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CONVEXITY AND VARIATION DIMINISHING PROPERTY FOR BERNSTEIN POLYNOMIALS IN HIGHER DIMENSIONS

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1. Introduction

Let $f(x)$ be a real function defined in $Q := [0, 1]$. The Bernstein polynomial associated with f is defined by

$$B^n(f; x) := \sum_{i=0}^{n} f_i \cdot \binom{n}{i} x^i (1-x)^{n-i}, \qquad x \in Q, \tag{1}$$

where $f_i := f(i/n)$ for $i = 0, 1, \ldots, n$.

Joining $\big((i-1)/n, f_{i-1}\big)$ and $(i/n, f_i)$ by a line segment, for $i = 1, 2, \ldots, n$, we get a piecewise linear continuous function which is denoted by $\hat{f}_n(x)$ and called the *Bézier polygon of* $B^n(f; x)$.

It is easy to show that:

(i) If $\hat{f}_n(x)$ is convex in Q then so is $B^n(f; x)$.

(ii) If $f(x)$ is convex in Q then

$$B^n(f; x) \geqslant B^{n+1}(f; x), \qquad n = 1, 2, \ldots, \qquad \text{for } x \in Q. \tag{2}$$

(iii) The variation diminishing property holds, i.e.

$$V[B^n; Q] \leqslant V_1[\hat{f}_n; Q], \tag{3}$$

with equality if and only if $\hat{f}_n(x)$ is convex or concave. Here

$$V[B^n; Q] := \int_Q \left|\frac{d^2 B^n}{dx^2}\right| dx, \tag{4}$$

$$V_1[\hat{f}_n; Q] := n \sum_{i=0}^{n-2} |\Delta^2 f_i|, \tag{5}$$

$\Delta f_i := f_{i+1} - f_i, \quad i = 0, 1, \ldots, n-1.$

G. Chang, P. Davis and J. Hoschek (see [1], [3]) extended these results to the Bernstein polynomials over triangles. In the present paper we attempt to extend these results to the Bernstein polynomials over a k-dimensional simplex Q. In this case the definition of a variation like (5) called now the *variation of the Bézier net* is more complicated, but in the case of $k = 1, 2$ it coincides with the variations as given in [2], [3]. We also give an answer to the problem whether the variation of the Bézier net is equal to $\mathrm{Var}(\Delta f_n)$. Here the Laplace operator Δ is taken in the distribution sense (see a remark in Section 3). These problems were suggested to me by Professor Z. Ciesielski.

2. Notation

Let us begin with some definitions. Let $Q := [P_1, P_2, \ldots, P_{k+1}]$ be the k-dimensional simplex with vertices $P_1, P_2, \ldots, P_{k+1}$, where $P_i \in R^k$ for $i = 1, 2, \ldots, k+1$. By Q_α $(1 \leqslant \alpha \leqslant k+1)$ we denote the $(k-1)$-dimensional face of the simplex Q which does not contain the vertex P_α. By $W := [T_1, T_2, \ldots, T_{k+1}]$ we denote the simplex with the vertices $T_i = (0, \ldots, 0, 1, 1, \ldots, 1)$, where $i = 1, 2, \ldots, k+1$, and by $K_n(W)$ the subdivision of the simplex W such that a simplex Ω belongs to $K_n(W)$ if and only if

$$\Omega = \Delta^n_{i_1 i_2 \ldots i_k} + v$$

where

$$\Delta^n_{i_1 i_2 \ldots i_k} = \{(x_1, \ldots, x_k) \in R^k;\ 0 \leqslant x_{i_1} \leqslant x_{i_2} \leqslant \ldots \leqslant x_{i_k} \leqslant 1/n\},$$

$i_1, i_2, \ldots, i_k$ is a permutation of $\{1, 2, \ldots, k\}$, and the coordinates of the vector $v = (v_1, v_2, \ldots, v_k) \in R^k$ satisfy the following conditions:

(i) $nv_i \in N = \{0, 1, 2, \ldots\}$ for $i = 1, 2, \ldots, k$,

(ii) $0 \leqslant v_1 \leqslant v_2 \leqslant \ldots \leqslant v_k < 1$.

Now, let $L\colon R^k \to R^k$ be an affine transformation such that

$$L(T_i) = P_i \quad \text{for } i = 1, 2, \ldots, k+1.$$

Then L transforms the subdivision $K_n(W)$ of the simplex W to some subdivision of the simplex Q, which we denote by $S_n(Q)$. It is known that each point $P \in Q$ can be uniquely expressed as

$$P = \sum_{i=1}^{k+1} u_i P_i$$

with $u_1 + u_2 + \ldots + u_{k+1} = 1$, $u_i \geqslant 0$, $1 \leqslant i \leqslant k+1$.

The numbers $u_1, u_2, \ldots, u_{k+1}$ are called the *barycentric coordinates* of P with respect to the simplex Q. We identify the point $P \in Q$ with its barycentric coordinates and write $P = (u_1, u_2, \ldots, u_{k+1})$. For $n \in N$ and a given function $f\colon Q \to R$ we define

$$F_n := \left\{ f_\beta ; f_\beta := f\left(\frac{\beta}{n}\right),\ \beta \in N^{k+1},\ |\beta| = n \right\}$$

where

$$\beta = (\beta_1, \beta_2, \ldots, \beta_{k+1}) \quad \text{and} \quad |\beta| := \sum_{i=1}^{k+1} \beta_i.$$

We also define

$$B^n(f; P) = \sum_{|\beta|=n} f_\beta J_\beta^n(P) \quad \text{for } P \in Q \tag{6}$$

where

$$J_\beta^n(P) := \frac{n!}{\beta!} u^\beta := \frac{n!}{\beta_1!\,\beta_2!\ldots\beta_{k+1}!} u_1^{\beta_1} u_2^{\beta_2} \ldots u_{k+1}^{\beta_{k+1}} \tag{7}$$

and $P = (u_1, u_2, \ldots, u_{k+1})$.

The polynomials $J_\beta^n(P)$ are called the *basic Bernstein polynomials.* $B^n(f; P)$ is called the *n-th Bernstein polynomial over the simplex Q*. Setting

$$\tilde{P}_\beta := (P, f_\beta) := \left(\frac{\beta_1}{n}, \frac{\beta_2}{n}, \ldots, \frac{\beta_{k+1}}{n}, f_\beta\right),$$

we get a point on the surface associated with the function $f(P)$. The points $\tilde{P}_\beta$ with $|\beta| = n$ are called the *Bézier points of $B^n(f; P)$*. For any simplex $\Omega \in S_n(Q)$ we have $k+1$ Bézier points. Joining them by a k-dimensional plane we obtain a continuous piecewise linear surface $\hat{f}_n(P)$, which is called the *Bézier net of $B^n(f; P)$* over the simplex Q.

Let E_i $(1 \leqslant i \leqslant k+1)$ be the partial shift operator defined by

$$E_i f_\beta := f_{\beta + \hat{e}_i}$$

where $|\beta| = n-1$, $\hat{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^{k+1}$.

For convenience we introduce

$$D_{ij}(f_\beta) := -(E_i - E_{i+1})(E_j - E_{j+1}) f_\beta \tag{8}$$

where $|\beta| = n-2$, $1 \leqslant i < j \leqslant k+1$ and $E_{k+2} := E_1$.

3. Results

The first theorem generalizes a result of G. Chang and P. Davis [1].

THEOREM 1. *The convexity of the Bézier net $\hat{f}_n(P)$ over Q is equivalent to the following inequalities:*

$$D_{ij}(f_\beta) \geqslant 0 \tag{9}$$

for $1 \leqslant i < j \leqslant k+1$, $|\beta| = n-2$.

The proof is standard and will be omitted.

The next theorem is also a multidimensional analogue of a theorem of G. Chang and P. Davis.

THEOREM 2. (i) *If the Bézier net* $\hat{f}_n(P)$ *over* Q *is convex then so is the Bernstein polynomial* $B^n(f; P)$.

(ii) *If* $f(P)$ *is convex in* Q *then we have*

$$B^n(f; P) \geqslant B^{n+1}(f; P) \quad \text{for } P \in Q,\ n = 1, 2, \ldots \tag{10}$$

Proof. (i) The proof is based on the well-known convexity test for any function in $C^2(Q)$ and the following

LEMMA 1. *For the Bernstein polynomial* $B^n(f; P)$ *we have the equality*

$$\frac{\partial^2 B^n(f; P)}{\partial x_r\, \partial x_{r'}} = \frac{n(n-1)}{k^2 |Q|} \sum_{|\beta| = n-2} \sum_{1 \leqslant i < j \leqslant k+1} \Big(\sum_{\alpha = i+1}^{j} n_\alpha |Q_\alpha|, e_r \Big) \times \Big(\sum_{\alpha = i+1}^{j} n_\alpha |Q_\alpha|, e_{r'} \Big) D_{ij}(f_\beta) J^n_\beta(P) \quad \text{for } r, r' = 1, 2, \ldots, k \tag{11}$$

where $P \in Q$, $e_r = (0, \ldots, 0, \overset{r}{1}, 0, \ldots, 0) \in R^k$, $r = 1, 2, \ldots, k$, $|Q| := \mathrm{vol}_k Q$, $|Q_\alpha| := \mathrm{vol}_{k-1}(Q_\alpha)$, $(\cdot, \cdot)$ *is the scalar product in* R^k, *and* n_α *is the unit outward normal vector to* Q_α.

The tedious proof will be omitted.

(ii) G. Chang and P. Davis ([1]) proved the statement for $k = 2$. For $k > 2$ the proof is similar.

Now, we define a *variation of the Bernstein polynomial* $B^n(f; P)$ by

$$V[B^n; Q] := \int_Q |\Delta B^n| \tag{12}$$

where for simplicity B^n stands for $B^n(f; P)$ and Δ is the Laplacian. The variation $V_1[\hat{f}_n, Q]$ of the Bézier net $\hat{f}_n(P)$ is defined by

$$V_1[\hat{f}_n; Q] := \frac{n!(k-1)!}{k(n+k-2)!\,|Q|} \sum_{|\beta| = n-2} \sum_{1 \leqslant i < j \leqslant k+1} |D_{ij}(f_\beta)| \Big\| \sum_{\alpha = i+1}^{j} n_\alpha |Q_\alpha| \Big\|^2 \tag{13}$$

where $\|\cdot\|^2 = (\cdot, \cdot)$.

For $k = 2$ the variation $V_1[\hat{f}_n; Q]$ of the Bézier net coincides with the variation introduced by T. Goodman [3]. Our next theorem generalizes the result obtained by G. Chang and J. Hoschek [2].

THEOREM 3. *For the Bernstein polynomial over the simplex* Q *the variation diminishing property holds. More precisely, we have*

$$V[B^n; Q] \leqslant V_1[\hat{f}_n; Q] \tag{14}$$

with equality if and only if $\hat{f}_n(P)$ *is either convex or concave.*

For $k = 1, 2$ the theorem was proved by G. Chang and J. Hoschek [2]. For $k > 2$ the proof is similar.

Since

$$V[B^n; Q] = \operatorname{Var}(\Delta B^n) \tag{15}$$

where $\operatorname{Var}(\Delta B^n)$ is the total variation of the measure

$$\nu(A) := \int_A \Delta B^n,$$

the question arises whether the equality

$$V_1[\hat{f}_n; Q] = \operatorname{Var}(\Delta \hat{f}_n) \tag{16}$$

holds.

Remark. Here the Laplace operator Δ is taken in the distribution sense and because $\Delta \hat{f}_n$ is a measure (not necessarily non-negative), the total variation $\operatorname{Var}(\Delta \hat{f}_n)$ is well defined.

The following lemma solves this problem for $k = 1, 2$.

LEMMA 2. *For $k = 1, 2$ the equality* (16) *holds.*

Proof. For $k = 1$ the proof is very simple.

If $k = 2$ the simplex $Q = [P_1, P_2, P_3]$ with vertices $P_1 = P_1(x_1, y_1)$, $P_2 = P_2(x_2, y_2)$, $P_3 = P_3(x_3, y_3)$ is a triangle. Without loss of generality we can assume that

$$\det[P_1 - P_3, P_2 - P_3] := \det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} > 0.$$

The Stokes theorem yields

$$\begin{aligned} (\Delta \hat{f}_n)(\varphi) &= \int_Q \hat{f}_n \Delta\varphi \\ &= -\sum_{\Omega\in S_n(Q)} \left(\int_{\partial\Omega} \frac{\partial \hat{f}_n}{\partial x} m_1 \varphi\, d\sigma + \int_{\partial\Omega} \frac{\partial \hat{f}_n}{\partial y} m_2 \varphi\, d\sigma \right) \\ &= -\sum_{\Omega\in S_n(Q)} \sum_{r=1}^{3} \left(\int_{\Omega_r} \frac{\partial \hat{f}_n}{\partial x} m_1 \varphi\, d\sigma + \int_{\Omega_r} \frac{\partial \hat{f}_n}{\partial y} m_2 \varphi\, d\sigma \right) \end{aligned} \tag{17}$$

where $\varphi \in D(\operatorname{Int} Q)$, the space of test functions, (m_1, m_2) is the unit outward normal vector to the boundary $\partial\Omega$, and σ is the Lebesgue measure on $\partial\Omega$. First we consider the sum

$$\sum_{\Omega\in S_n(Q)} \int_{\Omega_1} \frac{\partial \hat{f}_n}{\partial x} m_1 \varphi\, d\sigma. \tag{18}$$

If $\Omega_1 \not\subset Q$ (for $\Omega \in S_n(Q)$) we can find a second triangle Ω' such that $\Omega_1 = \Omega'_1$.

Thus the sum (18) is equal to

$$\sum_{\substack{\Omega\in S_n(Q)\\ \Omega_1\neq Q_1}}\left(\int_{\Omega_1}\frac{\partial\hat{f}_n}{\partial x}m_1\,\varphi\,d\sigma+\int_{\Omega_1'}\frac{\partial\hat{f}_n}{\partial x}m_1'\,\varphi\,d\sigma\right). \tag{19}$$

Since $(m_1', m_2') = -(m_1, m_2)$ the expression (19) is equal to

$$\frac{1}{2}\sum_{\substack{\Omega\in S_n(Q)\\ \Omega_1\neq Q_1}}\left(\frac{\partial\hat{f}_n|_{\Omega}}{\partial x}-\frac{\partial\hat{f}_n|_{\Omega'_{m_1}}}{\partial x}\right)\int_{\Omega_1}\varphi\,d\sigma \tag{20}$$

where $\hat{f}_n|_{\Omega'}$, $\hat{f}_n|_{\Omega}$ are the restrictions of $\hat{f}_n$ to Ω', Ω, respectively. We can assume that the barycentric coordinates of the vertices of the triangles $\Omega = [U_1, U_2, U_3]$, $\Omega' = [U_1', U_2', U_3'] \in S_n(Q)$ ($\Omega_1 \not\subset Q_1$, $\Omega_1 = \Omega_1'$) are the following:

$$U_1=\left(\frac{i}{n},\frac{j}{n},\frac{k}{n}\right),\quad U_2=\left(\frac{i-1}{n},\frac{j+1}{n},\frac{k}{n}\right),\quad U_3=\left(\frac{i-1}{n},\frac{j}{n},\frac{k+1}{n}\right),$$

$$U_1'=\left(\frac{i-2}{n},\frac{j+1}{n},\frac{k+1}{n}\right),\quad U_2'=\left(\frac{i-1}{n},\frac{j+1}{n},\frac{k}{n}\right),\quad U_3'=\left(\frac{i-1}{n},\frac{j}{n},\frac{k+1}{n}\right),$$

for $i+j+k = n$, $j, k \geqslant 0$, $i \geqslant 2$.

Now by elementary calculations we get

$$\frac{\partial\hat{f}_n|_{\Omega}}{\partial x}=-n\frac{\det\begin{bmatrix} y_1-y_3 & y_2-y_3\\ f_{(i,j,k)}-f_{(i-1,j,k+1)} & f_{(i-1,j+1,k)}-f_{(i-1,j,k+1)}\end{bmatrix}}{\det[P_1-P_3\quad P_2-P_3]}, \tag{21}$$

$$\frac{\partial\hat{f}_n|_{\Omega'}}{\partial x}=n\frac{\det\begin{bmatrix} y_2-y_1 & y_2-y_3\\ f_{(i-2,j+1,k+1)}-f_{(i-1,j,k+1)} & f_{(i-1,j+1,k)}-f_{(i-1,j,k+1)}\end{bmatrix}}{\det[P_1-P_3\quad P_2-P_3]}, \tag{22}$$

$$m_1=\frac{y_3-y_2}{|Q_1|},\quad m_2=\frac{x_2-x_3}{|Q_1|}. \tag{23}$$

Thus the sum (20) is equal to

$$\sum_{\substack{i+j+k=n\\ j,k\geqslant 0\\ i\geqslant 2}}\frac{n(y_2-y_3)^2}{2|Q|\,|Q_1|}\left(f_{(i-1,j+1,k)}+f_{(i-1,j,k+1)}-f_{(i,j,k)}-f_{(i-2,j+1,k+1)}\right)\int_{\Omega_1}\varphi\,d\sigma \tag{24}$$

$$=-\sum_{\substack{i+j+k=n\\ j,k\geqslant 0\\ i\geqslant 2}}\frac{n(y_2-y_3)^2}{2|Q|\,|Q_1|}D_{13}f_{(i-2,j,k)}\int_{\Omega_1}\varphi\,d\sigma$$

$$=-\sum_{\substack{i+j+k=n-2\\ i,j,k\geqslant 0}}\frac{n(y_2-y_3)^2}{2|Q|\,|Q_1|}D_{13}f_{(i,j,k)}\int_{\Omega_1}\varphi\,d\sigma.$$

In a similar way we obtain

$$(25)\quad \sum_{\Omega\in S_n(Q)} \int_{\Omega_1} \frac{\partial \hat{f}_n}{\partial y} m_2 \varphi\, d\sigma = -\sum_{i+j+k=n-2} \frac{n(x_2-x_3)^2}{2|Q|\,|Q_1|} D_{13} f_{(i,j,k)} \int_{\Omega_1} \varphi\, d\sigma.$$

Hence

$$(26)\quad \sum_{\Omega\in S_n(Q)} \int_{\Omega_1} \left(\frac{\partial \hat{f}_n}{\partial x} m_1 + \frac{\partial \hat{f}_n}{\partial y} m_2\right) \varphi\, d\sigma = -\sum_{i+j+k=n-2} \frac{n|Q_1|}{2|Q|} D_{13} f_{(i,j,k)} \int_{\Omega_1} \varphi\, d\sigma.$$

Now it is easy to check that

$$(27)\quad \sum_{\Omega\in S_n(Q)} \int_{\Omega_2} \left(\frac{\partial \hat{f}_n}{\partial x} m_1 + \frac{\partial \hat{f}_n}{\partial y} m_2\right) \varphi\, d\sigma = -\sum_{i+j+k=n-2} \frac{n|Q_2|}{2|Q|} D_{12} f_{(i,j,k)} \int_{\Omega_2} \varphi\, d\sigma,$$

$$(28)\quad \sum_{\Omega\in S_n(Q)} \int_{\Omega_3} \left(\frac{\partial \hat{f}_n}{\partial x} m_1 + \frac{\partial \hat{f}_n}{\partial y} m_2\right) \varphi\, d\sigma = -\sum_{i+j+k=n-2} \frac{n|Q_3|}{2|Q|} D_{23} f_{(i,j,k)} \int_{\Omega_3} \varphi\, d\sigma.$$

By (26), (27), (28) the variation of f_n is equal to

$$\begin{aligned}
&\operatorname{Var}(\Delta \hat{f}_n) \\
&= \frac{1}{2|Q|} \sum_{i+j+k=n-2} \left(|Q_1|^2 |D_{13} f_{(i,j,k)}| + |Q_2|^2 |D_{12} f_{(i,j,k)}| + |Q_3|^2 |D_{23} f_{(i,j,k)}|\right) \\
&= V_1[\hat{f}_n; Q].
\end{aligned}$$

In case $k > 2$ equality (16) does not hold. More precisely the two variations $\operatorname{Var}(\Delta \hat{f}_n)$, $V_1[\hat{f}_n; Q]$ are incomparable.

EXAMPLE 1. Let

$$f_\beta = \begin{cases} 1 & \text{for } \beta = n\hat{e}_i, \\ 0 & \text{for } \beta \neq n\hat{e}_i, \end{cases} \qquad i = 1, 2, \ldots, k+1,\ |\beta| = n.$$

One can prove that the following equalities hold:

$$V_1[f_n; Q] = \frac{n!\,(k-1)!}{k|Q|(n+k-2)!} \sum_{j=1}^{k+1} |D_{j,j+1}(f_{\beta_{j+1}})| \cdot |Q_{j+1}|^2,$$

$$\operatorname{Var}(\Delta \hat{f}_n) = \frac{1}{k|Q|\,n^{k-2}} \sum_{j=1}^{k+1} |D_{j,j+1}(f_{\beta_{j+1}})| \cdot |Q_{j+1}|^2,$$

where $\beta_j = (n-2)\hat{e}_j$ for $j = 1, 2, \ldots, k+1$ and $D_{k+1,k+2} := D_{1,k+1}$, $\beta_{k+2} := \beta_1$, $Q_{k+2} := Q_1$.

Now, it is easy to see that

$$V_1[\hat{f}_n, Q] > \operatorname{Var}(\Delta \hat{f}_n) \quad \text{for } k = 3, 4, \ldots,\ n = 2, 3, \ldots$$

Because the Bézier net is convex (by Theorem 1) we obtain (by Theorem 3)

$$\mathrm{Var}(\Delta B^n) = V_1[\hat{f}_n; Q] > \mathrm{Var}(\Delta \hat{f}_n).$$

Hence, the multidimensional version of the variation diminishing property for the Laplacian does not hold for $k > 2$.

EXAMPLE 2. Let $n = p(k+1)+2$, $k, p \in N$, $p \geqslant 3$. Define

$$f_\beta = \begin{cases} 1 & \text{for } \beta = (p, p, \ldots, p, p+2) \in N^{k+1}, \\ 0 & \text{for } \beta \neq (p, p, \ldots, p, p+2) \in N^{k+1}, \end{cases} \quad |\beta| = n.$$

One can prove that the following equalities hold

$$V_1[\hat{f}_n; Q] = \frac{n!(k-1)!}{k|Q|(n+k-2)!} \sum_{|\beta|=n-2} \sum_{1 \leqslant i < j \leqslant k+1} |D_{ij}(f_\beta)| \Big\| \sum_{\alpha=i+1}^{j} n_\alpha |Q_\alpha| \Big\|^2,$$

$$\mathrm{Var}(\Delta \hat{f}_n) = \frac{(k-1)!}{k|Q|n^{k-2}} \sum_{|\beta|=n-2} \sum_{1 \leqslant i < j \leqslant k+1} |D_{ij}(f_\beta)| \Big\| \sum_{\alpha=i+1}^{j} n_\alpha |Q_\alpha| \Big\|^2.$$

Now, it is easy to see that

$$V_1[\hat{f}_n; Q] < \mathrm{Var}(\Delta \hat{f}_n) \quad \text{for } n = 2, 3, \ldots, k = 3, 4, \ldots$$

Fortunately, we can estimate the variation $\mathrm{Var}(\Delta \hat{f}_n)$ by $V_1[\hat{f}_n; Q]$.

LEMMA 3. *For the Bézier net $\hat{f}_n$ we have the inequalities*

$$(29) \qquad \frac{(n+k-2)!}{n^{k-2}n!} V_1[\hat{f}_n; Q] \geqslant \mathrm{Var}(\Delta \hat{f}_n) \geqslant \frac{1}{(k-1)!} V_1[\hat{f}_n; Q],$$

$k = 1, 2, \ldots, n = 2, 3, \ldots$

The tedious proof will be omitted.

Theorem 3 and the above lemma yield the "multidimensional version of the variation diminishing property for the Laplacian".

THEOREM 4. *For the Bernstein polynomial $B_n(f; P)$ over the k-dimensional simplex Q the following inequality holds*:

$$(30) \qquad \mathrm{Var}(\Delta B^n) \leqslant (k-1)! \, \mathrm{Var}(\Delta \hat{f}_n),$$

where for large n the constant $(k-1)!$ is best possible.

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