongs to Q_1 and, thus, to $\overline{\mathrm{co}}(Q_1 \cap S_1)$ for j sufficiently large. Q.E.D.

Theorem 2.1 contains, in particular, the following existence result which has been proved in [1] with the aid of the Schauder–Tychonoff fixed point theorem.

COROLLARY 2.1. *Let $g: J \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and such that $g(t, s, s) = f(t, s)$ for all $(t, s) \in J \times \mathbf{R}^n$. Assume there exist a convex closed subset Q_1 of $C^0(J, \mathbf{R}^n)$ and a bounded closed subset S_1 of $S \cap Q_1$ which make the problem*

$$\dot{x}(t) = g(t, x(t), q(t)), \qquad t \in J,$$
$$x \in S_1,$$

uniquely solvable for each $q \in Q_1$.

Then problem (2.1) *admits a solution.*

Proof. It suffices to observe that assumptions (c_2) and (c_3) of Theorem 2.1 are obviously satisfied since S_1 is contained in $S \cap Q_1$. Q.E.D.

The next result which is concerned with boundary value problems associated with scalar differential equations, is an easy consequence of Theorem 2.1 and, although a little involved in its general formulation, it permits one (see Example 2.1 and 2.2 below) to deal directly with the scalar equation instead of transforming the equation into a system.

For any nonnegative integer $i \in N$, we will denote by $C^i(J)$ the Fréchet space of all C^i functions $x: J \to \mathbf{R}$ with the topology of uniform convergence on compact subintervals of J of the functions and of all their derivatives up to the order i.

Clearly, a subset $A \subset C^i(J)$ is bounded if and only if there exists a positive continuous function $\varphi\colon J \to \boldsymbol{R}$ such that

$$|x^{(j)}(t)| \leqslant \varphi(t) \quad \text{for all } x \in A,\ 0 \leqslant j \leqslant i,\ t \in J.$$

However, it is known that to show the boundedness in $C^i(J)$ of a subset A it suffices to prove the existence of a positive continuous function $\psi;\ J \to \boldsymbol{R}$ such that

$$(|x(t)| + |x^{(i)}(t)|) \leqslant \psi(t) \quad \text{for all } x \in A,\ t \in J.$$

Observe also that $C^i(J)$ can be embedded as a closed subspace in $C^0(J, \boldsymbol{R}^{i+1})$ by means of the map $x \mapsto (x, x^{(1)}, \ldots, x^{(i)})$.

THEOREM 2.2. *Consider the boundary value problem*

$$\text{(2.4)} \qquad \begin{aligned} &x^{(n)}(t) = f(t, x(t), x^{(1)}(t), \ldots, x^{(n-1)}(t)), \qquad t \in J,\\ &x \in S, \end{aligned}$$

where $f\colon J \times \boldsymbol{R}^n \to \boldsymbol{R}$ *is continuous and* S *is a subset of* $C^0(J)$.

Let $g\colon J \times \boldsymbol{R}^k \times \boldsymbol{R}^l \times [0, 1] \to \boldsymbol{R}$ $(k \leqslant n,\ l \leqslant n)$ *be a continuous function such that*

$$g(t; s_0, \ldots, s_{k-1}; s_0, \ldots, s_{l-1}; 1) = f(t; s_0, \ldots, s_{n-1})$$

$$\text{for all } (t; s_0, \ldots, s_{n-1}) \in J \times \boldsymbol{R}^n.$$

Assume that

(γ_1) *there exist a convex closed bounded subset* Q_1 *of* $C^{l-1}(J)$ *and a bounded closed subset* S_1 *of* $C^{k-1}(J)$ *contained in* S *such that the problem*

$$\text{(2.5)} \qquad \begin{aligned} &x^{(n)}(t) = g(t, x(t), \ldots, x^{(k-1)}(t), q(t), \ldots, q^{(l-1)}(t), \lambda), \qquad t \in J,\\ &x \in S_1, \end{aligned}$$

is uniquely solvable for each $(q, \lambda) \in Q_1 \times [0, 1]$;

(γ_2) *the solution of* (2.5) *corresponding to any* $(q, 0)$ *belongs to* Q_1;

(γ_3) *if* $\{(x_j, \lambda_j)\}_{j \in \boldsymbol{N}}$ *is a sequence in* $C^{n-1}(J) \cap S_1 \times [0, 1]$, *with* $\lambda_j \to \lambda \in [0, 1)$ *and* x_j *converging in* $C^{n-1}(J)$ *to a solution* $x \in Q_1$ *of*

$$\text{(2.6)} \qquad \begin{aligned} &x^n(t) = g(t, x(t), \ldots, x^{(k-1)}(t), x(t), \ldots, x^{(l-1)}(t), \lambda), \qquad t \in J,\\ &x \in S, \end{aligned}$$

then x_j *belongs to* Q_1 *for* j *large enough.*

Then problem (2.4) *has a solution (in* $Q_1 \cap S_1$).

Proof. By (γ_2), the set $Q_1 \cap S_1$ is nonempty. So, since $Q_1 \cap S_1$ and S_1 are bounded in $C^{l-1}(J)$ and $C^{k-1}(J)$ respectively and g is continuous, it follows that the solutions $x \in S_1$ of (2.5) corresponding to pairs

$(q, \lambda) \in (Q_1 \cap S_1) \times [0, 1]$ and their derivatives of order n are uniformly bounded by a positive continuous function $\psi\colon J \to \mathbf{R}$. Therefore, as previously observed, this guarantees the existence of a positive continuous function $\varphi\colon J \to \mathbf{R}$ such that $|x^{(j)}(t)| \leqslant \varphi(t)$, $0 \leqslant j \leqslant n$, $t \in J$. Hence, the set

$$\hat{S}_1 = S_1 \cap \{x \in C^{n-1}(J)\colon |x^{(n-1)}(t)| \leqslant \varphi(t)\}$$

is a nonempty closed bounded subset of $C^{n-1}(J)$.

Let now $\hat{Q}_1$ denote the closure in C^{n-1} of the convex hull of $Q_1 \cap S_1$ and observe that, for any $(q, \lambda) \in \hat{Q}_1 \times [0, 1]$, problem (2.5) has a unique solution $x \in \hat{S}_1$. Moreover, by assumption (γ_3), if $\{(x_j, \lambda_j)\}_{j \in \mathbf{N}}$ is a sequence in $\hat{S}_1 \times [0, 1]$ with $\lambda_j \to \lambda \in [0, 1)$ and x_j converging (in $C^{n-1}(J)$) to a solution $x \in \hat{Q}_1$ of (2.6), then x_j belongs to $\hat{Q}_1$ for j large enough. Therefore, since $\hat{Q}_1$ and $\hat{S}_1$ can be embedded into $C^0(J, \mathbf{R}^n)$ via the mapping $x \mapsto (x, x^{(1)}, \ldots, x^{(n-1)})$, by regarding (2.4) as a first order differential system, the assertion of the theorem will follow directly from Theorem 2.1. Q.E.D.

We illustrate now Theorem 2.2 presenting some applications to differential equations on noncompact intervals.

EXAMPLE 2.1. Consider the second order boundary value problem

$$\begin{aligned} &\ddot{x}(t) + \dot{x}(t) = f(t, x(t)), \qquad t \in [0, +\infty), \\ &x(0) = 0, \quad \lim_{t \to +\infty} x(t) = 0, \end{aligned} \tag{2.7}$$

where $f\colon [0, +\infty) \times \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying the following hypotheses:

(H_1) *there exists a positive constant* M *such that* $f(t, s)_{s>0}$ *for* $|s| \geqslant M$ *and all* $t \geqslant 0$;

(H_2) *for all* $R > 0$,

$$\int^{\infty} \alpha_R(t)\, dt < \infty \quad \text{and} \quad \lim_{t \to \infty} \alpha_R(t) = 0,$$

where $\alpha_R(t) = \sup_{|s| \leqslant R} |f(t, s)|$.

Our aim is to prove the existence of a (classical) solution of (2.7). We will apply Theorem 2.2 with $n = 2$, $k = 2$, $l = 1$, $g\colon [0, \infty) \times \mathbf{R}^2 \times \mathbf{R} \times [0, 1] \to \mathbf{R}$ given by $g(t, s_0, s_1; r_0; \lambda) = \lambda f(t, r_0) - s_1$. In order to construct the subsets Q_1 and S_1, observe first that, if x is a solution of (2.7) such that $\max_{t \geqslant 0} |x(t)| = |x(t_0)| > M$, then $\dot{x}(t_0) = 0$ and, by (H_1), $\ddot{x}(t_0) \operatorname{sign} x(t_0) = f(t_0, x(t_0)) \operatorname{sign} x(t_0) > 0$, which is a contradiction. Hence, the possible solutions of (2.7) must satisfy the a priori bound

$$|x(t)| \leqslant M \quad \text{for all } t \geqslant 0.$$

Consider the convex closed subset of $C^0([0, \infty))$

$$Q_1 = \{q \in C^0([0, \infty)): |q(t)| \leqslant M+1 \text{ for } t \geqslant 0\}.$$

It is not hard to see that, for any $(q, \lambda) \in Q_1 \times [0, 1]$, the linear problem

$$\begin{aligned} &\ddot{x}(t) + \dot{x}(t) = \lambda f(t, q(t)), \quad t \geqslant 0, \\ &x(0) = 0, \quad \lim_{t \to +\infty} x(t) = 0, \end{aligned} \tag{2.8}$$

has a unique solution x given by

$$x(t) = -\lambda\Big[\int_t^\infty f(\tau, q(\tau))\,d\tau + e^{-t}\Big(\int_0^t e^\tau f(\tau, q(\tau))\,d\tau - \int_0^\infty f(\tau, q(\tau))\,d\tau\Big)\Big].$$

Hence

$$|x(t)| \leqslant (1-e^{-t}) \int_t^\infty \alpha_{M+1}(\tau)\,d\tau + e^{-t}\Big(\int_0^t e^\tau \alpha_{M+1}(\tau)\,d\tau + \int_0^t \alpha_{M+1}(\tau)\,d\tau\Big)$$

and $x(t) \equiv 0 \in Q_1$ for $\lambda = 0$ and all $q \in Q_1$.

Let $\gamma(t)$ denote the right-hand side of the above inequality. Clearly, $\gamma(0) = 0$ and, by standard calculations, one may check that $\lim_{t \to \infty} \gamma(t) = 0$.

Moreover,

$$\begin{aligned} |\dot{x}(t)| &= \Big|\lambda\Big(e^{-t} \int_0^\infty f(\tau, q(\tau))\,d\tau - e^{-t}\int_0^t e^\tau f(\tau, q(\tau))\,d\tau\Big)\Big| \\ &\leqslant \Big(e^{-t} \int_0^\infty \alpha_{M+1}(\tau)\,d\tau + e^{-t}\int_0^t e^\tau \alpha_{M+1}(\tau)\,d\tau\Big) = \gamma_1(t). \end{aligned}$$

Therefore the set

$$S_1 = \{x \in C^1([0, \infty)): |x(t)| \leqslant \gamma(t),\ |\dot{x}(t)| \leqslant \gamma_1(t)\}$$

is a (convex) bounded closed subset of $C^1([0, \infty))$ containing the solutions of (2.8) which correspond to pairs $(q, \lambda) \in Q_1 \times [0, 1]$.

In order to apply Theorem 2.2, it remains only to verify assumption (γ_3). Take a sequence $\{(x_j, \lambda_j)\}_{j \in N} \subset S_1 \times [0, 1]$, with $\lambda_j \to \lambda$ and x_j converging in C^1 to a solution x_λ of

$$\begin{aligned} &\ddot{x}(t) + \dot{x}(t) = \lambda f(t, x(t)), \quad t \geqslant 0, \\ &x(0) = 0, \quad \lim_{t \to \infty} x(t) = 0. \end{aligned}$$

Thus, since $\gamma(0) = 0 = \lim_{t \to \infty} \gamma(t)$, there exist t_0, $t_\infty \in [0, \infty)$ such that $|x_j(t)| \leqslant \gamma(t) \leqslant M+1$ for all $t \in [0, t_0) \cup (t_\infty, \infty)$ and all $j \in N$. On the other hand, the sequence $\{x_j\}_{j \in N}$ converges uniformly to x_λ in the compact interval

$[t_0, t_\infty]$ and an estimate similar to the one obtained above for the solutions of (2.7) shows that $|x_\lambda(t)| \leqslant M$ for all t. Therefore,

$$|x_j(t)| \leqslant M+1 \qquad \text{for } t \in [t_0, t_\infty) \text{ and } j > j_0,$$

which implies that x_j belongs to Q_1 for j large enough as required.

EXAMPLE 2.2. Let f: $[0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$ be continuous and such that:

(K_1) *there exists* $M > 0$ *such that* $f(t, s)_{s>0}$ *for* $|s| \geqslant M$ *and* $t \in [0, 1]$.

Consider the problem

$$\text{(2.9)} \qquad \begin{aligned} &t\ddot{x}(t) = f\big(t, x(t)\big), \qquad t \in (0, 1), \\ &\lim_{t \to 0^+} x(t) = \lim_{t \to 1^-} x(t) = 0. \end{aligned}$$

By a solution of (2.9) we mean a function $x \in C^2\big((0, 1)\big)$ satisfying the boundary condition

$$x(0^+) = \lim_{t \to 0^+} x(t) = 0, \qquad x(1^-) = \lim_{t \to 1^-} x(t) = 0.$$

As in Example 2.1, assumption (K_1) guarantees that all the possible solutions of (2.9) satisfy the a priori bound $|x(t)| \leqslant M$ for $t \in (0, 1)$. Again, a suitable choice of the convex set Q_1 turns out to be

$$Q_1 = \big\{q \in C^0\big((0, 1)\big)\text{: } |q(t)| \leqslant M+1 \text{ for } 0 < t < 1\big\}.$$

Observe that, for any $(q, \lambda) \in Q_1 \times [0, 1]$, the linear problem

$$\text{(2.10)} \qquad \begin{aligned} &t\ddot{x}(t) = \lambda f\big(t, q(t)\big), \qquad t \in (0, 1), \\ &x(0^+) = x(1^-) = 0, \end{aligned}$$

is uniquely solvable. To see this it suffices to observe that the solution of the Cauchy problem

$$\ddot{x}(t) = \lambda \frac{f\big(t, q(t)\big)}{t}, \qquad x(\tfrac{1}{2}) = 0, \qquad \dot{x}(\tfrac{1}{2}) = 0, \qquad t \in (0, 1),$$

is defined and uniformly continuous in the interval $(0, 1)$. Thus, it extends continuously to $[0, 1]$.

Let now α and β denote respectively the solutions of the boundary value problems

$$t\ddot{x}(t) = K, \qquad x(0^+) = x(1^-) = 0, \qquad t \in (0, 1),$$

and

$$t\ddot{x}(t) = -K, \qquad t \in (0, 1), \qquad \text{where } K = \sup_{\substack{t \in (0,1) \\ |s| \leqslant M+1}} |f(t, s)|.$$

$$x(0^+) = x(1^-) = 0,$$

A simple calculation shows that $\alpha(t) < 0 < \beta(t)$ for each $t \in (0, 1)$. Moreover, if x is any solution of (2.10) corresponding to some $(q, \lambda) \in Q_1 \times [0, 1]$, then $x - \alpha$ is concave and $x - \beta$ is convex, so that $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ for $t \in (0, 1)$.

Hence, setting $S_1 = \{x \in C^0((0, 1)) \colon \alpha(t) \leqslant x(t) \leqslant \beta(t)\}$ and recalling that $\alpha(0^+) = \alpha(1^-) = 0$, $\beta(0^+) = \beta(1^-) = 0$, by an argument analogous to the one used in Example 2.1, we obtain that if $\{(x_j, \lambda_j)\}_{j \in N}$ is a sequence in $S_1 \times [0, 1]$, with $\lambda_j \to \lambda \in [0, 1)$ and x_j converging to a solution (which, clearly, belongs to Q_1) of

$$t\ddot{x}(t) = \lambda f(t, x(t)), \qquad x(0^+) = x(1^-) = 0, \qquad t \in (0, 1),$$

then x_j belongs to Q_1 for j sufficiently large.

Consequently, in this problem (2.9), the existence of a solution follows directly from Theorem 2.2. Observe finally that the above argument yields the same conclusion even in the case when f is defined only in $(0, 1) \times \boldsymbol{R}$ provided f is assumed to send bounded sets into bounded sets.

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