CONVEXITY AND VARIATION DIMINISHING PROPERTY
FOR BERNSTEIN POLYNOMIALS IN HIGHER DIMENSIONS

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1. Introduction

Let $f(x)$ be a real function defined in $Q := [0, 1]$. The Bernstein polynomial associated with $f$ is defined by

$$(1) \quad B^n(f; x) := \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i}, \quad x \in Q,$$

where $f_i := f(i/n)$ for $i = 0, 1, \ldots, n$.

Joining $((i-1)/n, f_{i-1})$ and $(i/n, f_i)$ by a line segment, for $i = 1, 2, \ldots, n$, we get a piecewise linear continuous function which is denoted by $\tilde{f}_n(x)$ and called the Bézier polygon of $B^n(f; x)$.

It is easy to show that:
(i) If $\tilde{f}_n(x)$ is convex in $Q$ then so is $B^n(f; x)$.
(ii) If $f(x)$ is convex in $Q$ then

$$(2) \quad B^n(f; x) \geq B^{n+1}(f; x), \quad n = 1, 2, \ldots, \text{ for } x \in Q.$$

(iii) The variation diminishing property holds, i.e.

$$(3) \quad V[B^n; Q] \leq V_1[\tilde{f}_n; Q],$$

with equality if and only if $\tilde{f}_n(x)$ is convex or concave. Here

$$(4) \quad V[B^n; Q] := \int_0^1 \left| \frac{d^2 B^n}{dx^2} \right| dx,$$

$$(5) \quad V_1[\tilde{f}_n; Q] := n \sum_{i=0}^{n-2} |\Delta^2 f_i|,$$

$\Delta f_i := f_{i+1} - f_i, \quad i = 0, 1, \ldots, n-1.$
G. Chang, P. Davis and J. Hoschek (see [1], [3]) extended these results to the Bernstein polynomials over triangles. In the present paper we attempt to extend these results to the Bernstein polynomials over a \( k \)-dimensional simplex \( Q \). In this case the definition of a variation like (5) called now the \textit{variation of the Bézier net} is more complicated, but in the case of \( k = 1, 2 \) it coincides with the variations as given in [2], [3]. We also give an answer to the problem whether the variation of the Bézier net is equal to \( \text{Var}(Df_a) \). Here the Laplace operator \( \Delta \) is taken in the distribution sense (see a remark in Section 3). These problems were suggested to me by Professor Z. Ciesielski.

2. Notation

Let us begin with some definitions. Let \( Q := \{P_1, P_2, \ldots, P_{k+1}\} \) be the \( k \)-dimensional simplex with vertices \( P_1, P_2, \ldots, P_{k+1} \), where \( P_i \in \mathbb{R}^k \) for \( i = 1, 2, \ldots, k+1 \). By \( Q_\alpha \) (\( 1 \leq \alpha \leq k+1 \)) we denote the \((k-1)\)-dimensional face of the simplex \( Q \) which does not contain the vertex \( P_\alpha \). By \( W := \{T_1, T_2, \ldots, T_{k+1}\} \) we denote the simplex with the vertices \( T_i = (0, \ldots, 0, 1, \ldots, 1) \), where \( i = 1, 2, \ldots, k+1 \), and by \( K_n(W) \) the subdivision of the simplex \( W \) such that a simplex \( \Omega \) belongs to \( K_n(W) \) if and only if

\[
\Omega = \Delta^n_{i_1i_2\ldots i_k} + v
\]

where

\[
\Delta^n_{i_1i_2\ldots i_k} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k; 0 \leq x_{i_1} \leq x_{i_2} \leq \ldots \leq x_{i_k} \leq 1/n\},
\]

\( i_1, i_2, \ldots, i_k \) is a permutation of \( \{1, 2, \ldots, k\} \), and the coordinates of the vector \( v = (v_1, v_2, \ldots, v_k) \in \mathbb{R}^k \) satisfy the following conditions:

(i) \( nv_i \in \mathbb{N} = \{0, 1, 2, \ldots\} \) for \( i = 1, 2, \ldots, k \),

(ii) \( 0 \leq v_1 \leq v_2 \leq \ldots \leq v_k < 1 \).

Now, let \( L: \mathbb{R}^k \rightarrow \mathbb{R}^k \) be an affine transformation such that

\[
L(T_i) = P_i \quad \text{for} \quad i = 1, 2, \ldots, k+1.
\]

Then \( L \) transforms the subdivision \( K_n(W) \) of the simplex \( W \) to some subdivision of the simplex \( Q \), which we denote by \( S_n(Q) \). It is known that each point \( P \in Q \) can be uniquely expressed as

\[
P = \sum_{i=1}^{k+1} u_i P_i
\]

with \( u_1 + u_2 + \ldots + u_{k+1} = 1 \), \( u_i \geq 0 \), \( 1 \leq i \leq k+1 \).

The numbers \( u_1, u_2, \ldots, u_{k+1} \) are called the \textit{barycentric coordinates} of \( P \) with respect to the simplex \( Q \). We identify the point \( P \in Q \) with its barycentric coordinates and write \( P = (u_1, u_2, \ldots, u_{k+1}) \). For \( n \in \mathbb{N} \) and a given
function $f : Q \to R$ we define

$$F_n := \left\{ f_\beta : f_\beta := f \left( \frac{\beta}{n} \right), \beta \in \mathbb{N}^{k+1}, |\beta| = n \right\}$$

where

$$\beta = (\beta_1, \beta_2, \ldots, \beta_{k+1}) \quad \text{and} \quad |\beta| := \sum_{i=1}^{k+1} \beta_i.$$

We also define

$$(6) \quad B^n(f ; P) = \sum_{|\beta| = n} f_\beta J^n_\beta(P) \quad \text{for } P \in Q$$

where

$$(7) \quad J^n_\beta(P) := \frac{n!}{\beta_1! \beta_2! \ldots \beta_{k+1}!} u_1^{\beta_1} u_2^{\beta_2} \ldots u_{k+1}^{\beta_{k+1}}$$

and $P = (u_1, u_2, \ldots, u_{k+1})$.

The polynomials $J^n_\beta(P)$ are called the basic Bernstein polynomials. $B^n(f ; P)$ is called the $n$-th Bernstein polynomial over the simplex $Q$. Setting

$$\tilde{f}_\beta := (P, f_\beta) := \left( \frac{\beta_1}{n}, \frac{\beta_2}{n}, \ldots, \frac{\beta_{k+1}}{n}, f_\beta \right),$$

we get a point on the surface associated with the function $f(P)$. The points $\tilde{f}_\beta$ with $|\beta| = n$ are called the Bézier points of $B^n(f ; P)$. For any simplex $\Omega \in S_n(Q)$ we have $k+1$ Bézier points. Joining them by a $k$-dimensional plane we obtain a continuous piecewise linear surface $\tilde{f}_n(P)$, which is called the Bézier net of $B^n(f ; P)$ over the simplex $Q$.

Let $E_i$ ($1 \leq i \leq k+1$) be the partial shift operator defined by

$$E_i f_\beta := f_{\beta + \hat{e}_i}$$

where $|\beta| = n - 1$, $\hat{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{k+1}$.

For convenience we introduce

$$(8) \quad D_{ij}(f_\beta) := -(E_i - E_{i+1})(E_j - E_{j+1}) f_\beta$$

where $|\beta| = n - 2$, $1 \leq i < j \leq k+1$ and $E_{k+2} := E_1$.

3. Results

The first theorem generalizes a result of G. Chang and P. Davis [1].

**Theorem 1.** The convexity of the Bézier net $\tilde{f}_n(P)$ over $Q$ is equivalent to the following inequalities:

$$(9) \quad D_{ij}(f_\beta) \geq 0$$

for $1 \leq i < j \leq k+1$, $|\beta| = n - 2$. 
The proof is standard and will be omitted.

The next theorem is also a multidimensional analogue of a theorem of G. Chang and P. Davis.

**Theorem 2.** (i) If the Bézier net \( \hat{f}_n(P) \) over \( Q \) is convex then so is the Bernstein polynomial \( B^n(f; P) \).

(ii) If \( f(P) \) is convex in \( Q \) then we have

\[
B^n(f; P) \geq B^{n+1}(f; P) \quad \text{for } P \in Q, \ n = 1, 2, \ldots
\]

**Proof.** (i) The proof is based on the well-known convexity test for any function in \( C^2(Q) \) and the following

**Lemma 1.** For the Bernstein polynomial \( B^n(f; P) \) we have the equality

\[
\frac{\partial^2 B^n(f; P)}{\partial x_r \partial x_{r'}} = \frac{n(n-1)}{k^2 |Q|} \sum_{|\beta| = n-2} \sum_{1 \leq i < j \leq k+1} \sum_{a=1}^{j} n_a |Q_a|, e_r \times (\sum_{a=1}^{j} n_a |Q_a|, e_r) D_{ij}(f_\beta) J_{\beta}^r(P)
\]

for \( r, r' = 1, 2, \ldots, k \), where \( P \in Q, \ e_r = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^k, \ r = 1, 2, \ldots, k, \ |Q| := \text{vol}_k Q, \ |Q_a| := \text{vol}_{k-1} (Q_a), \ (\cdot, \cdot) \) is the scalar product in \( \mathbb{R}^k \), and \( n_a \) is the unit outward normal vector to \( Q_a \).

The tedious proof will be omitted.

(ii) G. Chang and P. Davis ([1]) proved the statement for \( k = 2 \). For \( k > 2 \) the proof is similar.

Now, we define a *variation of the Bernstein polynomial* \( B^n(f; P) \) by

\[
V[B^n; Q] := \int_{Q} |\Delta B^n|
\]

where for simplicity \( B^n \) stands for \( B^n(f; P) \) and \( \Delta \) is the Laplacian. The variation \( V_1[\hat{f}_n; Q] \) of the Bézier net \( \hat{f}_n(P) \) is defined by

\[
V_1[\hat{f}_n; Q] := \frac{n!(k-1)!}{k(n+k-2)! |Q|} \sum_{|\beta| = n-2} \sum_{1 \leq i < j \leq k+1} |D_{ij}(f_\beta)| \sum_{a=1}^{j} n_a |Q_a|^2
\]

where \(|\cdot|^2 = (\cdot, \cdot)\).

For \( k = 2 \) the variation \( V_1[\hat{f}_n; Q] \) of the Bézier net coincides with the variation introduced by T. Goodman [3]. Our next theorem generalizes the result obtained by G. Chang and J. Hoschek [2].

**Theorem 3.** For the Bernstein polynomial over the simplex \( Q \) the variation diminishing property holds. More precisely, we have

\[
V[B^n; Q] \leq V_1[\hat{f}_n; Q]
\]

with equality if and only if \( \hat{f}_n(P) \) is either convex or concave.
For $k = 1, 2$ the theorem was proved by G. Chang and J. Hoschek [2]. For $k > 2$ the proof is similar.

Since

$$V[B^*; Q] = \text{Var}(\Delta B^*)$$

where $\text{Var}(\Delta B^*)$ is the total variation of the measure

$$\nu(A) := \int_A \Delta B^*,$$

the question arises whether the equality

$$V_1[\hat{f}_n; Q] = \text{Var}(\Delta \hat{f}_n)$$

holds.

**Remark.** Here the Laplace operator $\Delta$ is taken in the distribution sense and because $\Delta \hat{f}_n$ is a measure (not necessarily non-negative), the total variation $\text{Var}(\Delta \hat{f}_n)$ is well defined.

The following lemma solves this problem for $k = 1, 2$.

**Lemma 2.** For $k = 1, 2$ the equality (16) holds.

**Proof.** For $k = 1$ the proof is very simple.

If $k = 2$ the simplex $Q = [P_1, P_2, P_3]$ with vertices $P_1 = P_1(x_1, y_1), P_2 = P_2(x_2, y_2), P_3 = P_3(x_3, y_3)$ is a triangle. Without loss of generality we can assume that

$$\det[P_1 - P_3, P_2 - P_3] := \det\begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} > 0.$$

The Stokes theorem yields

$$V[\Delta \hat{f}_n](\phi) = \int_Q \Delta \phi$$

$$= -\sum_{\alpha \in S_n(Q)} \left( \int_{\partial Q} \frac{\partial \phi}{\partial x} m_1 \phi \, d\sigma + \int_{\partial Q} \frac{\partial \phi}{\partial y} m_2 \phi \, d\sigma \right)$$

$$= -\sum_{\alpha \in S_n(Q)} \sum_{r=1}^3 \left( \int_{\alpha_r} \frac{\partial \phi}{\partial x} m_1 \phi \, d\sigma + \int_{\alpha_r} \frac{\partial \phi}{\partial y} m_2 \phi \, d\sigma \right)$$

where $\phi \in D(\text{Int} Q)$, the space of test functions, $(m_1, m_2)$ is the unit outward normal vector to the boundary $\partial Q$, and $\sigma$ is the Lebesgue measure on $\partial Q$. First we consider the sum

$$\sum_{\alpha \in S_n(Q)} \int_{\alpha_1} \frac{\partial \phi}{\partial x} m_1 \phi \, d\sigma.$$

If $\Omega_1 \neq Q$ (for $\Omega \in S_n(Q)$) we can find a second triangle $\Omega'$ such that $\Omega_1 = \Omega_1$. 

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Thus the sum (18) is equal to

\[ \sum_{\Omega \in S_n(Q)} \left( \int_{\Omega_1} \frac{\partial f}{\partial x} m_1 \varphi \, d\sigma + \int_{\Omega_1} \frac{\partial f}{\partial x} m_2' \varphi \, d\sigma \right). \]  

Since \((m_1', m_2') = -(m_1, m_2)\) the expression (19) is equal to

\[ \frac{1}{2} \sum_{\Omega \in S_n(Q)} \left( \frac{\partial f_{\text{rel}}}{\partial x} - \frac{\partial f_{\text{rel}}'}{\partial x} \right) \int_{\Omega_1} \varphi \, d\sigma \]

where \(f_{\text{rel}}, f_{\text{rel}}'\) are the restrictions of \(f\) to \(\Omega', \Omega\), respectively. We can assume that the barycentric coordinates of the vertices of the triangles \(\Omega = [U_1, U_2, U_3], \, \Omega' = [U'_1, U'_2, U'_3] \in S_n(Q) (\Omega_1 \not\in \Omega, \Omega_1 = \Omega_1)\) are the following:

\[
U_1 = \left( \frac{i}{n}, \frac{j}{n}, \frac{k}{n} \right), \quad U_2 = \left( \frac{i-1}{n}, \frac{j+1}{n}, \frac{k}{n} \right), \quad U_3 = \left( \frac{i-1}{n}, \frac{j}{n}, \frac{k+1}{n} \right),
\]

\[
U'_1 = \left( \frac{i-2}{n}, \frac{j+1}{n}, \frac{k+1}{n} \right), \quad U'_2 = \left( \frac{i-1}{n}, \frac{j+1}{n}, \frac{k}{n} \right), \quad U'_3 = \left( \frac{i-1}{n}, \frac{j}{n}, \frac{k+1}{n} \right)
\]

for \(i + j + k = n, \, j, k \geq 0, \, i \geq 2\).

Now by elementary calculations we get

\[ \frac{\partial f_{\text{rel}}}{\partial x} = -n \frac{\det \left[ \begin{array}{cc} y_1 - y_3 & y_2 - y_3 \\ f_{(i-1,j,k+1)} - f_{(i,j,k+1)} & f_{(i,j,k+1)} - f_{(i-1,j,k+1)} \end{array} \right]}{\det \left[ \begin{array}{cc} P_1 & P_3 \\ P_2 & P_3 \end{array} \right]}, \]

\[ \frac{\partial f_{\text{rel}}'}{\partial x} = n \frac{\det \left[ \begin{array}{cc} y_2 - y_1 & y_2 - y_3 \\ f_{(i-2,j+1,k+1)} - f_{(i,j,k+1)} & f_{(i-1,j,k+1)} - f_{(i-1,j,k+1)} \end{array} \right]}{\det \left[ \begin{array}{cc} P_1 & P_3 \\ P_2 & P_3 \end{array} \right]}, \]

\[ m_1 = \frac{y_3 - y_2}{|\Omega_1|}, \quad m_2 = \frac{x_2 - x_3}{|\Omega_1|}.
\]

Thus the sum (20) is equal to

\[ \sum_{\Omega_1} \frac{n(y_2 - y_3)^2}{2|\Omega||\Omega_1|} (f_{(i-1,j+1,k)} + f_{(i-1,j,k+1)} - f_{(i,j,k+1)} - f_{(i-2,j+1,k+1)}) \int_{\Omega_1} \varphi \, d\sigma \]

\[ = - \sum_{\Omega_1} \frac{n(y_2 - y_3)^2}{2|\Omega||\Omega_1|} D_{13} f_{(i-2,j,k)} \int_{\Omega_1} \varphi \, d\sigma \]

\[ = - \sum_{\Omega_1} \frac{n(y_2 - y_3)^2}{2|\Omega||\Omega_1|} D_{13} f_{(i,j,k)} \int_{\Omega_1} \varphi \, d\sigma. \]
In a similar way we obtain

\[ (25) \sum_{\alpha \in \mathcal{S}_d(Q)} \int_{\alpha_1} \left( \frac{\partial^2}{\partial x} m_1 + \frac{\partial^2}{\partial y} m_2 \right) \varphi \, d\sigma = - \sum_{i+j+k=n-2} \frac{n(x_2-x_3)^2}{2|Q| |Q_1|} D_{13} f_{(i,j,k)} \int_{\alpha_1} \varphi \, d\sigma. \]

Hence

\[ (26) \sum_{\alpha \in \mathcal{S}_d(Q)} \int_{\alpha_1} \left( \frac{\partial^2}{\partial x} m_1 + \frac{\partial^2}{\partial y} m_2 \right) \varphi \, d\sigma = - \sum_{i+j+k=n-2} \frac{n|Q_1|}{2|Q|} D_{13} f_{(i,j,k)} \int_{\alpha_1} \varphi \, d\sigma. \]

Now it is easy to check that

\[ (27) \sum_{\alpha \in \mathcal{S}_d(Q)} \int_{\alpha_2} \left( \frac{\partial^2}{\partial x} m_1 + \frac{\partial^2}{\partial y} m_2 \right) \varphi \, d\sigma = - \sum_{i+j+k=n-2} \frac{n|Q_2|}{2|Q|} D_{12} f_{(i,j,k)} \int_{\alpha_2} \varphi \, d\sigma, \]

\[ (28) \sum_{\alpha \in \mathcal{S}_d(Q)} \int_{\alpha_3} \left( \frac{\partial^2}{\partial x} m_1 + \frac{\partial^2}{\partial y} m_2 \right) \varphi \, d\sigma = - \sum_{i+j+k=n-2} \frac{n|Q_3|}{2|Q|} D_{23} f_{(i,j,k)} \int_{\alpha_3} \varphi \, d\sigma. \]

By (26), (27), (28) the variation of \( f_n \) is equal to

\[ \text{Var}(\Delta \hat{f}_n) \]

\[ = \frac{1}{2|Q|} \sum_{i+j+k=n-2} (|Q_1|^2 |D_{13} f_{(i,j,k)}| + |Q_2|^2 |D_{12} f_{(i,j,k)}| + |Q_3|^2 |D_{23} f_{(i,j,k)}|) \]

\[ = V_1 [\hat{f}_n; Q]. \]

In case \( k > 2 \) equality (16) does not hold. More precisely the two variations \( \text{Var}(\Delta \hat{f}_n), V_1 [\hat{f}_n; Q] \) are incomparable.

**Example 1.** Let

\[ f_\beta = \begin{cases} 1 & \text{for } \beta = n \hat{e}_i, \\
0 & \text{for } \beta \neq n \hat{e}_i, \end{cases} \quad i = 1, 2, \ldots, k+1, |\beta| = n. \]

One can prove that the following equalities hold:

\[ V_1 [f_\beta; Q] = -\frac{n! (k-1)!}{k |Q| (n+k-2)!} \sum_{j=1}^{k+1} |D_{j,j+1} (f_{\beta_{j+1}})| \cdot |Q_{j+1}|^2, \]

\[ \text{Var}(\Delta \hat{f}_n) = \frac{1}{k |Q| n^{k-2}} \sum_{j=1}^{k+1} |D_{j,j+1} (f_{\beta_{j+1}})| \cdot |Q_{j+1}|^2, \]

where \( \beta_j = (n-2) \hat{e}_j \) for \( j = 1, 2, \ldots, k+1 \) and \( D_{k+1,k+2} := D_{1,k+1}, \beta_{k+2} := \beta_1, Q_{k+2} := Q_1. \)

Now, it is easy to see that

\[ V_1 [\hat{f}_n, Q] > \text{Var}(\Delta \hat{f}_n) \quad \text{for } k = 3, 4, \ldots, n = 2, 3, \ldots \]
Because the Bézier net is convex (by Theorem 1) we obtain (by Theorem 3)
\[ \text{Var}(\Delta B^n) = V_1[\hat{f}_n; Q] > \text{Var}(\Delta \hat{f}_n). \]

Hence, the multidimensional version of the variation diminishing property for the Laplacian does not hold for \( k > 2 \).

**Example 2.** Let \( n = p(k + 1) + 2, k, p \in \mathbb{N}, p \geq 3 \). Define
\[ f_\beta = \begin{cases} 1 & \text{for } \beta = (p, p, \ldots, p, p + 2) \in \mathbb{N}^{k+1}, |eta| = n. \\ 0 & \text{for } \beta \neq (p, p, \ldots, p, p + 2) \in \mathbb{N}^{k+1}, \end{cases} \]

One can prove that the following equalities hold
\[ V_1[\hat{f}_n; Q] = \frac{n!(k-1)!}{k|Q|(n+k-2)!} \sum_{|\beta|=n-2} \sum_{1 \leq i < j \leq k+1} |D_{ij}(f_\beta)| \parallel \sum_{a=i+1}^j n_a |Q_a| \parallel^2, \]
\[ \text{Var}(\Delta \hat{f}_n) = \frac{(k-1)!}{k|Q|n^{k-2}} \sum_{|\beta|=n-2} \sum_{1 \leq i < j \leq k+1} |D_{ij}(f_\beta)| \parallel \sum_{a=i+1}^j n_a |Q_a| \parallel^2. \]

Now, it is easy to see that
\[ V_1[\hat{f}_n; Q] < \text{Var}(\Delta \hat{f}_n) \quad \text{for } n = 2, 3, \ldots, k = 3, 4, \ldots \]

Fortunately, we can estimate the variation \( \text{Var}(\Delta \hat{f}_n) \) by \( V_1[\hat{f}_n; Q] \).

**Lemma 3.** For the Bézier net \( \hat{f}_n \) we have the inequalities
\[ \frac{(n+k-2)!}{n^{k-2}n!} V_1[\hat{f}_n; Q] \geq \text{Var}(\Delta \hat{f}_n) \geq \frac{1}{(k-1)!} V_1[\hat{f}_n; Q], \quad k = 1, 2, \ldots, n = 2, 3, \ldots \]

The tedious proof will be omitted.

Theorem 3 and the above lemma yield the "multidimensional version of the variation diminishing property for the Laplacian".

**Theorem 4.** For the Bernstein polynomial \( B_n(f; P) \) over the \( k \)-dimensional simplex \( Q \) the following inequality holds:
\[ \text{Var}(\Delta B^n) \leq (k-1)! \text{Var}(\Delta \hat{f}_n), \]
where for large \( n \) the constant \( (k-1)! \) is best possible.

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