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On non-localized oriented angles

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In the plane Euclidean geometry there are several distinct notions referred to by the same name: *angle*. One of the most important ones and with the largest area of applications is the notion of an oriented angle introduced with the aid of the following

DEFINITION 1. An *oriented angle* is an ordered pair of rays with a common origin. First of these rays is called the *initial arm* of the angle, the other is called the *terminal arm*, and their common origin is called the *vertex* of the angle.

If we denote by $PQ^{\to}$ the ray issuing from P and passing through the point Q, then the oriented angle $\langle AB^{\to}, AC^{\to}\rangle$ will be denoted by $\sphericalangle BAC^{\to}$.

We meet oriented angles in particular in the plane trigonometry, not in the part concerned with solving the triangles, but in the more theoretical one, devoted to the investigation of the trigonometric functions (and, in particular, reduction and summation formulas).

For oriented angles we may introduce an equivalence relation by the following

DEFINITION 2. Two oriented angles are *equivalent* iff there exists an even isometry [1] of the plane which transforms one of these angles onto the other (of course, in such a manner that the initial arm of one angle is mapped onto the initial arm of the other).

There are far reaching analogies between the notion of an oriented angle and that of a localized vector. In particular:

1. a localized vector is an ordered pair of points, an oriented angle is an ordered pair of rays;

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(1) The group of the even isometries of the plane consists of all the rotations and parallel translations of this plane. One defines the groups of parallel translations and rotations, as usual, by means of the analytical formulas (1) and (2) (with additional condition (3)), respectively.

2. for both these notions one defines special equivalence relations.

The latter remark suggests that the procedure leading to the definition of a non-localized vector (as an abstraction class of equivalent localized vectors) may be applied as well to oriented angles. Namely, we may adopt the following (²)

Definition 3. *Non-localized oriented angle* generated by an oriented angle $\sphericalangle ABC^{\rightarrow}$ is the set of all the oriented angles $\sphericalangle PQR^{\rightarrow}$ equivalent to the angle $\sphericalangle ABC^{\rightarrow}$.

In the present paper we shall use lower case bold-face Greek letters (e.g. $\boldsymbol{\alpha}, \boldsymbol{\beta}$) to denote non-localized oriented angles. The set of all non-localized oriented angles will be denoted by $\mathscr{A}$.

The notion of a non-localized oriented angle, though as a rule omitted in the mathematical considerations, is very convenient (³) and it is just this notion that we have in mind when in the elementary geometry we compare or add angles.

This notion is also closely connected with that of the rotation. A rotation of the plane may be uniquely determined by specifying the centre of rotation and so-called "angle of rotation"; the latter is nothing else as just a certain non-localized oriented angle. This follows from the fact that every non-identical rotation is an isometry which has exactly one invariant point S (the centre of rotation) and has the property that for all points $A \neq S$ of the plane and for their images A' the oriented angles $\sphericalangle ASA'^{\rightarrow}$ are equivalent, i.e., they belong to the same non-localized oriented angle $\boldsymbol{\alpha}$. This non-localized oriented angle $\boldsymbol{\alpha}$ is called the angle of rotation, or else, we say that the mapping considered is the rotation around the centre S by the angle $\boldsymbol{\alpha}$.

The connection described between non-localized oriented angles and rotations allows one to notice further analogies between non-localized oriented angles and non-localized vectors. Namely, similarly as there is a one-to-one correspondence between the set of all the parallel translations and the set of all the non-localized vectors, there is also a similar correspondence between the set of all the rotations of the plane around a fixed centre and the set $\mathscr{A}$.

This correspondence allows us (the coordinate system being fixed) to assign to every non-localized oriented angle a pair of real numbers which are an analogue of the components of a non-localized vector. To this purpose let us fix a (rectangular) Cartesian coordinate system on the

(²) Analogously one defines the notion of a non-localized non-oriented angle (cf. [2], p. 109).

(³) Cf. [3], p. 17: "angle".

plane. In this coordinate system the translation by a vector $\boldsymbol{v}$ is described by the formula

(1) $$x' = x + a, \quad y' = y + b.$$

The pair of numbers $\langle a, b\rangle$ is called the *components* of the non-localized vector $\boldsymbol{v}$.

Analogously, the rotation around the origin of the coordinate system by a non-localized oriented angle $\boldsymbol{a}$ is described by the formula

(2) $$x' = ax + by, \quad y' = -bx + ay,$$

in which the coefficients a, b fulfil the additional condition

(3) $$a^2 + b^2 = 1.$$

Similarly as formula (1) establishes a one-to-one correspondence between the collection of all pairs of real numbers $\langle a, b\rangle$ and the set of all non-localized vectors, also formula (2) establishes a one-to-one correspondence between the collection of all pairs of real numbers $\langle a, b\rangle$ fulfilling condition (3) and the set of all non-localized oriented angles $\mathscr{A}$. Consequently, making a profit of this analogy, we may call the pair of the coefficients $\langle a, b\rangle$ occurring in formula (2) the *components* of the non-localized oriented angle $\boldsymbol{a}$ (in a fixed Cartesian coordinate system).

We know from the elementary geometry that the successive performance of two rotations around a point S by angles $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively, results in the rotation around the same centre S by a certain angle $\boldsymbol{a}_3$, where the angle $\boldsymbol{a}_3$ depends only on the angles $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, but not on the particular choice of the centre of rotation S. Hence we may define the addition of non-localized oriented angles as follows:

DEFINITION 4. The *sum* $\boldsymbol{a}_1 + \boldsymbol{a}_2$ of non-localized oriented angles $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is the non-localized oriented angle $\boldsymbol{a}_3$ such that the composition of the rotations by the angles $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ around an arbitrary centre S is the rotation around S by the angle $\boldsymbol{a}_3$.

Here again we have an analogy with non-localized vectors, the addition of which may be defined in the following way: if we are given two translations, by vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, respectively, then their composition is the translation by a vector $\boldsymbol{a}_3$ called the sum of the non-localized vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

It can be easily shown that the addition of the non-localized oriented angles (similarly as the addition of the non-localized vectors) is associative and commutative. Similarly as one defines the non-localized null vector and non-localized opposite vector, we may define the non-localized oriented null angle and non-localized oriented opposite angle. Making use of the fact that if angles θ and β belonging to $\mathscr{A}$ fulfil the condition $\beta+\theta=\beta$, then for every non-localized oriented angle a the equality

$$(4) \qquad a+\theta = a$$

holds, we may adopt the following

DEFINITION 5. A non-localized oriented angle θ is called the *null angle* iff it fulfils condition (4) for every $a \in \mathscr{A}$.

It is easily seen that in the set $\mathscr{A}$ there exists exactly one null angle and so we may introduce for it the fixed symbol θ.

It can be also easily shown that, for every $a \in \mathscr{A}$, there exists a unique angle $\beta \in \mathscr{A}$ such that

$$(5) \qquad a+\beta = \theta,$$

which allows us to adopt the following

DEFINITION 6. If $a \in \mathscr{A}$, then the unique angle $\beta \in \mathscr{A}$ fulfilling (5) is called the *opposite angle to* a.

What has been said above implies the following

THEOREM 1. *The set $\mathscr{A}$ with the operation of the addition of angles forms a commutative group; the neutral element of this group is the null angle θ, and the inverse element of an angle from $\mathscr{A}$ is its opposite angle.*

Regarding the components of the non-localized oriented angles we have the following

THEOREM 2. *In a fixed Cartesian coordinate system the sum of angles with the components $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ has the components $\langle a_1a_2-b_1b_2, a_1b_2+b_1a_2\rangle$.*

In order to prove Theorem 2 it is enough, making use of formula (2), to find the formula for the superposition of two rotations.

One can also prove the following

THEOREM 3. *Independently of the choice of a Cartesian coordinate system the null angle θ always has the components $\langle 1, 0\rangle$.*

A similar property is found with the flat non-localized oriented angle. This is the non-localized oriented angle generated by any oriented angle $\sphericalangle ABC^{\to}$, where B is a point lying between A and C. Namely, in every coordinate system this angle has the components $\langle -1, 0\rangle$. Thus we may adopt

DEFINITION 7. The symbol π will denote the *non-localized oriented flat angle*, i.e. the angle whose components (in every Cartesian coordinate system) are $\langle -1, 0\rangle$.

Now we shall introduce a certain relation which is a pseudo-order in the set $\mathscr{A}$. First of all let us fix a certain length unit and let PQ denote the distance between the points P and Q.

It is well known from the elementary geometry that if we are given two non-localized oriented angles α_1 and α_2, and two pairs of distinct points S, A_0 and R, B_0, and if A_i denotes the image of the point A_0 under the rotation of the plane around the centre S by the angle α_i, and B_i denotes the image of the point B_0 under the rotation of the plane around the centre R by the angle α_i $(i = 1, 2)$, then the inequality

$$A_0A_1 < A_0A_2 \tag{6}$$

holds if and only if we have $B_0B_1 < B_0B_2$.

This fact allows us to adopt the following

DEFINITION 8. We say that a non-localized oriented angle α_2 is *larger* than α_1, and we write $\alpha_1 \dashv\vdash \alpha_2$, iff for every pair of distinct points S and A_0 condition (6) is fulfilled, where A_i denotes the image of the point A_0 under the rotation of the plane around the centre S by the angle α_i $(i = 1, 2)$.

It must be stressed, however, that the addition of angles is not monotonic with respect to the relation $\dashv\vdash$, i.e. the condition $\alpha_1 \dashv\vdash \alpha_2$ does not imply $\alpha_1 + \beta \dashv\vdash \alpha_2 + \beta$.

The main problem with which we are going to deal in the present paper is the question of defining a magnitude of non-localized oriented angles. As is known from the elementary geometry, the most commonly used one is the radian measure of angles, or other proportional measures (e.g., the measure in degrees). For a determination of a non-localized oriented angle also its trigonometric functions can be used, e.g. the cosine and sine. The numbers a and b, called in the present paper the components of an angle, are just the cosine and sine of this angle.

In the metric geometry there is an essential difference between the definition of the trigonometric functions of an angle and the definition of the measure of an angle. The trigonometric functions of an angle $\alpha \in \mathscr{A}$ generated by a localized oriented angle $\sphericalangle ABC^{\rightarrow}$ may be expressed as algebraic functions of the distances of the points A, B, C. So, e.g.,

$$\cos\alpha = \frac{BA^2 + BC^2 - AC^2}{2BA \cdot BC} \quad (^4).$$

In the case of the radian measure (or any other measure) of an angle the situation is different. Such a measure cannot be expressed as an algebraic function of the distances of points, since the circular functions are transcendental. Therefore all usual definitions of the radian measure

(4) The sine and the remaining trigonometric functions also can be expressed as algebraic functions of distances, although the necessity of taking into account the orientation of the angle causes some complications.

require a direct or indirect use of limit processes (sum of a series, integral, the arc length of a curve, or the limit of diadic approximations of the measure of the angle).

The problem which we have put forward in the present paper is to find a definition of a measure of angles which might be formulated in an entirely elementary manner. More precisely, we have aimed at finding an axiomatic definition of a measure of angles, a definition which would define the measure possibly uniquely and could be formulated without referring to limit processes ([5]).

At first let us establish what do we understand by a *measure of angles* and what elementary properties should such a measure have.

The measure we are looking for is a function φ which maps the set $\mathscr{A}$ onto a set $\mathscr{Y}$ referred to as the range of φ. The function φ should be single-valued, i.e. it should fulfil the condition

$$(*) \qquad a_1 \neq a_2 \text{ implies } \varphi(a_1) \neq \varphi(a_2).$$

Moreover, in the range $\mathscr{Y}$ of the measure φ an addition should be defined and the function φ should fulfil the condition

$$(**) \qquad \varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$$

for every a_1, a_2 belonging to $\mathscr{A}$. In equation (**) the sign $+$ on the left-hand side denotes the sum of non-localized oriented angles introduced in Definition 4, whereas the sign $+$ on the right-hand side denotes the addition that must be defined in the set $\mathscr{Y}$.

Since the function φ is a one-to-one map of the set $\mathscr{A}$ onto the set $\mathscr{Y}$, it follows from condition (**) in view of Theorem 1 that the set $\mathscr{Y}$ must be a commutative group with respect to the addition defined there, and we have even the following

THEOREM 4. *The groups*:

$\mathscr{A}$ — *of non-localized oriented angles, and*

$\mathscr{Y}$ — *of the values of a measure φ of these angles, fulfilling* (*) *and* (**) (*together with the operations of addition defined therein*), *are isomorphic.*

Now we shall show that the range $\mathscr{Y}$ of a measure φ fulfilling (*) and (**) cannot be a subgroup of the additive group of real numbers, for the latter is ordered (by the relation $<$), whereas we have the following

THEOREM 5. *If a function φ defined on the set $\mathscr{A}$ fulfils conditions* (*) *and* (**), *then its range $\mathscr{Y}$ cannot be an ordered group, and hence it cannot be a subgroup of any ordered group, in particular of the additive group of real numbers.*

([5]) Such a definition is given for non-oriented (convex) angles in [2], p. 123-124.

Proof. Let us note that the flat angle π (cf. Definition 7) fulfils the relation

$$\pi+\pi = \theta \tag{7}$$

and $\pi \neq \theta$. By Definition 5 we have also

$$\pi+\theta = \pi. \tag{8}$$

If the group $\mathscr{Y}$ could be ordered, then, according to Theorem 4, also the isomorphic group $\mathscr{A}$ could be ordered. We shall show, by an indirect proof, that it is impossible. So let us suppose that there exists a relation $<$ ordering the group $\mathscr{A}$. Then we must have either $\theta < \pi$ or $\pi < \theta$. In the first case, adding the angle π to both the sides of the inequality, we obtain

$$\pi+\theta < \pi+\pi,$$

or, according to (7) and (8), $\pi < \theta$, a contradiction. A similar argument leads to a contradiction also in the other case.

Now let us consider the measure of angles most frequently used, viz. the radian measure, which in the sequel will be denoted by φ_0. It assigns to a non-localized oriented angle α the set of all real numbers of the form

$$a+2k\pi, \tag{9}$$

where a is a certain real number, uniquely determined for a given α, fulfilling the inequality

$$-\pi < a \leqslant \pi, \tag{10}$$

and k runs over the set of all integers. Thus the function φ_0 assigns to an angle α a whole class of real numbers congruent modulo 2π. The range of φ_0, i.e. the set of all those classes, is the quotient group of the additive group of real numbers $\mathscr{R}$ by the relation of congruence modulo 2π. In the sequel the symbol $\mathscr{R}_m$ will denote the quotient group of the additive group of real numbers $\mathscr{R}$ by the relation of congruence modulo m. Thus the measure φ_0 has the range $\mathscr{R}_{2\pi}$.

By Theorem 4 the ranges of all possible measures of angles (fulfilling (*) and (**)) are isomorphic with each other. Therefore we may confine ourselves to considering measures with a fixed range only. In fact, suppose that $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are two groups isomorphic with $\mathscr{A}$, and hence with each other, and let γ be an isomorphism of $\mathscr{Y}_1$ onto $\mathscr{Y}_2$. If φ_1 is a measure of angles with the range $\mathscr{Y}_1$, then $\varphi_2(\alpha) = \gamma(\varphi_1(\alpha))$ also is a measure of angles (i.e., fulfils (*) and (**)) and has the range $\mathscr{Y}_2$.

In the light of the above remark it is no restriction to consider only the measures with the range $\mathscr{R}_{2\pi}$.

Among other possible ranges we mention the groups $\mathscr{R}_m$ (in particular, $\mathscr{R}_{360}$ is the range of the measure in degrees); and also the group $\mathscr{E}$

of the complex numbers of the absolute value 1, with the operation of multiplication. The latter group has an obvious geometrical interpretation. Finally, we would like to call the reader's attention to the group $\mathscr{I}$ consisting of all the real numbers a fulfilling (10), with the operation $+$ defined as follows:

$$a+b=\begin{cases}a+b & \text{if } -\pi<a+b\leqslant\pi,\\ a+b-2\pi & \text{if } a+b>\pi,\\ a+b+2\pi & \text{if } a+b\leqslant-\pi.\end{cases} \tag{11}$$

All these groups $\mathscr{R}_m$, $\mathscr{E}$, $\mathscr{I}$ are isomorphic with $\mathscr{R}_{2\pi}$. In particular, an isomorphism between $\mathscr{I}$ and $\mathscr{R}_{2\pi}$ is established by the function τ which assigns to an $a\in\mathscr{I}$ the class (9):

$$\tau(a)=\{a+2k\pi\}. \tag{12}$$

Our choice of $\mathscr{R}_{2\pi}$ as the standard range is motivated by the fact that the most important measure of angles, the radian measure φ_0, has just the range $\mathscr{R}_{2\pi}$. We know that φ_0 is a function with the domain $\mathscr{A}$ and the range $\mathscr{R}_{2\pi}$ fulfilling conditions (*) and (**). We may ask in how far it is determined by these conditions.

Problem 1. Decide whether $\varphi(a)=\varphi_0(a)$ is the only function with domain $\mathscr{A}$ and range $\mathscr{R}_{2\pi}$ fulfilling conditions (*) and (**).

Unfortunately, the answer to this problem is negative, as may be seen from the following

Theorem 6. *There exist infinitely many distinct functions φ fulfilling the conditions specified in Problem* 1.

Proof. Let $\mathscr{H}$ be an arbitrary fixed Hamel basis of the set of real numbers such that $2\pi\in\mathscr{H}$ and let f_0 be an arbitrary permutation of $\mathscr{H}$ (a one-to-one map of $\mathscr{H}$ onto itself) such that

$$f_0(2\pi)=2\pi. \tag{13}$$

As is well known ([1], p. 35), f_0 can be uniquely extended to an additive function f on the whole set of real numbers:

$$f(x+y)=f(x)+f(y). \tag{14}$$

We shall show that $f(x)=0$ implies $x=0$. In fact, x may be uniquely written in the form

$$x=\sum r_i h_i,\quad h_i\in\mathscr{H},\ r_i \text{ rational}, \tag{15}$$

and we have

$$f(x)=f\left(\sum r_i h_i\right)=\sum r_i f_0(h_i). \tag{16}$$

But, since f_0 is a permutation of $\mathscr{H}$, the set of $f_0(h_i)$ occurring in (16) is linearly independent over the set of rationals [6].

Thus the value (16) may be zero if and only if all r_i are zero, which implies $x = 0$. Hence it follows that f is invertible, for $f(x) = f(y)$ implies by (14) $f(x-y) = 0$, whence $x = y$.

The range of f is the whole set of real numbers. For suppose that we are given an x written in form (15). Since f_0 is a permutation of $\mathscr{H}$, we can find $h_i' \in \mathscr{H}$ such that $f_0(h_i') = h_i$, whence for $x' = \sum r_i h_i'$ we have

$$f(x') = \sum r_i f_0(h_i') = \sum r_i h_i = x.$$

Now let us take an $a \in \mathscr{A}$ and let

$$\varphi_0(a) = \{a+2k\pi\}, \tag{17}$$

where k runs over the set of integers. We write

$$\varphi_1(a) = \{f(a)+2k\pi\}. \tag{18}$$

The right-hand side of (18) is independent of the choice of a in (17). In fact, if $\{a_1+2k\pi\} = \{a_2+2k\pi\}$, then $a_1-a_2 = 2l\pi$, where l is an integer, and by (14) $f(a_1)-f(a_2) = f(a_1-a_2) = f(2l\pi) = lf(2\pi) = lf_0(2\pi) = 2l\pi$, whence $\{f(a_1)+2k\pi\} = \{f(a_2)+2k\}$. Consequently φ_1 is by (17) and (18) unambiguously defined on $\mathscr{A}$ and its range is $\mathscr{R}_{2\pi}$, since the range of f is the whole set of reals. The function φ_1 fulfils (*) in virtue of the invertibility of f and φ_0, and fulfils (**) by (14) and the additivity of φ_0.

There are as many functions φ_1 as permutations of the (uncountable) set $\mathscr{H}$ fulfilling (13), i.e., $2^{\mathfrak{c}}$.

Since the answer to Problem 1 is negative, we must seek further conditions [7] which imposed on the function $\varphi(a)$ allow us to eliminate superfluous solutions. This could be easily achieved if we could impose onto φ the condition of the monotonicity. However, in view of Theorem 5, this is impossible, since there exists no relation ordering the group $\mathscr{R}_{2\pi}$. Anyhow we shall introduce a certain relation in $\mathscr{R}_{2\pi}$ which, being not an order (nor even a pseudo-order), nevertheless will allow us to compare the elements of $\mathscr{R}_{2\pi}$ [8]. We start with the following

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[6] The basis $\mathscr{H}$, as well as every its subset, is linearly independent over the rationals.

[7] The problem of determining the angular measure by means of functional equations is not novel. In 1933 S. Gołąb considered analogous problems in general spaces (cf. [4]).

[8] The relation which we are going to define is a pseudo-order in the *set* $\mathscr{R}_{2\pi}$, but it is not a pseudo-order in the *group* $\mathscr{R}_{2\pi}$, because the addition in this group is not monotonic with respect to the relation $\prec$.

DEFINITION 9. The *norm* of an $a \in \mathscr{R}_{2\pi}$ ([9]) is the real number $\|a\|$ defined by

$$\|a\| = \min_{x \in a} |x|. \tag{19}$$

It is readily seen that for any $a \in \mathscr{R}_{2\pi}$ we have the inequalities

$$0 \leqslant \|a\| \leqslant \pi.$$

Instead of comparing the elements of the group $\mathscr{R}_{2\pi}$ we shall compare their norms with the aid of the usual relation of majority for real numbers.

Taking into account Definitions 8 and 9 we easily see that the conditions $\boldsymbol{a}_1 \dashv \boldsymbol{a}_2$ and $\|\varphi_0(\boldsymbol{a}_1)\| < \|\varphi_0(\boldsymbol{a}_2)\|$ are equivalent. Thus the function $\varphi(\boldsymbol{a}) = \varphi_0(\boldsymbol{a})$ fulfils for all $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathscr{A}$ the condition

$$\boldsymbol{a}_1 \dashv \boldsymbol{a}_2 \text{ implies } \|\varphi(\boldsymbol{a}_1)\| < \|\varphi(\boldsymbol{a}_2)\|. \tag{***}$$

Now we may ask whether conditions (*), (**) and (***) determine the radian measure uniquely, i.e., we have the following

PROBLEM 2. Decide whether $\varphi(\boldsymbol{a}) = \varphi_0(\boldsymbol{a})$ is the only function with domain $\mathscr{A}$ and range $\mathscr{R}_{2\pi}$ fulfilling conditions (*), (**) and (***).

Now, the answer to Problem 2 is essentially positive: the function φ is determined up to the sign. Moreover, it is superfluous to assume that the range of φ is $\mathscr{R}_{2\pi}$, it is enough to assume that the range is contained in $\mathscr{R}_{2\pi}$. Thus we shall prove the following theorem which is the main result of the present paper:

THEOREM 7. *There exist exactly two distinct functions φ which are defined on $\mathscr{A}$, take values in the group $\mathscr{R}_{2\pi}$ and fulfil for every $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathscr{A}$ conditions (*), (**) and (***). These functions differ only by the sign.*

Proof. Let $\varphi(\boldsymbol{a})$ be a function fulfilling the conditions of the theorem and let us write

$$g(x) = \tau^{-1}\Big(\varphi\big(\varphi_0^{-1}(\tau(x))\big)\Big), \tag{20}$$

where τ is the isomorphism of $\mathscr{I}$ onto $\mathscr{R}_{2\pi}$ defined by (12). The function g has the domain $\mathscr{I}$ and its range is contained in $\mathscr{I}$. Conditions (*) and (**) and the analogous properties of φ_0 and τ imply that

$$g(x_1) \neq g(x_2) \quad \text{for } x_1 \neq x_2,\ x_1, x_2 \in \mathscr{I}, \tag{21}$$

and

$$g(x_1 + x_2) = g(x_1) + g(x_2) \quad \text{for } x_1, x_2 \in \mathscr{I}. \tag{22}$$

Further, in view of (12) and (19) we have

$$\|\tau(a)\| = |a|. \tag{23}$$

([9]) Note that if $a \in \mathscr{R}_{2\pi}$, then a is a set of real numbers.

Relation (23) and condition (***) for φ_0 and φ imply in view of (20) that

(24) $$|x_1| < |x_2| \text{ implies } |g(x_1)| < |g(x_2)|.$$

Setting $x_1 = x_2 = 0$ in (22) we obtain by (11)

(25) $$g(0) = g(0) + g(0),$$

whence (10)

(26) $$g(0) = 0.$$

Similarly, setting $x_1 = x_2 = \pi$, we have by (11), (22) and (26)

(27) $$g(\pi) + g(\pi) = 0,$$

whence either

(28) $$g(\pi) = 0,$$

or

(29) $$g(\pi) = \pi.$$

But in view of (21) and (26), relation (28) is impossible and thus necessarily (29) holds. Finally, setting $x_1 = x_2 = \frac{1}{2}\pi$, we get by (11), (22) and (29)

(30) $$g(\tfrac{1}{2}\pi) + g(\tfrac{1}{2}\pi) = \pi,$$

whence either $g(\frac{1}{2}\pi) = \frac{1}{2}\pi$, or $g(\frac{1}{2}\pi) = -\frac{1}{2}\pi$, at any case

(31) $$|g(\tfrac{1}{2}\pi)| = \tfrac{1}{2}\pi.$$

Relations (24) and (31) imply that we have

(32) $$|g(x)| < \tfrac{1}{2}\pi \quad \text{for } |x| < \tfrac{1}{2}\pi;$$

and since for $|x| < \frac{1}{2}\pi$ operation (11) reduces to the usual addition, we obtain by (22)

(33) $$g(x_1 + x_2) = g(x_1) + g(x_2) \quad \text{for } |x_1| < \tfrac{1}{2}\pi, |x_2| < \tfrac{1}{2}\pi.$$

It follows from (33) and (24) (cf. [1], p. 43-46) that for $|x| < \frac{1}{2}\pi$

(34) $$g(x) = cx$$

with a certain real constant c, and it is readily seen from (32) that $|c| \leqslant 1$. For $\frac{1}{2}\pi \leqslant |x| < \pi$ we obtain from (22) and (34) in view of (11)

$$g(x) = g(\tfrac{1}{2}x + \tfrac{1}{2}x) = g(\tfrac{1}{2}x) + g(\tfrac{1}{2}x) = \tfrac{1}{2}cx + \tfrac{1}{2}cx = cx,$$

i.e., (34) holds for all x with $|x| < \pi$. Relation (31) yields now $|c| = 1$ and taking into account (29) we obtain (11) $g(x) = x$ for all $x \in \mathscr{J}$ or $g(x)$

(10) In order to solve quickly equations like (25), (27) or (30) one can replace $\mathscr{J}$ by the isomorphic group $\mathscr{E}$. Then (25) is equivalent to $z = z^2$, $z \in \mathscr{E}$, (27) is equivalent to $z^2 = 1$, $z \in \mathscr{E}$, and (30) is equivalent to $z^2 = -1$, $z \in \mathscr{E}$.

(11) Here $-x$ denotes the element inverse to x with respect to operation (11). This is identical with the usual opposite number except for $x = \pi$, for which $-\pi = \pi$.

$= -x$ for all $x \in \mathscr{I}$. This implies in view of (20) that either $\varphi(a) = \varphi_0(a)$ for $a \in \mathscr{A}$, or $\varphi(a) = -\varphi_0(a)$ for $a \in \mathscr{A}$, which was to be proved.

As we know, the rotations of the Euclidean plane may be performed in two directions referred to as orientations of the plane (clockwise and anti-clockwise). Depending on which of these orientations is distinguished as positive, we may fix the radian measure of oriented angles in two possible ways. According to tradition we usually choose the anti-clockwise orientation as positive, however, there is nothing hampering us from accepting the clockwise orientation as positive. Both the radian measures obtained in this way fulfil all the conditions of Theorem 7, so we have

THEOREM 8. *Both functions whose existence results from Theorem 7 are radian measures of the oriented non-localized angles with two possible orientations of the plane accepted as positive.*

The results of the present paper can be applied in the teaching in secondary schools. One of the serious difficulties is presented by the problem of introducing the notion of the radian measure of *oriented* angles which would be correct and at the same time short and easily understandable. And such a measure is necessary for building up the trigonometry.

The above Theorem 8 may serve as a simple axiomatic definition of that measure. However, in order to make it accessible to the pupils' level, we must introduce some modifications, in particular to remove the quotient group $\mathscr{R}_{2\pi}$. Our proposal reads as follows.

We introduce the following notions:

1. non-localized oriented angles (Definition 3);
2. their sum (Definition 4);
3. the comparison of angles (Definition 8);

and then we give the following

DEFINITION 10. The *basic radian measure* of non-localized oriented angles is any function f with the following properties:

(i) f is a one-to-one function assigning to non-localized oriented angles real numbers from the interval $(-\pi, +\pi]$;

(ii) for any two angles a and β there exists an integer k such that $f(a+\beta) = f(a)+f(\beta)+2k\pi$;

(iii) if $a \dashv \beta$, then $|f(a)| < |f(\beta)|$.

After this definition we give either as an axiomat ([12]), or as a theorem, the proof of which is omitted, the following

THEOREM 9. *There exist exactly two different basic radian measures*

([12]) Note that in Z. Krygowska's textbook [5] for the first course there occurs the axiome WV concerning the natural ordering of points on the line, which is quite analogous to our Theorem 9 below.

of non-localized oriented angles: $f_1(a)$ *and* $f_2(a)$; *moreover, for every angle* a *with the exception of the flat angle, we have* $f_1(a) = -f_2(a)$.

Afterwards we can introduce the *generalized measure* of non-localized oriented angles by the rule that, the basic measure f being fixed, we assign to any angle a not only $f(a)$, but all the numbers of the form $f(a)+2k\pi$.

Besides the method described above there is also another way of axiomatic introducing the radian measure of oriented angles. It consists on introducing first non-oriented angles and their addition and the measure of those angles (cf. [2]), and then extending that measure onto oriented angles. This method, however, requires a rather involved argument to prove that the extended measure also satisfies equation (**).

References

[1] J. Aczél, *Lectures on functional equations and their applications*, New York, London 1966.

[2] K. Borsuk i W. Szmielew, *Podstawy geometrii*, Warszawa 1955.

[3] L. Chambadal, *Dictionnaire de mathématiques modernes*, Paris (Larousse) 1969.

[4] S. Gołąb, *O metryce kątowej ogólnych przestrzeni. I. Zasada Chaslesa*, Wiadomości Mat. XXXVIII (1935), p. 1-12.

[5] Z. Krygowska i J. Maroszkowa, *Geometria dla klasy I liceum ogólnokształcącego*, Warszawa 1967.

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