ANNALES
POLONICI MATHEMATICI
XL (1981)

On tempered integrals and derivatives of non-negative orders

by Krystyna Skórnik (Ruda Śląska)

Abstract. This paper deals with a class of distributions, having the property that the operation inverse to differentiation is unique. The tempered integrals and the tempered derivatives of these distributions belong to the same class. It is shown that if the operation D of tempered differentiation is uniquely invertible, then it can be extended to real non-negative orders such that the equality $D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f$ holds for all $\alpha, \beta \geq 0$. Some inequalities for tempered integrals are given.

In Analysis, the concept of derivative is susceptible of various interpretations. E.g., one can consider continuous derivatives, almost everywhere derivatives, distributional derivatives, tempered derivatives, and also algebraic derivatives. The inverse operation to any of those derivatives is not unique, in general. However, it can become unique, if we restrict ourselves to a subclass of derivable elements. E.g., the continuous derivative is uniquely invertible in the class of functions $e^{\alpha x}$ $(\alpha \in R)$. An interesting example is the class of hereditarily periodic functions and also the class of hereditarily periodic distributions (see [4]). In those classes the operation inverse to derivation exists and is unique. Similar classes can be formed in the case of tempered derivative (see Section 1). Such are e.g.: the class of smooth functions having a zero of infinite order at the origin; the class of distributions (not necessarily tempered), having value 0 at the origin; the class of tempered distributions having value 0 at the origin.

In this note we consider some other distribution classes having property that the inverse operation is unique. The tempered integral and the tempered derivative of a distribution in such a class belongs to the same class.

In the paper we also show that if the tempered derivative D is uniquely invertible in a class of distributions, then it can be extended to derivatives of any non-negative real order so that, for all real k and r, the following equation holds:

$$D^k D^r f = D^{k+r} f.$$

The point at which the distributions under consideration are supposed to have value 0 need not be the origin. By fixing this point beyond the origin we can easily obtain other classes with similar properties.

The main results will be preceded by a few remarks on tempered integrals and tempered derivatives.

The Euclidean space is denoted by R. We use the following notation: N = set of all positive integers; P = set of all non-negative integers.

1. Tempered integrals and tempered derivatives. By the *n-th tempered derivative of a distribution f*, defined in R, we mean the distribution

$$D^n f(x) = e^{-x^2/4}(e^{x^2/4} f(x))^{(n)} \quad (n \in N). \tag{1}$$

This definition can be found in the book [1].

It is easy to check that

$$D^0 f = f, \quad D^n D^m f = D^{n+m} f \quad (n, m \in N).$$

The tempered derivative is a linear operation, i.e., such that

$$D^n(f+g) = D^n f + D^n g, \quad D^n(cf) = cD^n f$$

for any number c and $n \in N$.

According to [1], the *n-th tempered integral* of a continuous function f in R is defined by the formulae

$$Sf(x) = e^{-x^2/4} \int_0^x f(t) e^{t^2/4} dt, \tag{2}$$

$$S^n f = S(S^{n-1} f) \quad (n \in N). \tag{3}$$

Additionally it is assumed that $S^0 f = f$.

Tempered integral S^n is a linear operation.

It is easy to check that

$$D^n S^n f = f \quad (n \in N). \tag{4}$$

However, the formula $S^n D^n f = f$ does not hold, even for $n = 1$.

It is interesting that for tempered integrals the following property holds:

For any given $r \geqslant 0$, $n \in N$ there are numbers $\gamma = \gamma(r, n)$ and $\gamma_1 = \gamma_1(r, n)$ such that

$$\left|S^n |x|^r\right| \leqslant \gamma \cdot |x|^{r-n}, \tag{5}$$

$$|S^n \hat{x}^r| \leqslant \gamma_1 \cdot \hat{x}^{r-n}, \tag{6}$$

where $\hat{x} = 1 + |x|$.

In fact. We introduce the auxiliary functions

$$\gamma(r, x) = e^{-x^2/4} x^{1-r} \int_0^x e^{t^2/4} t^r dt \quad (x \geqslant 0)$$

and we assume that

$$\gamma(r) = \sup_{0 < x < \infty} \gamma(r, x).$$

Using l'Hôspital's rule, it is easy to show that $\gamma(r, x)$ tends to 2, as $x \to \infty$, independently of the value of the real number r. This implies that the function $\gamma(r)$ takes a finite positive value for every real r.

We now prove that for any given $r \in R$ there exists a number $\gamma(r)$ such that

$$|S|x|^r| \leqslant \gamma(r) \cdot |x|^{r-1}. \tag{7}$$

It is easy to verify that for $x \geqslant 0$

$$S|x|^r = |x|^{r-1}\gamma(r, x).$$

This implies inequality (7) for $x \geqslant 0$. Since both sides of (7) are even functions of x, (7) holds for every $x \in R$.

From (7), by induction, inequality (5) follows.

The proof of inequality (6) is similar to the proof of (5).

For the tempered integral, the following Cauchy formula holds:

$$S^n f(x) = e^{-x^2/4} \int_0^x \frac{(x-t)^{n-1}}{\Gamma(n)} f(t) e^{t^2/4} dt \quad (n \in N). \tag{8}$$

The Cauchy formula can be extended for any positive real number α.

By the *α-th tempered integral* ($\alpha > 0$, $\alpha \in R$) of a continuous function f in R, we mean

$$S^\alpha f(x) = e^{-x^2/4} \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t) e^{t^2/4} dt \quad (\alpha > 0). \tag{9}$$

S^α is clearly a linear operation, i.e., $S^\alpha(f+g) = S^\alpha f + S^\alpha g$ and $S^\alpha(cf) = cS^\alpha f$ for any number c.

THEOREM 1. *If f is a continuous function, then for any $\alpha \geqslant 0$, $\beta \geqslant 0$ ($\alpha, \beta \in R$) the following equality holds*:

$$S^\alpha S^\beta f = S^{\alpha+\beta} f.$$

Proof. Using formula (9), the Dirichlet Theorem for double integral, and the substitution $\sigma = \tau - t$, we obtain

$$\begin{aligned} S^\alpha(S^\beta f(x)) &= e^{-x^2/4} \int_0^x \frac{(x-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \int_0^\tau \frac{(\tau-t)^{\beta-1}}{\Gamma(\beta)} e^{t^2/4} f(t)\, dt \\ &= e^{-x^2/4} \int_0^x e^{t^2/4} f(t)\, dt \int_t^x \frac{(x-\tau)^{\alpha-1}(\tau-t)^{\beta-1}}{\Gamma(\alpha)\cdot\Gamma(\beta)} d\tau \\ &= e^{-x^2/4} \int_0^x e^{t^2/4} f(t)\, dt \int_0^{x-t} \frac{(x-\sigma-t)^{\alpha-1}\sigma^{\beta-1}}{\Gamma(\alpha)\cdot\Gamma(\beta)} d\sigma. \end{aligned}$$

By applying the substitution $z = \sigma/(x-t)$ and then the equation

$$\frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 (1-z)^{\alpha-1} z^{\beta-1}\,dz,$$

the last iterated integral becomes

$$\begin{aligned}e^{-x^2/4}\int_0^x e^{t^2/4}f(t)\,dt\int_0^1 &\frac{(1-z)^{\alpha-1}(x-t)^{\alpha-1}z^{\beta-1}(x-t)^{\beta}}{\Gamma(\alpha)\cdot\Gamma(\beta)}\,dz\\ &= e^{-x^2/4}\int_0^x (x-t)^{\alpha+\beta-1}e^{t^2/4}f(t)\,dt\int_0^1\frac{(1-z)^{\alpha-1}z^{\beta-1}}{\Gamma(\alpha)\cdot\Gamma(\beta)}\,dz\\ &= e^{-x^2/4}\int_0^x\frac{(x-t)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}e^{t^2/4}f(t)\,dt = S^{\alpha+\beta}f(x).\end{aligned}$$

In the sequel we shall use the symbol $E(\alpha)$ for the greatest integer less or equal to α, i.e., $E(\alpha) = \max_{i\leqslant\alpha} i$ $(i\in N)$.

It is interesting that for the tempered integral of order α $(\alpha \geqslant 0,\ \alpha\in R)$ the following property holds:

For any given $r \geqslant 0$, $\alpha \geqslant 0$ $(r,\alpha\in R)$ there are numbers $\bar{\gamma} = \bar{\gamma}(r,\alpha+1)$ and $\bar{\gamma}_1 = \bar{\gamma}_1(r,\alpha+1)$ such that

$$(10)\qquad \left|S^{\alpha+1}|x|^r\right| \leqslant \bar{\gamma}(r,\alpha+1)\cdot|x|^{r-E(\alpha+1)},$$

$$(11)\qquad |S^{\alpha+1}\hat{x}^r| \leqslant \bar{\gamma}_1\cdot\hat{x}^{r-E(\alpha+1)},$$

where $\hat{x} = 1+|x|$ [1].

In fact. Let $n < \alpha < n+1$ $(n\in N)$. For $x \geqslant 1$ we have

$$\begin{aligned}\left|S^{\alpha+1}|x|^r\right| &= e^{-x^2/4}\int_0^x\frac{(x-t)^{\alpha}}{\Gamma(\alpha+1)}t^r e^{t^2/4}\,dt\\ &\leqslant e^{-x^2/4}\int_0^{x-1}\frac{(x-t)^{n+1}}{\Gamma(\alpha+1)}t^r e^{t^2/4}\,dt + e^{-x^2/4}\int_{x-1}^x\frac{(x-t)^n}{\Gamma(\alpha+1)}t^r e^{t^2/4}\,dt\\ &\leqslant \frac{\Gamma(n+2)}{\Gamma(\alpha+1)}\left|S^{n+2}|x|^r\right| + \frac{\Gamma(n+1)}{\Gamma(\alpha+1)}\left|S^{n+1}|x|^r\right|.\end{aligned}$$

---

[1] **Added in proof.** The following stronger lemma is actually true: *For any given $r > -1$, $\alpha \geqslant 0$ $(r,\alpha\in R)$ there are numbers $\gamma = \gamma(r,\alpha)$, $\gamma_1 = \gamma_1(r,\alpha)$ such that*

$$\left|S^{\alpha}|x|^r\right| \leqslant \gamma(r,\alpha)|x|^{r-\alpha},\qquad |S^{\alpha}\hat{x}^r| \leqslant \gamma_1(r,\alpha)\hat{x}^{r-\alpha},$$

where $\hat{x} = 1+|x|$ (see [5]). The proof is similar.

For $x \leqslant -1$ we obtain

$$|S^{\alpha+1}|x|^r| = e^{-x^2/4} \int_x^0 \frac{(t-x)^\alpha}{\Gamma(\alpha+1)} (-t)^r e^{t^2/4}\, dt$$

$$\leqslant e^{-x^2/4} \int_x^{x+1} \frac{(t-x)^n}{\Gamma(\alpha+1)} (-t)^r e^{t^2/4}\, dt + e^{-x^2/4} \int_{x+1}^0 \frac{(t-x)^{n+1}}{\Gamma(\alpha+1)} (-t)^r e^{t^2/4}\, dt$$

$$\leqslant \frac{\Gamma(n+1)}{\Gamma(\alpha+1)} \cdot |S^{n+1}|x|^r| + \frac{\Gamma(n+2)}{\Gamma(\alpha+1)} \cdot |S^{n+2}|x|^r|.$$

Let $0 < \alpha < 1$. If $x \geqslant 1$, we have

$$|S^{\alpha+1}|x|^r| \leqslant e^{-x^2/4} \int_0^{x-1} \frac{(x-t)^0}{\Gamma(\alpha+1)} t^r e^{t^2/4}\, dt + e^{-x^2/4} \int_{x-1}^{x} \frac{(x-t)^1}{\Gamma(\alpha+1)} t^r e^{t^2/4}\, dt$$

$$\leqslant \frac{\Gamma(1)}{\Gamma(\alpha+1)} |S|x|^r| + \frac{\Gamma(2)}{\Gamma(\alpha+1)} |S^2|x|^r|.$$

Similarly we can prove that the above inequality holds for $x \leqslant -1$. Therefore, for every $|x| \geqslant 1$, $\alpha \geqslant 0$ and $n \leqslant \alpha \leqslant n+1$ ($n = 0, 1, 2, \ldots$) we have

$$|S^{\alpha+1}|x|^r| \leqslant \frac{\Gamma(n+1)}{\Gamma(\alpha+1)} |S^{n+1}|x|^r| + \frac{\Gamma(n+2)}{\Gamma(\alpha+1)} |S^{n+2}|x|^r|.$$

Hence, by (5), it follows that there exist numbers $\gamma = \gamma(r, n+1)$, $\gamma_0 = \gamma_0(r, n+2)$, such that

$$|S^{\alpha+1}|x|^r| \leqslant \frac{\Gamma(n+1)}{\Gamma(\alpha+1)} \gamma(r, n+1)|x|^{r-n-1} + \frac{\Gamma(n+2)}{\Gamma(\alpha+1)} \gamma_0(r, n+2)|x|^{r-n-2}$$

$$\leqslant \frac{\Gamma(n+2)}{\Gamma(\alpha+1)} \max(\gamma, \gamma_0)|x|^{r-n-1}\left(1 + \frac{1}{|x|}\right) \leqslant \bar{\gamma}(r, \alpha+1)|x|^{r-E(\alpha+1)}.$$

It is easy to check that for $|x| < 1$ and $n \leqslant \alpha \leqslant n+1$ ($n = 0, 1, 2, \ldots$) the following inequality is true:

$$|S^{\alpha+1}|x|^r| \leqslant \frac{\Gamma(n+1)}{\Gamma(\alpha+1)} |S^{n+1}|x|^r| \leqslant \bar{\gamma}(r, \alpha+1)|x|^{r-E(\alpha+1)}.$$

Consequently, inequality (10) holds for all $x \in R$.

As in the proof of (10), we can show that the undernamed inequality holds:

$$|S^{\alpha+1}\hat{x}^r| \leqslant \frac{\Gamma(n+1)}{\Gamma(\alpha+1)} |S^{n+1}\hat{x}^r| + \frac{\Gamma(n+2)}{\Gamma(\alpha+1)} |S^{n+2}\hat{x}^r|$$

for $x \in R$ and $n \leqslant \alpha \leqslant n+1$,

$n = 0, 1, 2, \ldots$ Hence, by (6), it follows that there are $\gamma_1 = \gamma_1(r, n+1)$, $\gamma_{10} = \gamma_{10}(r, n+2)$ such that

$$|S^{\alpha+1}\hat{x}^r| \leqslant \frac{\Gamma(n+1)}{\Gamma(\alpha+1)} \cdot \gamma_1(r, n+1)\hat{x}^{r-n-1} + \frac{\Gamma(n+2)}{\Gamma(\alpha+1)} \cdot \gamma_{10}(r, n+2)\hat{x}^{r-n-2}$$
$$\leqslant \frac{\Gamma(n+2)}{\Gamma(\alpha+1)} \max(\gamma_1, \gamma_{10}) \cdot \hat{x}^{r-n-1}(1+\hat{x}^{-1})$$
$$\leqslant \frac{2\Gamma(n+2)}{\Gamma(\alpha+1)} \max(\gamma_1, \gamma_{10})\hat{x}^{r-n-1}$$
$$= \bar{\gamma}_1(r, \alpha+1)\hat{x}^{r-E(\alpha+1)}.$$

Thus the following inequality holds for $\alpha \geqslant 0$:

$$|S^{\alpha+1}\hat{x}^r| \leqslant \bar{\gamma}_1 \cdot \hat{x}^{r-E(\alpha+1)}.$$

2.1. Functions rapidly decreasing at zero. We say that a *function f*, defined in R, is *rapidly decreasing at* 0, iff for every $r > 0$ ($r \in R$) there is $M > 0$ such that $|f(x)| \leqslant M \cdot |x|^r$ for $|x| \leqslant 1$.

Remark 1. The set of all functions rapidly decreasing at 0 is a linear space.

THEOREM 2. *If f is a continuous function, rapidly decreasing at* 0, *then every distributional derivative of f has value* 0 *at the point* 0.

This theorem follows from Łojasiewicz' theorem (see [1]–[3]) which says: A distribution $f(x)$ has value c at a point x_0 if and only if there exist an integer $k \geqslant 0$ and a continuous function $F(x)$ such that $F^{(k)}(x) = f(x)$ and

$$\lim_{x \to x_0} \frac{F(x)}{(x-x_0)^k} = \frac{c}{k!}.$$

THEOREM 3. *If f is a continuous function, rapidly decreasing at* 0, *then for any $\alpha \in R$, $\alpha \geqslant 0$, the tempered integral $S^\alpha f$ is also continuous and rapidly decreasing at* 0.

Proof. If f is a continuous function, then the tempered integral $S^\alpha f$ is also continuous. Since f is rapidly decreasing at 0, it follows that for every $r > 0$ ($r \in R$) there is $M > 0$ such that

$$(12) \qquad |f(x)| \leqslant M|x|^r \quad \text{for } |x| \leqslant 1.$$

Applying definition (9) we obtain

$$|S^\alpha f| \leqslant |S^\alpha |f|| = e^{-x^2/4}\left|\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}|f(t)|e^{t^2/4}\,dt\right|$$
$$\leqslant \frac{1}{\Gamma(\alpha)}e^{-x^2/4}e^{x^2/4}\left|\int_0^x (x-t)^{\alpha-1}|f(t)|\,dt\right|$$
$$= \frac{1}{\Gamma(\alpha)}\left|\int_0^x (x-t)^{\alpha-1}|f(t)|\,dt\right|.$$

If $0 \leqslant x \leqslant 1$, then by (12) we have

$$\left|\int_0^x (x-t)^{\alpha-1}|f(t)|\,dt\right| = \int_0^x (x-t)^{\alpha-1}|f(t)|\,dt \leqslant M_r\int_0^x (x-t)^{\alpha-1}t^r\,dt$$
$$\leqslant M_r x^r\int_0^x (x-t)^{\alpha-1}\,dt = M_r\frac{x^{r+\alpha}}{\alpha} = M_r\frac{|x|^{r+\alpha}}{\alpha}.$$

If $-1 \leqslant x < 0$, then by (12) we obtain

$$\left|\int_0^x (x-t)^{\alpha-1}|f(t)|\,dt\right| \leqslant M_r\int_x^0 (t-x)^{\alpha-1}(-t)^r\,dt = M_r\int_0^{-x}(-z-x)^{\alpha-1}z^r\,dz$$
$$\leqslant M_r(-x)^r\int_0^{-x}(-z-x)^{\alpha-1}\,dz$$
$$= M_r\frac{(-x)^{r+\alpha}}{\alpha} = M_r\frac{|x|^{r+\alpha}}{\alpha}.$$

Thus for all $|x| \leqslant 1$ we get

$$|S^\alpha f| \leqslant M_r\frac{|x|^{r+\alpha}}{\alpha\cdot\Gamma(\alpha)} = M_r\frac{|x|^{r+\alpha}}{\Gamma(\alpha+1)} = M_{(\alpha,r)}|x|^{r+\alpha}.$$

This means that the tempered integral $S^\alpha f$ is also rapidly decreasing at 0.

2.2. Distributions rapidly decreasing at 0. We say that a *distribution* f, defined in R, is *rapidly decreasing at* 0 iff f is a tempered derivative of finite order of a continuous function F which is rapidly decreasing at 0. More precisely, a distribution f is rapidly decreasing at 0 iff there are a continuous function F rapidly decreasing at 0 and a positive integer l such that $D^l F = f$.

We have the following

THEOREM 4. *If a distribution f is rapidly decreasing at* 0, *then every tempered derivative $D^k f$ $(k \in P)$ of f has value* 0 *at the point* 0.

Proof. Let f be a distribution rapidly decreasing at 0. Then $f = D^l F$ for some $l \in P$ and a continuous function F which is rapidly decreasing at 0. Let k be a fixed non-negative integer. Then

$$D^k f = D^{k+l} F = e^{-x^2/4}(e^{x^2/4} F(x))^{(k+l)}.$$

Since $F(x)$ is a continuous function, rapidly decreasing at 0, it is easy to check that also

$$e^{x^2/4} F(x)$$

is a continuous function rapidly decreasing at 0. By Theorem 2, $(e^{x^2/4} F(x))^{(k+l)}$ has value 0 at the point 0. Hence, by Theorem 3.5.3 (see [1]), the product

$$e^{-x^2/4}(e^{x^2/4} F(x))^{(k+l)}$$

has the required property. Thus $D^k f$ $(k \in P)$ has value 0 at the point 0.

THEOREM 5. *For every distribution f rapidly decreasing at 0 there exists a unique distribution g rapidly decreasing at 0 such that $Dg = f$.*

Proof. By assumption it follows that $f = D^l F$ for some $l \in P$ and for some continuous function F rapidly decreasing at 0.

If $l > 0$, then $f = D^l F = D(D^{l-1} F) = Dg$, where $D^{l-1} F = g$. It is easy to see that g is a distribution rapidly decreasing at 0.

If $l = 0$, then $f = F = DSF$. The function F is rapidly decreasing at 0 thus, by Theorem 3, the function SF is rapidly decreasing at 0.

This means that for every distribution f rapidly decreasing at 0 there exists a distribution g rapidly decreasing at 0 such that $Dg = f$. We shall show that g is the only distribution with these properties.

In fact, suppose that there are two such distributions g_1 and g_2. Because $Dg_1 = Dg_2 = f$, therefore $D(g_1 - g_2) = 0$. Hence we obtain $g_1(x) - g_2(x) = ce^{-x^2/4}$. By Theorem 4 distributions g_1 and g_2 have the distributional value 0 at 0 and so $ce^0 = 0$. Thus $c = 0$.

The proof of Theorem 5 is finished.

THEOREM 6. *The set of all distributions rapidly decreasing at 0 is a linear space.*

Proof. Let f and g be distributions rapidly decreasing at 0, i.e., $f = D^l F$, $g = D^r G$ for some $l, r \in P$ and for some continuous functions F and G rapidly decreasing at 0. Let $l \geqslant r$ and $l = r + m$, where $m \in P$. Consequently $D^r G = D^r D^m S^m G = D^{r+m} S^m G$. Thus $f + g = D^l F + D^r G = D^l(F + S^m G)$. By Theorem 3 and Remark 1, $F + S^m G$ is a continuous function rapidly decreasing at 0. Thus $f + g$ is a distribution rapidly decreasing at 0. It is easy to see that cf is a distribution rapidly decreasing at 0, for any number c.

2.3. Tempered integrals of order $\alpha \geqslant 0$ ($\alpha \in R$) of a distribution rapidly decreasing at 0. Let f be a distribution rapidly decreasing at 0, i.e., $f = D^l F$, where F is a function rapidly decreasing at 0 and $l \in P$.

By the *α-th tempered integral ($\alpha \geqslant 0$, $\alpha \in R$) of a distribution f* rapidly decreasing at 0 and defined in R we mean

$$S^\alpha f = D^l S^\alpha F \quad (\alpha \geqslant 0,\ \alpha \in R). \tag{13}$$

The tempered integral $S^\alpha f$ does not depend on the choice of l and F. In fact, let $f = D^l F$, $f = D^r F_0$ and let $l \geqslant r$, i.e., $l = r+m$, where $m \in P$. Therefore $D^r F_0 = D^r D^m S^m F_0 = D^l S^m F_0$. By Theorem 5 we have $F = S^m F_0$. Consequently, $S^\alpha f = D^r S^\alpha F_0 = D^r D^m S^m S^\alpha F_0 = D^{r+m} S^\alpha S^m F_0 = D^l S^\alpha F$.

THEOREM 7. *The tempered integral $S^\alpha f$ ($\alpha \geqslant 0$, $\alpha \in R$) is a linear operator in the class of distributions rapidly decreasing at 0.*

Proof. Let f and g be distributions rapidly decreasing at 0. This means that $f = D^l F$, $g = D^r G$ for some continuous functions F and G rapidly decreasing at 0 and for some $l, r \in P$. Let $l \geqslant r$ and $l = r+m$, where $m \in P$. Then $f+g = D^l(F+S^m G)$. We note that $F+S^m G$ is a continuous function rapidly decreasing at 0. Hence, by equality (13) and Theorem 1, $S^\alpha(f+g) = D^l S^\alpha(F+S^m G) = D^l S^\alpha F + D^{r+m} S^m S^\alpha G = S^\alpha f + S^\alpha g$.

It is easy to check that $S^\alpha(cf) = cS^\alpha f$ for any number c.

THEOREM 8. *If f is a distribution rapidly decreasing at 0, then for any $\alpha \geqslant 0$, $\beta \geqslant 0$ ($\alpha, \beta \in R$) the following equality holds*:

$$S^\alpha S^\beta f = S^{\alpha+\beta} f.$$

Proof. From the assumption it follows that there are $l \in P$ and a continuous function F rapidly decreasing at 0 such that $f = D^l F$. Applying equality (13) we have $S^\beta f = D^l S^\beta F$. By Theorem 3 the function $G = S^\beta F$ is continuous and rapidly decreasing at 0, consequently $g = D^l G$ is a distribution rapidly decreasing at 0. Hence, by equality (13) and Theorem 1, we obtain $S^\alpha g = D^l S^\alpha G = D^l S^\alpha S^\beta F = D^l S^{\alpha+\beta} F = S^{\alpha+\beta} f$.

2.4. Tempered derivatives of order $\alpha \geqslant 0$ ($\alpha \in R$) of a distribution rapidly decreasing at 0. Let f be a distribution rapidly decreasing at 0, i.e., $f = D^l F$ for some continuous function F rapidly decreasing at 0 and for some $l \in P$.

By the *α-th tempered derivative ($\alpha \geqslant 0$, $\alpha \in R$) of a distribution f* rapidly decreasing at 0, defined in R, we mean

$$D^\alpha f = D^{p+l} S^{p-\alpha} F, \tag{14}$$

where p is an integer such that $0 \leqslant p-1 < \alpha \leqslant p$.

The tempered derivative $D^\alpha f$ ($\alpha \geqslant 0$, $\alpha \in R$) does not depend on the choice of l and F. In fact, let $f = D^l F$, $f = D^r F_0$ and let $l \geqslant r$, i.e., $l = r+m$, where $m \in P$. Therefore $D^r F_0 = D^r D^m S^m F_0 = D^l S^m F_0$. By Theorem

5 we have $F = S^m F_0$; consequently $D^\alpha f = D^{p+r} S^{p-\alpha} F_0 = D^{p+r+m} S^m S^{p-\alpha} F_0$ $= D^{p+l} S^{p-\alpha} S^m F_0 = D^{p+l} S^{p-\alpha} F$.

THEOREM 9. *The tempered derivative $D^\alpha f$ ($\alpha \geqslant 0$, $\alpha \in R$) is a linear operation in the class of distributions rapidly decreasing at* 0.

Proof. Let f and g be distributions rapidly decreasing at 0, i.e., $f = D^l F$, $g = D^r G$ for some functions F and G rapidly decreasing at 0 and for some $l, r \in P$. Let $l \geqslant r$ and $l = r+m$, where $m \in P$. Then $f+g$ $= D^l(F+S^m G)$. From Theorem 3 and Remark 1 it follows that $F+S^m G$ is a continuous function rapidly decreasing at 0. Hence, by equality (14) and Theorem 1, we have

$$D^\alpha(f+g) = D^{p+l} S^{p-\alpha}(F+S^m G) = D^{p+l} S^{p-\alpha} F + D^{p+r+m} S^m S^{p-\alpha} G$$
$$= D^\alpha f + D^{p+r} S^{p-\alpha} G = D^\alpha f + D^\alpha g$$

(p is an integer such that $0 \leqslant p-1 < \alpha \leqslant p$).

It is easy to check that $D^\alpha(cf) = cD^\alpha f$ for any number c.

THEOREM 10. *If f is a distribution rapidly decreasing at* 0, *then for any $\alpha \geqslant 0$, $\beta \geqslant 0$ ($\alpha, \beta \in R$) the following equality holds*:

$$D^\alpha D^\beta f = D^{\alpha+\beta} f.$$

Proof. From the assumption it follows that there are $l \in P$ and a continuous function F rapidly decreasing at 0 such that $f = D^l F$. Applying equality (14) we obtain $D^\beta f = D^{\bar{p}+l} S^{\bar{p}-\beta} F$, where $\bar{p}$ is an integer such that $0 \leqslant \bar{p}-1 < \beta \leqslant p$. By Theorem 3, a function $G = S^{\bar{p}-\beta} F$ is rapidly decreasing at 0; thus $g = D^{\bar{p}+l} G$ is a distribution rapidly decreasing at 0. Using (14) we have $D^\alpha D^\beta f = D^\alpha g = D^{p+\bar{p}+l} S^{p-\alpha} G = D^{p+\bar{p}+l} S^{p-\alpha} S^{\bar{p}-\beta} F$, where p is an integer such that $0 \leqslant p-1 < \alpha \leqslant p$. Hence, by Theorem 1,

$$D^\alpha D^\beta f = D^{p+\bar{p}+l} S^{p+\bar{p}-(\alpha+\beta)} F = D^{\alpha+\beta} f,$$

where $0 \leqslant p+\bar{p}-1 < \alpha+\beta \leqslant p+\bar{p}$ if $0 \leqslant \alpha_0+\beta_0 < 1$,

or

$$D^\alpha D^\beta f = D^{p+\bar{p}-1+l} S^{p+\bar{p}-1-(\alpha+\beta)} F = D^{\alpha+\beta} f,$$

$0 \leqslant p+\bar{p}-2 < \alpha+\beta \leqslant p+\bar{p}-1$ if $1 \leqslant \alpha_0+\beta_0 < 2$

for $\alpha = p-\alpha_0$, $0 \leqslant \alpha_0 < 1$ and $\beta = \bar{p}-\beta_0$, $0 \leqslant \beta_0 < 1$.

Remark 2. The operators S^α and D^α constitute a commutative group, with composition as group operation, which is isomorphic the additive group of real numbers.

References

[1] P. Antosik, J. Mikusiński and R. Sikorski, *Theory of distribution. The sequential approach*, Elsevier, Amsterdam; PWN, Warszawa 1973.
[2] S. Łojasiewicz, *Sur la valeur d'une distribution dans un point*, Bull. Polon. Acad. Sci. Cl. III, 4 (1956), p. 239-242.
[3] — *Sur la valeur et le limite d'une distribution dans un point*, Studia Math. 16 (1957), p. 1 36.
[4] K. Skórnik, *Hereditarily periodic distributions*, ibidem 43 (1972), p. 245-272.
[5] — *On the tempered integrals and derivatives of a class of generalized functions* (in Russian), Dokl. A.N. SSSR 254 (1980).

Reçu par la Rédaction le 16.7.1977