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On a class of quasi-conformal mappings with invariant boundary points, II

Applications and generalizations

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§ 5. Applications to some functionals.

12. We proceed to applications of Theorems 5 and 6 (see the first part of this paper, [12]). We apply these theorems in order to find the regions of variability of the functionals $F(z; w) = \log(w/z)$, $F(z; w) = w-z$ and $F(w_1, w_2) = \log(w_1-w_2)$; z, z_1, z_2 being fixed, $w = f(z)$, $w_1 = f(z_1)$, $w_2 = f(z_2)$, and f running over E_Q. We also obtain the Hölder constant and exponent for the class E_Q.

We begin with the functional $F(z; w) = \log(w/z)$ where the branch is chosen so that $\log 1 = 0$.

THEOREM 7. (i) *For any $f \in E_Q$ and $z \in \Delta$, $z \neq 0$, we have*

$$-\frac{1}{2}\left(Q-\frac{1}{Q}\right)\log\frac{1}{|z|} \leqslant \arg\frac{f(z)}{z} \leqslant \frac{1}{2}\left(Q-\frac{1}{Q}\right)\log\frac{1}{|z|},$$

where $\arg\big(f(z)/z\big) = 0$ *for* $z = 1$.

(ii) *Moreover, the condition*

$$\arg\frac{f(z)}{z} = \frac{1}{2}\left(Q-\frac{1}{Q}\right)\log\frac{1}{|z|}\sin\varphi \qquad (-\tfrac{1}{2}\pi \leqslant \varphi \leqslant \tfrac{1}{2}\pi) \tag{24}$$

implies

$$1-\frac{1}{2}\left(Q+\frac{1}{Q}\right)-\frac{1}{2}\left(Q-\frac{1}{Q}\right)\cos\varphi \leqslant \log\left|\frac{f(z)}{z}\right|/\log\frac{1}{|z|} \leqslant 1-\frac{1}{2}\left(Q+\frac{1}{Q}\right)+\frac{1}{2}\left(Q-\frac{1}{Q}\right)\cos\varphi . \tag{25}$$

All the given estimates are sharp for any $z \in \Delta$, $z \neq 0$, and $Q \in \langle 1, +\infty)$. Given φ, $-\frac{1}{2}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi$, the only extremal functions for every z in (25) *are: $f(s) = |s|^{\beta_1} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$ for the upper bound, and $f(s) = |s|^{\beta_2} e^{i\arg s}$ $(s \neq 0)$, $f(0) = 0$ for the lower bound, where $\beta_1 = \frac{1}{2}(Q+1/Q) - \frac{1}{2}(Q-1/Q)e^{i\varphi}$, $\beta_2 = \frac{1}{2}(Q+1/Q)+\frac{1}{2}(Q-1/Q)e^{-i\varphi}$, and the branch of $\arg\big(f(s)/s\big)$ is chosen in each case so that $\arg f(1) = 0$.*

(iii) *Furthermore,* (24) *and* (25) *give all points of the variability region of the functional* $F(z; w) = \log(w/z)$, *where* $\log 1 = 0$, $w = f(z)$, f *ranges over* E_Q, *and* z ($z \in \Delta$, $z \neq 0$) *is fixed.*

Proof. The bounds for $\arg\big(f(z)/z\big)$ are given in Theorem 2. They may also be immediately obtained from Theorem 5.

We proceed to prove (ii). We use the notation of Theorem 6. Applying this theorem to $F(\zeta; \omega) = \log(\omega/\zeta)$, $\log 1 = 0$, we have to assume $-\frac{1}{2}(Q-1/Q)\log(1/|z|) < \tau < \frac{1}{2}(Q-1/Q)\log(1/|z|)$. Clearly $D_1 = \{\zeta\colon 0 < |\zeta| \leqslant 1\}$, $D_{Q,1} = \{\omega\colon 0 < |\omega| \leqslant 1\}$, and $f(s) \equiv s$ is not an extremal function in our problem. We have

$$F^{(\lambda)}(\zeta; \omega) = \frac{1}{2}\log\frac{\omega\overline{\omega}}{\zeta\overline{\zeta}} - \frac{1}{2}i\lambda\log\frac{\overline{\zeta}\omega}{\zeta\overline{\omega}}.$$

Hence $wF^{(\lambda)}_{\omega}(z; w) = \frac{1}{2}(1-i\lambda) \neq 0$, and

$$\beta(z; \lambda, \varepsilon) = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) - \frac{1}{2}\varepsilon\Big(Q-\frac{1}{Q}\Big)(1+i\lambda)/|1+i\lambda|,$$

where $\varepsilon = 1$ or -1. Consequently

$$\frac{1}{2}\varepsilon\Big(Q-\frac{1}{Q}\Big)\frac{\lambda(\tau, \varepsilon)}{|1+i\lambda(\tau, \varepsilon)|}\log\frac{1}{|z|} = \tau.$$

Setting $\tau = \frac{1}{2}(Q-1/Q)\log(1/|z|)\sin\varphi$, $-\frac{1}{2}\pi < \varphi < \frac{1}{2}\pi$, which agrees with (24), we get $\lambda(\tau, \varepsilon) = \varepsilon\tan\varphi$. Hence, given φ, the maximum and minimum are attained for the functions given in Theorem 7, and there are no other functions for which the maximum or minimum is attained for every z. These functions give the bounds in (25). We verify directly that (ii) remains true for $\varphi = \frac{1}{2}\pi$ and $-\frac{1}{2}\pi$. This is also a consequence of Theorem 1.

In order to prove (iii) we notice that given φ, $-\frac{1}{2}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi$, and t, $-1 \leqslant t \leqslant 1$, the function $f(s, t) = |s|^{\gamma(t)}e^{i\arg s}$ ($s \neq 0$), $f(0, t) = 0$, $\arg f(1, t) = 0$, with $\gamma(t) = \frac{1}{2}(Q+1/Q) - \frac{1}{2}(Q-1/Q)(t\cos\varphi + i\sin\varphi)$ belongs to E_Q and satisfies (24) with $f(z, t)$ substituted for $f(z)$ and $|f(z, t)/z|/\log(1/|z|) = 1 - \frac{1}{2}(Q+1/Q) + \frac{1}{2}(Q-1/Q)t\cos\varphi$. The proof is completed.

13. In this section the functional $F(z; w) = w-z$ is considered.

THEOREM 8. *The region of variability of the functional* $F(z; w) = w-z$, z ($z \in \Delta$, $z \neq 0$) *being fixed,* $w = f(z)$ *and* f *running over* E_Q, *is bounded by a curve which is determined by the following system of equations:*

$$(26)\quad \begin{aligned} u(\varphi) &= |z|^{\frac{1}{2}(Q+1/Q)-\frac{1}{2}(Q-1/Q)\cos\varphi}\cos\Big(\arg z - \frac{1}{2}\Big(Q-\frac{1}{Q}\Big)\log\frac{1}{|z|}\sin\varphi\Big) - \\ &\qquad - |z|\cos\arg z, \\ v(\varphi) &= |z|^{\frac{1}{2}(Q+1/Q)-\frac{1}{2}(Q-1/Q)\cos\varphi}\sin\Big(\arg z - \frac{1}{2}\Big(Q-\frac{1}{Q}\Big)\log\frac{1}{|z|}\sin\varphi\Big) - \\ &\qquad - |z|\sin\arg z, \end{aligned}$$

where $u(\varphi)+iv(\varphi)$ *denote points of the curve*; $-\pi<\varphi\leqslant\pi$. *The only extremal function for which* $F(z;w)=u(\varphi)+iv(\varphi)$ *at* z *for every* z *is determined as follows*:

$$f(s)=|s|^{\frac{1}{2}(Q+1/Q)-\frac{1}{2}(Q-1/Q)\exp i\varphi}e^{i\arg s}\quad \textit{for}\quad s\in\Delta,\ s\neq 0,$$

$$f(s)=0\quad \textit{for}\quad s=0,$$

where the branch of $\arg\big(f(s)/s\big)$ *is chosen in each case so that* $\arg f(1)=0$.

Proof. Theorem 7 implies that for any boundary point of the region of variability under consideration there exists a real number φ, $-\frac{1}{2}\pi<\varphi\leqslant\frac{1}{2}\pi$, such that if we denote this point by $u(\varphi)+iv(\varphi)$, $u(\varphi)$ and $v(\varphi)$ being real, then we have either

$$u(\varphi)+iv(\varphi)+z=|z|^{\frac{1}{2}(Q+1/Q)-\frac{1}{2}(Q-1/Q)\exp i\varphi}e^{i\arg z} \tag{27}$$

or

$$u(\varphi)+iv(\varphi)+z=|z|^{\frac{1}{2}(Q+1/Q)+\frac{1}{2}(Q-1/Q)\exp(-i\varphi)}e^{i\arg z}. \tag{28}$$

Conversely, given φ, $-\frac{1}{2}\pi<\varphi\leqslant\frac{1}{2}\pi$, the point $u(\varphi)+iv(\varphi)$ determined by (27) is a boundary point of the region of variability under consideration, and the point $u(\varphi)+iv(\varphi)$ determined by (28) is a boundary point as well. Moreover, the same theorem gives us the corresponding extremal functions: $f(s)=|s|^{\beta_1}e^{i\arg s}$ $(s\neq 0)$, $f(0)=0$ in case of (27), and $f(s)=|s|^{\beta_2}e^{i\arg s}$ $(s\neq 0)$, $f(0)=0$ in case of (28), where β_1 and β_2 are defined in the quoted theorem, and the branch of $\arg\big(f(s)/s\big)$ is chosen in each case so that $\arg f(1)=0$. Hence Theorem 8 follows.

Theorem 8 may be applied in order to obtain an analogue of the following well-known result of Shah Tao-sching [15]: For any $f\in S_Q$, $Q\in(1,+\infty)$ and $z\in\Delta$ we have

$$|f(z)-z|/\log Q<(1/4\pi^2)\{\Gamma(\tfrac{1}{4})\}^4\approx 4.4\,. \tag{29}$$

The estimate is sharp. It seems to be easier, however, to derive this analogue as a consequence of Theorem 3. As we now deal with applications of Theorems 5 and 6, it seems more convenient to place the corresponding result in the next paragraph.

14. Now we proceed to consider the functional $F(w_1,w_2)=\log(w_1-w_2)$ where the branch is chosen so that $\log 1=0$.

THEOREM 9. (i) *For any* $f\in E_Q$ *and* $z_1,z_2\in\Delta$, $z_1\neq z_2$, $|z_1|\geqslant|z_2|>0$, *we have*

$$\arg\left(|z_1|^{\frac{1}{2}\varepsilon_0 i(Q-1/Q)}e^{i\arg z_1}\left\{1-1/A_2\left(\left|\frac{z_1}{z_2}\right|,\arg\frac{z_1}{z_2}\right)\right\}\right)$$
$$\leqslant\arg\big(f(z_1)-f(z_2)\big)\leqslant\arg\left(|z_1|^{-\frac{1}{2}\varepsilon_0 i(Q-1/Q)}e^{i\arg z_1}\left\{1-1/A_1\left(\left|\frac{z_1}{z_2}\right|,\arg\frac{z_1}{z_2}\right)\right\}\right),$$

where $\varepsilon_0 = 1$ *or* -1, A_1 *and* A_2 *are uniquely determined by the equations*

$$A_n = \left|\frac{z_1}{z_2}\right|^{\frac{1}{2}(Q+1/Q)+\frac{1}{2}\varepsilon_1(-1)^n i(Q-1/Q)\exp i\arg(1-A_n)} e^{i\arg(z_1/z_2)} \qquad (n = 1, 2),$$

$\varepsilon_1 = 1$ *or* -1, $\arg(f(z_1)-f(z_2))\to 0$ *for* $z_1 = 1$, $z_2 \to 0$ *with the correspondingly chosen branches of the estimating functions, and* $A_1 = A_2 = \exp i\arg(z_1/z_2)$ *for* $|z_1| = |z_2|$.

(ii) *Moreover, the condition*

$$(30)\qquad \arg(f(z_1)-f(z_2)) = \arg\left(|z_1|^{-\frac{1}{2}\varepsilon_0 i(Q-1/Q)\sin\varphi} e^{i\arg z_1}\left\{1-1/B\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi\right)\right\}\right),$$

where B *is uniquely determined by the equation*

$$B = \left|\frac{z_1}{z_2}\right|^{\frac{1}{2}(Q+1/Q)-\frac{1}{2}\varepsilon_1(Q-1/Q)\exp i\{\varphi+\arg(1-B)\}} e^{i\arg(z_1/z_2)}$$

and $B = \exp i\arg(z_1/z_2)$ *for* $|z_1| = |z_2|$, *implies*

$$(31)\qquad |z_1|^{q_2(\varphi_2)}\left|1-1/B_2\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi_2\right)\right| \leqslant |f(z_1)-f(z_2)| \leqslant |z_1|^{q_1(\varphi_1)}\left|1-1/B_1\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi_1\right)\right|,$$

where $q_n(\varphi_n) = \frac{1}{2}(Q+1/Q)+\frac{1}{2}(-1)^n(Q-1/Q)\cos\varphi_n$ $(n = 1, 2)$, B_1 *and* B_2 *are uniquely determined by the equations*

$$B_n = \left|\frac{z_1}{z_2}\right|^{\frac{1}{2}(Q+1/Q)+\frac{1}{2}\varepsilon_1'(-1)^n(Q-1/Q)\exp i\{\varphi_n+\arg(1-B_n)\}} e^{i\arg(z_1/z_2)} \qquad (n = 1, 2),$$

$\varepsilon_1' = 1$ *or* -1, $B_1 = B_2 = \exp i\arg(z_1/z_2)$ *for* $|z_1| = |z_2|$, *and* φ_1, φ_2 *are uniquely determined as the solutions of the equations*

$$(32)\qquad \arg\left(|z_1|^{\frac{1}{2}(-1)^n i(Q-1/Q)\sin\varphi_n}\left\{1-1/B_n\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi_n\right)\right\}\right) = \arg\left(|z_1|^{-\frac{1}{2}\varepsilon_0 i(Q-1/Q)\sin\varphi}\left\{1-1/B\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi\right)\right\}\right)$$
$$(-\tfrac{1}{2}\pi \leqslant \varphi_n \leqslant \tfrac{1}{2}\pi,\ n = 1, 2).$$

All the given estimates are sharp for any $z_1, z_2 \in \Delta$, $z_1 \neq z_2$, $|z_1| \geqslant |z_2| > 0$, *and* $Q \in \langle 1, +\infty)$. *Given* φ, $-\frac{1}{2}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi$, *the only extremal functions in* (31) *are*:

$$f(s) = |s|^{\beta_{01}} e^{i\arg s} \qquad (|z_1| \leqslant |s| \leqslant 1),$$
$$f(s) = f(z_1)\,|s/z_1|^{\beta_{11}} e^{i\arg(s/z_1)} \qquad (|z_2| \leqslant |s| < |z_1|),$$
$$f(s) = f(z_2) f_1(s/z_2) \qquad (|s| < |z_2|)$$

for the upper bound, and

$$f(s) = |s|^{\beta_{02}} e^{i \arg s} \qquad (|z_1| \leqslant |s| \leqslant 1),$$
$$f(s) = f(z_1)\, |s/z_1|^{\beta_{12}} e^{i \arg(s/z_1)} \qquad (|z_2| \leqslant |s| < |z_1|),$$
$$f(s) = f(z_2) f_2(s/z_2) \qquad (|s| < |z_2|)$$

for the lower bound, where $\beta_{0n} = \frac{1}{2}(Q+1/Q) + \frac{1}{2}(-1)^n (Q-1/Q)\exp i\varphi_n$, $\beta_{1n} = \frac{1}{2}(Q+1/Q) + \varepsilon_1'(-1)^n(Q-1/Q)\exp i\{\varphi_n + \arg(1-B_n)\}$ $(n = 1, 2)$, f_1 *and* f_2 *are arbitrary functions of the class* E_Q, *and the branch of* $\arg\big(f(s)/s\big)$ *is chosen in each case for* $|z_1| \leqslant |s| \leqslant 1$ *so that* $\arg f(1) = 0$, *and for* $|z_2| \leqslant |s| < |z_1|$ *so that* $\arg\big(f(s)/s\big) \to \arg\big(f(z_1)/z_1\big)$ *as* $s \to z_1$.

(iii) *Furthermore,* (30) *and* (31) *give all points of the variability region of the functional* $F(w_1, w_2) = \log(w_1 - w_2)$, *where* $\log 1 = 0$, $w_1 = f(z_1)$, $w_2 = f(z_2)$, f *ranges over* E_Q, *and* z_1, z_2 $(z_1, z_2 \in \Delta, z_1 \neq z_2, |z_1| \geqslant |z_2| > 0)$ *are fixed.*

Proof. We start with proving (i). We use the notation of Theorem 5. We apply this theorem to the functional $-\frac{1}{2}i\log\big((w_1-w_2)/(\overline{w}_1-\overline{w}_2)\big)$ and assume that $D_1 = \{\zeta_1\colon 0 < |\zeta_1| \leqslant 1\}$, $D_2 = D_2^{(\zeta_1)} = \{\zeta_2\colon 0 < |\zeta_2| \leqslant |\zeta_1|,$ $\zeta_2 \neq \zeta_1\}$, $D_{1,Q} = \{\omega_1\colon 0 < |\omega_1| \leqslant 1\}$, $D_{2,Q} = D_{2,Q}^{(\omega_1)} = \{\omega_2\colon 0 < |\omega_2| \leqslant |\omega_1|,$ $\omega_2 \neq \omega_1\}$. Since $f(s) \equiv s$ is not an extremal function in our problem and conditions (18) take the form

$$-\frac{1}{2} i w_1 \frac{1}{w_1 - w_2} + \frac{1}{2} i w_2 \frac{1}{w_1 - w_2} = -\frac{1}{2} i \neq 0,$$
$$\frac{1}{2} i w_2 \frac{1}{w_1 - w_2} = -\frac{1}{2} i \frac{1}{1 - w_1/w_2} \neq 0,$$

the extremal functions are given by the formulae:

$$f(s) = |s|^{q} e^{i \arg s} \qquad (|z_1| \leqslant |s| \leqslant 1),$$
$$f(s) = f(z_1)\, |s/z_1|^{\beta_1} e^{i \arg(s/z_1)} \qquad (|z_2| \leqslant |s| < |z_1|),$$
$$f(s) = f(z_2) \hat{f}_1(s/z_2) \qquad (|s| < |z_2|)$$

and

$$f(s) = |s|^{\overline{q}} e^{i \arg s} \qquad (|z_1| \leqslant |s| \leqslant 1),$$
$$f(s) = f(z_1)\, |s/z_1|^{\beta_1'} e^{i \arg(s/z_1)} \qquad (|z_2| \leqslant |s| < |z_1|),$$
$$f(s) = f(z_2) \hat{f}_2(s/z_2) \qquad (|s| < |z_2|),$$

where $q = \frac{1}{2}(Q+1/Q) - \frac{1}{2}\varepsilon_0 i(Q-1/Q)$, $\varepsilon_0 = 1$ or -1, $\beta_1 = \frac{1}{2}(Q+1/Q) -$ $-\frac{1}{2}\varepsilon_1 i(Q-1/Q)\exp i\arg(1-A_1)$, $\beta_1' = \frac{1}{2}(Q+1/Q) + \frac{1}{2}\varepsilon_1 i(Q-1/Q)\exp i\arg \times$ $\times (1-A_2)$, $\varepsilon_1 = 1$ or -1, $\hat{f}_1$ and $\hat{f}_2$ are arbitrary functions of the class E_Q, and the branch of $\arg\big(f(s)/s\big)$ is chosen in each case for $|z_1| \leqslant |s| \leqslant 1$ so that $\arg f(1) = 0$, and for $|z_2| \leqslant |s| < |z_1|$ so that $\arg\big(f(s)/s\big) \to \arg\big(f(z_1)/z_1\big)$ as

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$s \to z_1$. Since we may choose ε_0 and ε_1 so that the first function corresponds to the maximum of the functional in question, and the second function to the minimum, (i) follows.

Now we proceed to prove (ii). We use the notation of Theorem 6. Applying this theorem to $F(w_1, w_2) = \log(w_1 - w_2)$, $\log 1 = 0$, we have to assume

$$\arg\Big(|z_1|^{\frac{1}{2}\varepsilon_0 i(Q-1/Q)} e^{i\arg z_1}\Big\{1 - 1/A_2\Big(\Big|\frac{z_1}{z_2}\Big|, \arg\frac{z_1}{z_2}\Big)\Big\}\Big)$$
$$< \tau < \arg\Big(|z_1|^{-\frac{1}{2}\varepsilon_0 i(Q-1/Q)} e^{i\arg z_1}\Big\{1 - 1/A_1\Big(\Big|\frac{z_1}{z_2}\Big|, \arg\frac{z_1}{z_2}\Big)\Big\}\Big).$$

Clearly, we may define $D_1, D_2, D_{1,Q}, D_{2,Q}$ as in the proof of (i). Next we observe that $f(s) \equiv s$ is not an extremal function in our problem and that

$$F^{(\lambda)}(\omega_1, \omega_2) = \tfrac{1}{2}\log\big((\omega_1 - \omega_2)(\bar{\omega}_1 - \bar{\omega}_2)\big) - \tfrac{1}{2} i\lambda \log\big((\omega_1 - \omega_2)/(\bar{\omega}_1 - \bar{\omega}_2)\big).$$

Hence $w_1 F_{\omega_1}(w_1, w_2) + w_2 F_{\omega_2}(w_1, w_2) = \frac{1}{2}(1 - i\lambda) \neq 0$, $w_2 F_{\omega_2}(w_1, w_2) = \frac{1}{2}(1 - i\lambda)(1 - w_1/w_2)^{-1} \neq 0$, where $w_k = f^{(\lambda, \varepsilon_0^*, \varepsilon_1^*)}(z_k)$, $k = 1, 2$; $\varepsilon_0^*, \varepsilon_1^* = 1$ or -1, and also

$$\beta_0(z_1, z_2; \lambda, \varepsilon_0^*, \varepsilon_1^*) = \frac{1}{2}\Big(Q + \frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_0^*\Big(Q - \frac{1}{Q}\Big)(1 + i\lambda)/|1 + i\lambda|,$$

$$\beta_1(z_1, z_2; \lambda, \varepsilon_0^*, \varepsilon_1^*) = \frac{1}{2}\Big(Q + \frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_1^*\Big(Q - \frac{1}{Q}\Big) \times$$
$$\times \exp i\arg(1 - w_1/w_2)(1 + i\lambda)/|1 + i\lambda|.$$

Consequently

$$\arg\Big(|z_1|^{\beta_0(z_1, z_2; \lambda(\tau, \varepsilon_0^*, \varepsilon_1^*), \varepsilon_0^*, \varepsilon_1^*)} e^{i\arg z_1}\Big\{1 - \Big|\frac{z_1}{z_2}\Big|^{\beta_1(z_1, z_2; \lambda(\tau, \varepsilon_0^*, \varepsilon_1^*), \varepsilon_0^*, \varepsilon_1^*)} e^{i\arg(z_1/z_2)}\Big\}\Big) = \tau.$$

Setting

$$\tau = \arg\Big(|z_1|^{-\frac{1}{2}\varepsilon_0 i(Q-1/Q)\sin\varphi} e^{i\arg z_1}\Big\{1 - 1/B\Big(\Big|\frac{z_1}{z_2}\Big|, \arg\frac{z_1}{z_2}, \varphi\Big)\Big\}\Big) \qquad (-\tfrac{1}{2}\pi < \varphi < \tfrac{1}{2}\pi),$$

which agrees with (30), we get (32) where $\varepsilon_1' = \varepsilon_0^* \varepsilon_1^*$ and

$$\varphi_n = \arg\big((-1)^{n-1}\varepsilon_0^*\{1 + i\lambda(\tau, (-1)^{n-1}\varepsilon_0^*, (-1)^{n-1}\varepsilon_1^*)\}\big) \qquad (n = 1, 2).$$

Hence, given φ, the maximum and minimum are attained for the functions given in Theorem 9, and there are no other functions for which the maximum or minimum is attained. These functions give the bounds in (31). We verify directly that (ii) remains true for $\varphi = \frac{1}{2}\pi$ and $-\frac{1}{2}\pi$.

In order to prove (iii) we notice that given φ, $-\frac{1}{2}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi$, and t, $-1 \leqslant t \leqslant 1$, there is a homotopy

$$f(s, t) = |s|^{\gamma_0(t)} e^{i \arg s} \qquad (|z_1| \leqslant |s| \leqslant 1),$$
$$f(s, t) = f(z_1, t)\, |s/z_1|^{\gamma_1(t)} e^{i \arg (s/z_1)} \qquad (|z_2| \leqslant |s| < |z_1|),$$
$$f(s, t) = f(z_2, t) f^*(s/z_2, t) \qquad (|s| < |z_2|),$$
$$\arg f(1, t) = 0, \quad \arg\big(f(s, t)/s\big) \to \arg\big(f(z_1, t)/z_1\big) \quad \text{as } s \to z_1,$$
$$f^* \in E_Q \quad \text{for} \quad -1 \leqslant t \leqslant 1,$$

such that $f(s, 1)$ and $f(s, -1)$ are the functions for which the functional in question attains its maximum and minimum, respectively, and that the function $f(s, t)$ satisfies (30) with $f(z_1, t)$, $f(z_2, t)$ substituted for $f(z_1)$, $f(z_2)$, and belongs to E_Q for any $t \in \langle -1, 1\rangle$. Indeed, to this end we notice that the equation

$$\arg\left(|z_1|^{-\frac{1}{2}i(Q-1/Q)t\sin\varphi(t)}\left\{1 - 1/C\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi(t), t\right)\right\}\right)$$
$$= \arg\left(|z_1|^{-\frac{1}{2}\varepsilon_0 i(Q-1/Q)\sin\varphi}\left\{1 - 1/B\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi\right)\right\}\right),$$

where C is uniquely determined by

$$C = \left|\frac{z_1}{z_2}\right|^{\frac{1}{2}(Q+1/Q) - \frac{1}{2}\varepsilon_1'(Q-1/Q)t \exp i\{\varphi(t) + \arg(1-C)\}} e^{i \arg(z_1/z_2)}$$

and $C = \exp i \arg(z_1/z_2)$ for $|z_1| = |z_2|$, has a unique solution $\varphi(t) \in \langle \varphi_1, \varphi_2\rangle$ or $\langle \varphi_2, \varphi_1\rangle$ for any $t \in \langle -1, 1\rangle$. Hence we define $\gamma_0(t)$ and $\gamma_1(t)$ by the formulae

$$\gamma_0(t) = \frac{1}{2}\left(Q + \frac{1}{Q}\right) - \frac{1}{2}\left(Q - \frac{1}{Q}\right) t \exp i\varphi(t)$$

and

$$\gamma_1(t) = \frac{1}{2}\left(Q + \frac{1}{Q}\right) - \frac{1}{2}\varepsilon_1'\left(Q - \frac{1}{Q}\right) t \exp i\left\{\varphi(t) + \arg\left(1 - C\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi(t), t\right)\right)\right\},$$

respectively. The proof is completed.

15. Theorem 9 enables us to obtain an analogue of the following well-known theorem of Mori [13]: For any $f \in S_Q$ and $z_1, z_2 \in \Delta$, $z_1 \neq z_2$, we have $16^{-Q}|z_1 - z_2|^Q \leqslant |f(z_1) - f(z_2)| \leqslant 16|z_1 - z_2|^{1/Q}$. The exponents Q, $1/Q$ are the best possible (f running over S_Q, and z_1, z_2 over Δ), and it is conjectured (cf. [8], p. 71) that the best possible constants are 16^{1-Q} and $16^{1-1/Q}$ which have to replace 16^{-Q} and 16, respectively (16 being the best possible constant independent of Q, as proved by Mori).

Theorem 10. *For any $f \in E_Q$ and $z_1, z_2 \in \Delta$, $z_1 \neq z_2$, we have*

$$2^{1-Q}|z_1-z_2|^Q \leqslant |f(z_1)-f(z_2)| \leqslant 2^{1-1/Q}|z_1-z_2|^{1/Q}.$$

The constants 2^{1-Q}, $2^{1-1/Q}$ and exponents Q, $1/Q$ are the best possible, f running over E_Q, and z_1, z_2 over Δ. Equality holds if and only if either $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ real), f ($f \in E_Q$) being arbitrary, or $z_1 = re^{i\vartheta}$, $z_2 = -re^{i\vartheta}$ (r, ϑ real, $0 < r < 1$), $f(s) = |s|^{1/Q}e^{i\arg s}$ ($r \leqslant |s| \leqslant 1$), $f(s) = f(r)f^*(s/r)$ ($|s| < r$), f^* ($f^* \in E_Q$) being arbitrary — in the case of the upper bound. Equality holds if and only if either $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ real), f ($f \in E_Q$) being arbitrary, or $z_1 = re^{i\vartheta}$, $z_2 = -re^{i\vartheta}$ (r, ϑ real, $0 < r < 1$), $f(s) = |s|^Q e^{i\arg s}$ ($r \leqslant |s| \leqslant 1$), $f(s) = f(r)f^*(s/r)$ ($|s| < r$), f^* ($f^* \in E_Q$) being arbitrary — in the case of the lower bound. Moreover, the estimates obtained are also the best possible estimates of the form

$$c_2|z_1|^{\gamma_2}|z_2|^{\delta_2}|z_1-z_2|^Q \leqslant |f(z_1)-f(z_2)| \leqslant c_1|z_1|^{\gamma_1}|z_2|^{\delta_1}|z_1-z_2|^{1/Q}$$

where $c_1, c_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ are allowed to be dependent on Q only.

Proof. Suppose that the constants c, q are the best possible in the estimate $|f(z_1)-f(z_2)| \leqslant c|z_1-z_2|^q$, f running over E_Q, and z_1, z_2 over Δ; $z_1 \neq z_2$. By Theorem 1 and $f(0) = 0$ we have $c \geqslant 1$ and $q \leqslant 1/Q$. Since this theorem implies the estimates given in Theorem 10 in case where $z_1 = 0$ or $z_2 = 0$, we may assume $z_1 \neq 0$ and $z_2 \neq 0$ without loss of generality.

Let us consider the expression $|f(z_1)-f(z_2)|^Q/|z_1-z_2|$. Suppose, e.g., that $|z_1| \geqslant |z_2|$. By Theorem 9 we have

$$\frac{|f(z_1)-f(z_2)|^Q}{|z_1-z_2|} \leqslant \max_{\varphi_1}\left\{|z_1|^{Qq_1(\varphi_1)-1}\left|1-1/B_1\left(\left|\frac{z_1}{z_2}\right|, \arg\frac{z_1}{z_2}, \varphi_1\right)\right|^Q \Big/ \left|1-\frac{z_2}{z_1}\right|\right\},$$

where q_1 and B_1 are defined in the quoted theorem. Clearly, the expression in the braces, considered as a function of $|z_2|$, $|z_1/z_2|$, $\arg(z_1/z_2)$ and φ_1 is of the form $|z_2|^{Qq_1(\varphi_1)-1}B_1^*(|z_1/z_2|, \arg(z_1/z_2), \varphi_1)$ where B_1^* is independent of $|z_2|$. Hence it attains its maximum with respect to $|z_2|$ when $|z_2| = 1$ for any $|z_1/z_2|$, $\arg(z_1/z_2)$, φ_1, $\cos\varphi_1 \neq 1$, and it is a constant with respect to $|z_2|$ when $\varphi_1 = 0$ for any $|z_1/z_2|$, $\arg(z_1/z_2)$. By $|z_2| \leqslant |z_1| \leqslant 1$ we get

$$|f(z_1)-f(z_2)|^Q/|z_1-z_2| \leqslant \{2-2\cos\arg(z_1/z_2)\}^{\frac{1}{2}Q-\frac{1}{2}} \leqslant 2^{Q-1}.$$

Equality holds here if and only if either $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ real), f ($f \in E_Q$) being arbitrary, or $z_1 = re^{i\vartheta}$, $z_2 = -re^{i\vartheta}$ (r, ϑ real, $0 < r < 1$), $f(s) = |s|^{1/Q}e^{i\arg s}$ ($r \leqslant |s| \leqslant 1$), $f(s) = f(r)f^*(s/r)$ ($|s| < r$), f^* ($f^* \in E_Q$) being arbitrary. Now, by Lemma 1, we may apply the result obtained to the inverse function, whence $|f(z_1)-f(z_2)| \geqslant 2^{1-Q}|z_1-z_2|^Q$ with equality if and only if either $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ real), f ($f \in E_Q$) being arbitrary, or $z_1 = re^{i\vartheta}$, $z_2 = -re^{i}$ (r, ϑ real, $0 < r < 1$), $f(s) = |s|^Q e^{i\arg s}$ ($r \leqslant |s| \leqslant 1$), $f(s) = f(r)f^*(s/r)$ ($|s| < r$), f^* ($f^* \in E_Q$) being arbitrary.

The functions $f(s) = |s|^{1/Q} e^{i\arg s}$ ($s \neq 0$), $f(0) = 0$ and $f(s) = |s|^Q e^{i\arg s}$ ($s \neq 0$), $f(0) = 0$ give also the last conclusion in Theorem 10, and thus our proof is completed.

§ 6. Distortion theorem.

16. Here we derive an analogue of (29) for the class E_Q, as has been announced in Section 13. The result is a consequence of Theorem 3.

THEOREM 11. *For any $f \in E_Q$, $Q \in (1, +\infty)$ and $z \in \Delta$ we have*

$$|f(z) - z|/\log Q < 1/e \approx 0.37\,.$$

The estimate is sharp.

Proof. The proof is similar to that occurring in Shah Tao-shing's result (see [15]).

Let us use the notation of Theorem 3. Applying this theorem we have

$$|f(z) - z| = \left|\int_0^1 \frac{\partial}{\partial t} g(z, t)\,dt\right| \leqslant \int_0^1 \left|\frac{\partial}{\partial t} g(z, t)\right| dt$$
$$< 2\int_0^1 \frac{1}{t}|g(z, t)| \int_{|g(z,t)|}^1 \frac{(1/r)\,|\nu^*(r, t)|}{1 - |\nu^*(r, t)|^2}\,dr\,dt\,.$$

On the other hand, it can easily be verified that

$$\nu\big(g^{-1}(w, t), t\big) = -\nu^*(w, t)\exp\big(2i\arg g_w^{-1}(w, t)\big)\,.$$

Hence, by (13),

$$\nu^*(r, t) = -\nu\big(g^{-1}(r, t), t\big)\exp\big(-2i\arg g_w^{-1}(w, t)\big) \tag{33}$$
$$= -t\mu\big(g^{-1}(r, t)\big)\exp\big(-2i\arg g_w^{-1}(w, t)\big)\,,$$

and, consequently,

$$|f(z) - z| < 2\int_0^1 \int_{|g(z,t)|}^1 \frac{1}{r}|g(z, t)|\,\frac{\big|\mu\big(g^{-1}(r, t)\big)\big|}{1 - t^2\big|\mu\big(g^{-1}(r, t)\big)\big|^2}\,dr\,dt\,.$$

Since f is a Q-quasi-conformal mapping, we have

$$\big|\mu\big(g^{-1}(r, t)\big)\big| \leqslant q = (Q - 1)/(Q + 1) \quad \text{a.e. in } \Delta \ (0 \leqslant r \leqslant 1\,,\ 0 \leqslant t \leqslant 1)\,.$$

Hence we obtain

$$|f(z)-z| < 2\int_0^1 \int_{|g(z,t)|}^1 \frac{1}{r}|g(z,t)|\frac{q}{1-q^2t^2}\,dr\,dt$$

$$= 2\int_0^1 \frac{q}{1-q^2t^2}|g(z,t)|\log\frac{1}{|g(z,t)|}\,dt$$

$$= \frac{2}{e}\int_0^1 \frac{q}{1-q^2t^2}\,dt = \frac{1}{e}\log\frac{1+q}{1-q} = \frac{1}{e}\log Q\,,$$

as desired.

Suppose now that there is a constant $c_0 < 1/e$ such that

$$(34)\qquad |f(z)-z|/\log Q \leqslant c_0$$

for every $f \in E_Q$. Then (34) holds, in particular, for any function f_0 such that its complex dilatation $\mu_0(z)$ equals $q = (Q-1)/(Q+1)$, $1 < Q < +\infty$, in the whole disc Δ. Similarly, (34) holds for every g_0, where the functions $w = g_0(z,t)$, $0 \leqslant t \leqslant 1$, correspond to $w = f_0(z)$ as described in Theorem 3; but of course in (34) we have to replace

$$Q = \frac{1+q}{1-q} = \{1+\operatorname*{ess\,sup}_{z\in\Delta}|\mu_0(z)|\}/\{1-\operatorname*{ess\,sup}_{z\in\Delta}|\mu_0(z)|\}$$

by

$$\{1+\operatorname*{ess\,sup}_{z\in\Delta}|\nu_0(z,t)|\}/\{1-\operatorname*{ess\,sup}_{z\in\Delta}|\nu_0(z,t)|\} = \frac{1+qt}{1-qt}\,,$$

where the functions ν_0 replace ν in Theorem 3. This is a consequence of conditions $\nu_0(z,t) = t\mu_0(z)$ analogous to (13). Thus we have

$$|g_0(z,t)-z|/\log\frac{1+qt}{1-qt} \leqslant c_0 < 1/e \qquad (0 \leqslant t \leqslant 1)\,,$$

and also

$$(35)\qquad \lim_{t\to 0+}\left\{|g_0(z,t)-z|/\log\frac{1+qt}{1-qt}\right\} \leqslant c_0 < 1/e\,,$$

since, as is easy to show, the limit in (35) exists.

On the other hand, applying Theorem 3 to f_0 we have

$$\lim_{t\to 0+}\frac{g_0(z,t)-z}{t} = 2\lim_{t\to 0+}\left\{\frac{g_0(z,t)}{t}\int_{|g(z,t)|}^1 \frac{(1/r)\nu_0^*(r,t)}{1-|\nu_0^*(r,t)|^2}\,dr\right\}.$$

Hence, by (33), by the relation $\mu_0(z) = q$ for $z \in \Delta$, and by a well-known test concerning the integration of functions depending on a parameter, we get

$$\lim_{t\to 0+} \frac{g_0(z,t)-z}{t}$$

$$= -2\lim_{t\to 0+}\left\{g_0(z,t)\int_{|g(z,t)|}^{1} \frac{(1/r)\mu_0(g_0^{-1}(r,t))}{1-t^2|\mu_0(g_0^{-1}(r,t))|^2}\exp\left(-2i\arg\left[\frac{\partial}{\partial w}g_0^{-1}(w,t)\right]_{z=r}\right)dr\right\}$$

$$= -2z\int_{|z|}^{1}\frac{q}{r}dr = -2qz\log\frac{1}{|z|}.$$

Consequently,

$$\lim_{t\to 0+}\frac{g_0(z,t)-z}{qt} = -2z\log\frac{1}{|z|}$$

and

$$\lim_{t\to 0+}\left\{[g_0(z,t)-z]/\log\frac{1+qt}{1-qt}\right\}$$

$$= \lim_{t\to 0+}\left\{qt/\log\frac{1+qt}{1-qt}\right\}\lim_{t\to 0+}\frac{g_0(z,t)-z}{qt} = -z\log\frac{1}{|z|},$$

which contradicts (35) if we set $z = 1/e$. Thus our proof is completed.

Remark 6. The previous considerations also imply that for $f \in E_Q$, $Q \in \langle 1, +\infty)$ and $z \in \Delta$ we have

$$|f(z)-z| \leqslant c\log Q\{1+o(1)\}|z|\log(1/|z|),$$

where the best possible value of c is 1, the symbol o being connected with $Q \to 1+$. An analogous result for the class S_Q has been obtained by Krzyż and Ławrynowicz [6] (a particular case of Theorem 3 in the quoted paper).

§ 7. The class E_Q^*.

17. The class E_Q^* is an analogue of E_Q for functions defined in the closed plane $\mathcal{E}$. We give here six equivalent definitions for E_Q^*. The proofs of equivalence are omitted since they are analogous to that given in the case of E_Q. Before defining E_Q^* we introduce the class S_Q^*, which is an analogue of S_Q.

DEFINITION 3. A function f is said to be of class S_Q^* if it maps $\mathcal{E}$ onto itself Q-quasi-conformally with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

There are no analogues of Definitions 1B and 1C for S_Q^*.

DEFINITION 4A. A function f is said to be of class E_Q^* if it belongs to S_Q^* and if $f(z) = e^{i\arg z}f(|z|)$ for $z \in \mathcal{E}$, $z \neq 0, \infty$.

DEFINITION 4B. $f \in E_Q^*$ if $f \in S_Q^*$ and $f(z) = e^{-i\alpha} f(e^{i\alpha} z)$ for $z \in \mathcal{E}$, $z \neq 0, \infty$, where α is a real number such that α/π is irrational.

DEFINITION 4C. $f \in E_Q^*$ if $f \in S_Q^*$ and if its complex dilatation μ satisfies the condition $\mu(z) = e^{-2i\alpha}\mu(e^{i\alpha}z)$ a.e. in $\mathcal{E}$, where α is a real number such that α/π is irrational.

DEFINITION 4D. A function belonging to S_Q^* is said to be of class E_Q^* if its complex dilatation μ satisfies $\mu(z) = e^{2i\arg z}\mu(|z|)$ a.e. in $\mathcal{E}$.

DEFINITION 4E. $f \in E_Q^*$ if $f \in S_Q^*$ and $zf_z(z) - \bar{z}f_{\bar{z}}(z) = f(z)$ a.e. in $\mathcal{E}$.

DEFINITION 4F. $f \in E_Q^*$ if it is given by the formulae

$$(36)\quad \begin{aligned} &f(z) = \exp\left(-\int_{|z|}^{1} \frac{1+\mu(r)}{1-\mu(r)}\frac{dr}{r} + i\arg z\right) \quad \text{for } z \in \mathcal{E},\ z \neq 0, \infty, \\ &f(z) = z \quad \text{for } z = 0, \infty, \end{aligned}$$

where μ is measurable with $\sup_{0<r<+\infty} |\mu(r)| < 1$ and $\operatorname{ess\,sup}_{0<r<+\infty} |\mu(r)| \leqslant \dfrac{Q-1}{Q+1}$.

Definition 4A immediately implies:

$$(37)\quad \begin{aligned} |f(z)| &= |f(|z|)| \equiv R(|z|), \\ \arg\big(f(z)/z\big) &= \arg f(|z|) \equiv \theta(|z|) \quad (z \neq 0, \infty), \end{aligned}$$

and the following

LEMMA 2. *$f \in E_Q^*$ implies $f^{-1} \in E_Q^*$, and* (37) *implies*

$$(38)\quad \begin{aligned} |f^{-1}(w)| &= R^{-1}(|w|), \\ \arg\big(f^{-1}(w)/w\big) &= \arg f^{-1}(|w|) = -\theta\big(R^{-1}(|w|)\big) \quad (w \neq 0, \infty), \end{aligned}$$

where $\arg\big(f^{-1}(w)/w\big) = -\arg\big(w/f^{-1}(w)\big)$.

We also notice the following trivial result, which gives the correspondence between E_Q and E_Q^*:

LEMMA 3. *If a function f belongs to E_Q^*, then the functions f_1 and f_2 defined by $f_1(z) = f(z)$, $f_2(z) = 1/\overline{f(1/\bar{z})}$ for $z \in \Delta$, $z \neq 0$ and $f_1(0) = f_2(0) = 0$ both belong to E_Q.*

Remark 7. Clearly, if $f \in E_Q$, then any function f^* defined by $f^*(z) = f(z)$ for $z \in \Delta$, $f^*(z) = e^{-i\alpha}/\overline{f(e^{i\alpha}/\bar{z})}$ for $z \notin \Delta$, $z \neq \infty$, $f(\infty) = \infty$, where α is real, belongs to E_Q^*.

Lemma 3 enables us to obtain easily some analogues of Theorems 1, 2, 7, 8 and 11 for the class E_Q^*. An analogue of Theorem 9, and of Theorem 10 as well, can be obtained in a similar way only in the case where either $|z_1| \leqslant 1$, $|z_2| \leqslant 1$ or $|z_1| \geqslant 1$, $|z_2| \geqslant 1$. In order to get the corresponding estimates also in the case where $|z_1| < 1$, $|z_2| > 1$ or $|z_1| > 1$, $|z_2| < 1$ we need either some analogues of Theorems 5 and 6, where functionals dependent on one arbitrary function belonging to E_Q are replaced by functionals dependent on two arbitrary functions belonging to E_Q, or some analogues of Theorems 5 and 6, where functionals depend on one arbitrary function belonging to E_Q^*. We choose the second way, and hence we have to prove first an analogue of Theorem 3 or Theorem 4. For the completeness of our considerations we give here both of these analogues.

18. We shall now give two theorems on parametric representation for the class E_Q^*.

THEOREM 12. *Suppose that $w = f(z)$ belongs to E_Q^* and has $\omega = \mu(z)$ as its complex dilatation. Moreover, suppose that the functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, belong to S_Q^* and have complex dilatations* (13). *Then $w = g(z, t)$, considered as a function of z and t, satisfies on $\mathcal{E}' \times \{t\colon 0 \leqslant t \leqslant 1\}$ equation* (14) *subject to the initial condition $g(z, 0) = z$, where ν^* is the complex dilatation of g^{-1}, and $\mathcal{E}' = \mathcal{E} \setminus \{\infty\}$.*

Remark 8. By Definition 4D the functions $w = g(z, t)$, $0 \leqslant t \leqslant 1$, belong to E_Q^*.

Proof. We apply the theorem on parametrization for the class S_Q^* in the form which is an analogue of the theorem on parametrization for the class S_Q quoted in Section 9 (cf. also [16]). By this theorem $w = g(z, t)$ satisfies on $\mathcal{E}' \times \{t\colon 0 \leqslant t \leqslant 1\}$ the equation

$$\frac{\partial w}{\partial t} = \frac{w(1-w)}{\pi} \iint\limits_{|\zeta| < +\infty} \frac{\psi(\zeta, t)}{\zeta(1-\zeta)(w-\zeta)} d\xi\, d\eta \qquad (\zeta = \xi + i\eta)$$

subject to the initial condition $g(z, 0) = z$, where ψ is defined by (16). By (13) we have, as in the proof of Theorem 3,

$$\psi(w, t) = -\frac{(1/t)\nu^*(w, t)}{1 - |\nu^*(w, t)|^2}.$$

Hence

$$\frac{\partial w}{\partial t} = -\frac{w(1-w)}{\pi t} \iint\limits_{|\zeta| < +\infty} \frac{\nu^*(\zeta, t)}{1 - |\nu^*(\zeta, t)|^2} \frac{d\xi\, d\eta}{\zeta(1-\zeta)(z-\zeta)}.$$

Now we apply Definition 4D to g (cf. Remark 8). As in the proof of Theorem 3, we get

$$\frac{\partial w}{\partial t} = -\frac{w(1-w)}{\pi t} \int_0^{+\infty} \int_{-\pi}^{\pi} \frac{e^{2i\vartheta}\nu^*(r, t)}{1 - |\nu^*(r, t)|^2} \frac{r d\vartheta}{re^{i\vartheta}(1 - re^{i\vartheta})(w - re^{i\vartheta})} dr$$

$$= \frac{2w}{t} \int_{|w|}^{1} \frac{(1/r)\nu^*(r, t)}{1 - |\nu^*(r, t)|^2} dr,$$

as desired.

Remark 9. Theorem 12 can also be proved directly by means of Definition 4F.

THEOREM 13. *Under the hypotheses of Theorem 12 the function* $w = g(z, t)$, *considered as a function of* z *and* t, *satisfies on* $\mathcal{E}' \times \{t\colon 0 \leqslant t \leqslant 1\}$ *equation* (17) *subject to the initial condition* $g(z, 0) = z$.

This is an immediate consequence of Definition 4F.

19. Now we give an analogue of Theorems 5 and 6 for the class E_Q^*. Proofs are omitted since they can be performed in the same way as the proofs of Theorems 5 and 6.

THEOREM 14. *Theorems* 5 *and* 6 *remain valid if the following changes are made.*

(i) Δ, E_Q *and the conditions* $|z_k| \leqslant |z_{k-1}|$ $(k = 1, \ldots, n)$, $z_0 = 1$ *are replaced everywhere by* $\mathcal{E}$, E_Q^* *and the conditions* $|z_k| \leqslant |z_{k-1}|$ $(k = 1, \ldots, n_0-1, n_0+1, \ldots, n+1)$, $z_0 = z_{n+1} = 1$, $0 \leqslant n_0 \leqslant n+1$, *respectively.*

(ii) *The function* f_1 *is defined by* $f_1(s) = f(s)$ *if* $|z_{n_0-1}| < |s| < |z_{n_0}|$, *by* $f_1(s) = f(z_{n_0-1})f^*(s/z_{n_0-1})$ *if* $|s| \leqslant |z_{n_0-1}|$, *and by* $f_1(s) = f(z_{n_0})/\overline{f^{**}(\bar{z}_{n_0}/\bar{s})}$ *if* $|s| \geqslant |z_{n_0}|$, *where* $f^*, f^{**} \in E_Q$.

(iii) *Conditions* (18) *in Theorem* 5 *are replaced by*

$$\sum_{k=m+1}^{n_0-1} f(z_k)F_{\omega_k}\big(z_1, \ldots, z_n; f(z_1), \ldots, f(z_n)\big) \neq 0 \qquad (m = 0, \ldots, n_0-2),$$

$$\sum_{k=n_0}^{m-1} f(z_k)F_{\omega_k}\big(z_1, \ldots, z_n; f(z_1), \ldots, f(z_n)\big) \neq 0 \qquad (m = n_0+1, \ldots, n+1),$$

and analogous conditions for $f^{(\lambda,\varepsilon)}$ *and* f *in Theorem* 6 *are replaced by*

$$\sum_{k=m+1}^{n_0-1} f^{(\lambda,\varepsilon)}(z_k)F_{\omega_k}^{(\lambda)}\big(z_1, \ldots, z_n; f^{(\lambda,\varepsilon)}(z_1), \ldots, f^{(\lambda,\varepsilon)}(z_n)\big) \neq 0 \qquad (m = 0, \ldots, n_0-2),$$

$$\sum_{k=n_0}^{m-1} f^{(\lambda,\varepsilon)}(z_k)F_{\omega_k}^{(\lambda)}\big(z_1, \ldots, z_n; f^{(\lambda,\varepsilon)}(z_1), \ldots, f^{(\lambda,\varepsilon)}(z_n)\big) \neq 0 \qquad (m = n_0+1, \ldots, n+1)$$

and

$$\sum_{k=m+1}^{n_0-1} f(z_k)F_{\omega_k}^{(\lambda_*(\tau))}\big(z_1, \ldots, z_n; f(z_1), \ldots, f(z_n)\big) \neq 0 \qquad (m = 0, \ldots, n_0-2),$$

$$\sum_{k=n_0}^{m-1} f(z_k)F_{\omega_k}^{(\lambda_*(\tau))}\big(z_1, \ldots, z_n; f(z_1), \ldots, f(z_n)\big) \neq 0 \qquad (m = n_0+1, \ldots, n+1),$$

respectively.

(iv-a) *Formulae* (19) *and* (20) *in Theorem* 5 *are replaced by*

$$f(s) = w_m |s/z_m|^{\beta_m(z_1, \ldots, z_n; \varepsilon_m)} e^{i \arg(s/z_m)}$$

for $|z_{m+1}| \leqslant |s| \leqslant |z_m|$ $(m = 0, \ldots, n_0-2)$ *and for* $|z_m| \leqslant |s| \leqslant |z_{m-1}|$ $(m = n_0+ +1, \ldots, n+1)$, *and*

$$\beta_m(z_1, \ldots, z_n; \varepsilon_m) = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_m\Big(Q-\frac{1}{Q}\Big)\exp\Big(-i\arg\sum_{k=m+1}^{n_0-1} w_k F_{\omega_k}(z_1, \ldots, z_n; w_1, \ldots, w_n)\Big) \quad (m = 0, \ldots, n_0-2),$$

$$\beta_m(z_1, \ldots, z_n; \varepsilon_m) = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_m\Big(Q-\frac{1}{Q}\Big)\exp\Big(-i\arg\sum_{k=n_0}^{m-1} w_k F_{\omega_k}(z_1, \ldots, z_n; w_1, \ldots, w_n)\Big) \quad (m = n_0+1, \ldots, n+1),$$

$\varepsilon_m = 1$ *or* -1, $w_0 = 1$, $w_1 = f(z_1), \ldots, w_n = f(z_n)$, $w_{n+1} = 1$, *where the branch of* $\arg(f(s)/s)$ *is chosen for* $|z_{m+1}| \leqslant |s| < |z_m|$ $(m = 0, \ldots, n_0-2)$ *and for* $|z_m| < |s| \leqslant |z_{m-1}|$ $(m = n_0+1, \ldots, n+1)$ *so that* $f(s) \to w_m$ *as* $s \to z_m$ $(m = 0, \ldots, n_0-2, n_0+1, \ldots, n+1)$, *respectively.*

(iv-b) *The formulae which determine the functions* $f^{(\lambda,\varepsilon)}$ *in Theorem* 6 *are replaced by* $f^{(\lambda,\varepsilon)}(s) = s$ *for* $|s| = 1$ *and by*

$$f^{(\lambda,\varepsilon)}(s) = w_m^{(\lambda,\varepsilon)} |s/z_m|^{\beta_m(z_1, \ldots, z_n; \lambda, \varepsilon)} e^{i\arg(s/z_m)}$$

for $|z_{m+1}| \leqslant |s| < |z_m|$ $(m = 0, \ldots, n_0-2)$ *and for* $|z_m| < |s| \leqslant |z_{m-1}|$ $(m = n_0+1, \ldots, n+1)$, *where*

$$\beta_m(z_1, \ldots, z_n; \lambda, \varepsilon) = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_m\Big(Q-\frac{1}{Q}\Big)\exp\Big(-i\arg\sum_{k=m+1}^{n_0-1} w_k^{(\lambda,\varepsilon)} F_{\omega_k}^{(\lambda)}(z_1, \ldots, z_n; w_1^{(\lambda,\varepsilon)}, \ldots, w_n^{(\lambda,\varepsilon)})\Big) \quad (m = 0, \ldots, n_0-2),$$

$$\beta_m(z_1, \ldots, z_n; \lambda, \varepsilon) = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_m\Big(Q-\frac{1}{Q}\Big)\exp\Big(-i\arg\sum_{k=n_0}^{m-1} w_k^{(\lambda,\varepsilon)} F_{\omega_k}^{(\lambda)}(z_1, \ldots, z_n; w_1^{(\lambda,\varepsilon)}, \ldots, w_n^{(\lambda,\varepsilon)})\Big) \quad (m = n_0+1, \ldots, n+1),$$

$w_0^{(\lambda,\varepsilon)} = 1$, $w_1^{(\lambda,\varepsilon)} = f^{(\lambda,\varepsilon)}(z_1), \ldots, w_n^{(\lambda,\varepsilon)} = f^{(\lambda,\varepsilon)}(z_n)$, $w_{n+1}^{(\lambda,\varepsilon)} = 1$, *and the branch of* $\arg(f^{(\lambda,\varepsilon)}(s)/s)$ *is chosen for* $|z_{m+1}| \leqslant |s| < |z_m|$ $(m = 0, \ldots, n_0-2)$ *and for*

$|z_m| < |s| \leqslant |z_{m-1}|$ $(m = n_0+1, \ldots, n+1)$ *so that* $f^{(\lambda,\varepsilon)}(s) \to w_m^{(\lambda,\varepsilon)}$ *as* $s \to z_m$ $(m = 0, \ldots, n_0-2, n_0+1, \ldots, n+1)$, *respectively. The extremal function* f *satisfies a condition of the same form* $f(s) = f^{(\lambda_0(\tau),\varepsilon^0)}(s)$ *but in* $\{s\colon |z_{n_0-1}| \leqslant |s| \leqslant |z_{n_0}|\}$ *instead of* $\{s\colon |z_n| \leqslant |s| \leqslant 1\}$.

20. In this section an analogue of Theorem 9 for the class E_Q^* is established. According to the conclusion of Section 17, which was a consequence of Lemma 3, we confine ourselves to the case where $|z_1| \leqslant 1$ and $|z_2| \geqslant 1$.

THEOREM 15. (i) *For any* $f \in E_Q^*$ *and* $z_1, z_2 \in \mathcal{E}$, $z_1 \neq z_2$, $0 < |z_1| \leqslant 1$, $1 \leqslant |z_2| < +\infty$, *we have*

$$\arg(|z_1|^{p_{12}}e^{i\arg z_1} - |z_2|^{p_{22}}e^{i\arg z_2}) \leqslant \arg\big(f(z_1) - f(z_2)\big) \leqslant \arg(|z_1|^{p_{11}}e^{i\arg z_1} - |z_2|^{p_{21}}e^{i\arg z_2}),$$

where p_{mn} *are uniquely determined by the equations*

$$p_{mn} = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) + \frac{1}{2}\varepsilon_m(-1)^n i\Big(Q-\frac{1}{Q}\Big)\exp i\arg(1-[z_1^{p_{1n}}/z_2^{p_{2n}}]^{2m-3}) \quad (m = 1,2;\ n = 1,2),$$

$\varepsilon_m = 1$ *or* -1 $(m = 1, 2)$, $\arg\big(f(z_1) - f(z_2)\big) \to 0$ *for* $z_1 \to 0$, $z_2 = 1$ *with the correspondingly chosen branches of the estimating functions and* $1^{p_{mn}} = 1$ $(m = 1, 2;\ n = 1, 2)$.

(ii) *Moreover, the condition*

$$\arg\big(f(z_1) - f(z_2)\big) = \arg(|z_1|^{q_1(\varphi)}e^{i\arg z_1} - |z_2|^{q_2(\varphi)}e^{i\arg z_2}), \tag{39}$$

where q_1, q_2 *are uniquely determined by the equations*

$$q_m = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) - \frac{1}{2}\varepsilon_m\Big(Q-\frac{1}{Q}\Big)\exp i\{\varphi + \arg(1-[z_1^{q_1}/z_2^{q_2}]^{2m-3})\} \quad (m = 1, 2)$$

and $1^{q_1} = 1^{q_2} = 1$, *implies*

$$\Big||z_1|^{q_{12}(\varphi_2)}e^{i\arg z_1} - |z_2|^{q_{22}(\varphi_2)}e^{i\arg z_2}\Big| \leqslant |f(z_1) - f(z_2)| \leqslant \Big||z_1|^{q_{11}(\varphi_1)}e^{i\arg z_1} - |z_2|^{q_{21}(\varphi_1)}e^{i\arg z_2}\Big|, \tag{40}$$

where q_{mn} *are uniquely determined by the equations*

$$q_{mn} = \frac{1}{2}\Big(Q+\frac{1}{Q}\Big) + \frac{1}{2}\varepsilon_m'(-1)^n\Big(Q-\frac{1}{Q}\Big)\exp i\{\varphi_n + \arg(1-[z_1^{q_{1n}}/z_2^{q_{2n}}]^{2m-3})\} \quad (m = 1, 2;\ n = 1, 2),$$

$\varepsilon'_m = 1$ *or* -1 $(m = 1, 2)$, $1^{q_{mn}} = 1$ $(m = 1, 2;\ n = 1, 2)$, *and* φ_1, φ_2 *are uniquely determined as the solutions of the equations*

$$\arg(|z_1|^{q_{1n}(\varphi_n)} e^{i \arg z_1} - |z_2|^{q_{2n}(\varphi_n)} e^{i \arg z_2})$$
$$= \arg(|z_1|^{q_1(\varphi)} e^{i \arg z_1} - |z_2|^{q_2(\varphi)} e^{i \arg z_2}) \quad (-\tfrac{1}{2}\pi \leqslant \varphi_n \leqslant \tfrac{1}{2}\pi;\ n = 1, 2)\,.$$

All the given estimates are sharp for any $z_1, z_2 \in \mathcal{E}$, $z_1 \neq z_2$, $0 < |z_1| \leqslant 1$, $1 \leqslant |z_2| < +\infty$, *and* $Q \in \langle 1, +\infty)$. *Given* φ, $-\frac{1}{2}\pi \leqslant \varphi \leqslant \frac{1}{2}\pi$, *the only extremal functions in* (40) *are:*

$$\begin{aligned}
f(s) &= |s|^{q_{11}} e^{i \arg s} && (|z_1| \leqslant |s| \leqslant 1)\,,\\
f(s) &= |s|^{q_{21}} e^{i \arg s} && (1 < |s| \leqslant |z_2|)\,,\\
f(s) &= f(z_1) f_1^*(s/z_1) && (|s| < |z_1|)\,,\\
f(s) &= f(z_2)/\overline{f_1^{**}(\bar{z}_2/\bar{s})} && (|s| > |z_2|\,,\ s \neq \infty)\,,\\
f(s) &= \infty && (s = \infty)
\end{aligned}$$

for the upper bound, and

$$\begin{aligned}
f(s) &= |s|^{q_{12}} e^{i \arg s} && (|z_1| \leqslant |s| < 1)\,,\\
f(s) &= |s|^{q_{22}} e^{i \arg s} && (1 < |s| \leqslant |z_2|)\,,\\
f(s) &= f(z_1) f_2^*(s/z_1) && (|s| < |z_1|)\,,\\
f(s) &= f(z_2)/\overline{f_2^{**}(\bar{z}_2/\bar{s})} && (|s| > |z_2|\,,\ s \neq \infty)\,,\\
f(s) &= \infty && (s = \infty)
\end{aligned}$$

for the lower bound, where $f_1^*, f_1^{**}, f_2^*, f_2^{**}$ *are arbitrary functions of the class* E_Q, *and the branch of* $\arg(f(s)/s)$ *is chosen in each case so that* $\arg f(1) = 0$.

(iii) *Furthermore,* (39) *and* (40) *give all points of the variability region of the functional* $F(w_1, w_2) = \log(w_1 - w_2)$, *where* $\log 1 = 0$, $w_1 = f(z_1)$, $w_2 = f(z_2)$, f *ranges over* E_Q^*, *and* z_1, z_2 $(z_1, z_2 \in \mathcal{E},\ z_1 \neq z_2,\ 0 < |z_1| \leqslant 1,\ 1 \leqslant |z_2| < +\infty)$ *are fixed.*

The proof is completely analogous to that of Theorem 9.

21. As an application of Theorem 15 we obtain an analogue of Theorem 10 for the class E_Q^*. It is clear that there area no constants and q such that $|f(z_1) - f(z_2)| \leqslant c|z_1 - z_2|^q$, f running over E_Q^*, even in the case where we confine ourselves to z_1 and z_2 running over the exterior of Δ. Nevertheless, we can find an estimate of the form $|f(z_1) - f(z_2)| \leqslant c_1 |z_1|^{\gamma_1} |z_2|^{\delta_1} |z_1 - z_2|^{1/Q}$ (where c_1, γ_1, δ_1 do not depend on z_1, z_2) in each of the cases: (i) $|z_1| \leqslant 1$, $|z_2| \leqslant 1$, (ii) $|z_1| \geqslant 1$, $|z_2| \geqslant 1$, (iii) $|z_1| \leqslant 1$, $|z_2| \geqslant 1$, (iv) $|z_1| \geqslant 1$, $|z_2| \leqslant 1$, where we assume $z_1 \neq 0$ if $\gamma_1 \neq 0$; $z_2 \neq 0$ if $\delta_1 \neq 0$; and $z_1 \neq z_2$ in each of the cases under consideration. The result is as follows.

THEOREM 16. *Suppose that* $f \in E_Q^*$, $z_1 \neq z_2$, *and consider the following cases*: (i) $|z_1| \leqslant 1$, $|z_2| \leqslant 1$, (ii) $|z_1| \geqslant 1$, $|z_2| \geqslant 1$, (iii) $|z_1| \leqslant 1$, $|z_2| \geqslant 1$, (iv) $|z_1| \geqslant 1$, $|z_2| \leqslant 1$. *Then in each of the cases in question we have a pair of estimates of the form*

$$c_2|z_1|^{\gamma_2}|z_2|^{\delta_2}|z_1-z_2|^Q \leqslant |f(z_1)-f(z_2)| \leqslant c_1|z_1|^{\gamma_1}|z_2|^{\delta_1}|z_1-z_2|^{1/Q}.$$

The best possible values of $c_1, c_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ *corresponding to the cases indicated above are*:

(i) $c_1 = 2^{1-1/Q}$, $c_2 = 2^{1-Q}$, $\gamma_1 = 0$, $\gamma_2 = 0$, $\delta_1 = 0$, $\delta_2 = 0$,

(ii) $c_1 = 2^{1-1/Q}$, $c_2 = 2^{1-Q}$, $\gamma_1 = Q-\frac{1}{Q}$, $\gamma_2 = \frac{1}{Q}-Q$, $\delta_1 = Q-\frac{1}{Q}$, $\delta_2 = \frac{1}{Q}-Q$,

(iii) $c_1 = 2^{1-1/Q}$, $c_2 = 2^{1-Q}$, $\gamma_1 = 0$, $\gamma_2 = 0$, $\delta_1 = Q-\frac{1}{Q}$, $\delta_2 = \frac{1}{Q}-Q$,

(iv) $c_1 = 2^{1-1/Q}$, $c_2 = 2^{1-Q}$, $\gamma_1 = Q-\frac{1}{Q}$, $\gamma_2 = \frac{1}{Q}-Q$, $\delta_1 = 0$, $\delta_2 = 0$.

In case (i) *equality holds if and only if either* $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ *real*), f ($f \in E_Q^*$) *being arbitrary, or* $z_1 = re^{i\vartheta}$, $z_2 = -re^{i\vartheta}$ (r, ϑ *real*, $0 < r < 1$), $f(s) = |s|^{1/Q}e^{i\arg s}$ ($r \leqslant |s| \leqslant 1$), $f(s) = f(r)f^*(s/r)$ ($|s| < r$), $f(s) = 1/\overline{f^{**}(1/\bar{s})}$ ($|s| > 1$, $s \neq \infty$), $f(\infty) = \infty$, f^* *and* f^{**} ($f^*, f^{**} \in E_Q$) *being arbitrary — in the case of the upper bound. Similarly, equality holds if either* $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ *real*), f ($f \in E_Q^*$) *being arbitrary, or* $z_1 = re^{i\vartheta}$, $z_2 = -re^{i\vartheta}$ (r, ϑ *real*, $0 < r < 1$), $f(s) = |s|^Q e^{i\arg s}$ ($r \leqslant |s| \leqslant 1$), $f(s) = f(r)f^*(s/r)$ ($|s| < r$), $f(s) = 1/\overline{f^{**}(1/\bar{s})}$ ($|s| > 1$, $s \neq \infty$), $f(\infty) = \infty$, f^* *and* f^{**} ($f^*, f^{**} \in E_Q^*$) *being arbitrary — in the case of the lower bound. In case* (ii) *equality holds if and only if* $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ *real*), f ($f \in E_Q^*$) *being arbitrary — in the case of the upper bound and in the case of the lower bound as well. In case* (iii) *equality holds if and only if either* $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ *real*), f ($f \in E_Q^*$) *being arbitrary, or* $z_2 = \infty$, z_1 *and* f ($|z_1| \leqslant 1$, $f \in E_Q^*$) *being arbitrary — in the case of the upper bound and lower bound as well. In case* (iv) *equality holds if and only if either* $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ *real*), f ($f \in E_Q^*$) *being arbitrary, or* $z_1 = \infty$, z_2 *and* f ($|z_2| \leqslant 1$, $f \in E_Q^*$) *being arbitrary — in the case of the upper bound and the lower bound as well.*

Proof. Theorem 16 is equivalent to Theorem 10 in case (i), and the cases (iii) and (iv) are equivalent, so it remains to consider the cases (ii) and (iii) only.

Suppose first (ii). By Lemma 3 and Theorem 10 we have $|1/f(z_1)-1/f(z_2)| \leqslant 2^{1-1/Q}|1/z_1-1/z_2|^{1/Q}$ with the equality if and only if either $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ real), f ($f \in E_Q^*$) being arbitrary, or $z_1 = re^{i\vartheta}$, $z_2 = -re^{i\vartheta}$ (r, ϑ real, $1 < r < +\infty$), $f(s) = |s|^{1/Q}e^{i\arg s}$ ($1 \leqslant |s| \leqslant r$), $f(s) = f_1^*(s)$ ($|s| < 1$), $f(s) = f(r)/\overline{f_1^{**}(r/\bar{s})}$ ($|s| > r$, $s \neq \infty$), $f(\infty) = \infty$, f_1^* and f_1^{**} ($f_1^*, f_1^{**} \in E_Q^*$)

being arbitrary. On the other hand, by Lemma 3 and Theorem 1 we have $|1/f(z_m)| \geqslant |1/z_m|^Q$ $(m = 1, 2)$. Equality holds here for $m = 1, 2$ if and only if either $|z_m| = 1$, f $(f \in E_Q^*)$ being arbitrary, or $|z_m| > 1$, $f(s) = |s|^Q e^{i\arg s}$ $(1 \leqslant |s| \leqslant |z_m|)$, $f(s) = f_2^*(s)$ $(|s| < 1)$, $f(s) = f(z_m)/\overline{f_2^{**}(\bar{z}_m/\bar{s})}$ $(|s| > r,\ s \neq \infty)$, $f(\infty) = \infty$, f_2^* and f_2^{**} $(f_2^*, f_2^{**} \in E_Q)$ being arbitrary, for $m = 1, 2$, respectively. Hence, by Lemma 2, we obtain both the desired estimates and all the extremal cases when (ii) holds. Hence it follows, in particular, that the values obtained for c_1 and c_2 are the best possible. We claim that the values obtained for $\gamma_1, \gamma_2, \delta_1, \delta_2$ are also the best possible. Indeed, by the symmetry to this end it is enough to prove that the best possible value of δ_1 is $Q - 1/Q$. Suppose therefore that there is a constant $\delta_0 = \delta_0(Q) < Q - 1/Q$ such that $|f(z_1) - f(z_2)| \leqslant 2^{1-1/Q}|z_1|^{Q-1/Q}|z_2|^{\delta_0}|z_1 - z_2|^{1/Q}$ for every $|z_1| \geqslant 1$, $|z_2| \geqslant 1$, $z_1 \neq z_2$, and $f \in E_Q^*$. Hence this estimate holds, in particular, for $z_1^{(n)} = 1$, $z_2^{(n)} = n$ and $f(s) = |s|^Q e^{i\arg s}$ $(s \neq 0, \infty)$, $f(s) = s$ $(s = 0, \infty)$, where $n = 1, 2, \ldots$ Consequently

$$\delta_0 \geqslant \lim_{n\to+\infty} \left\{ \log\left(2^{-1+1/Q} \frac{n^Q + 1}{(n+1)^{1/Q}}\right) \Big/ \log n \right\} = Q - \frac{1}{Q},$$

which contradicts $\delta_0 < Q - 1/Q$, and thus the proof is completed in case (ii).

Suppose next (iii). Since Theorem 16 is a consequence of Lemma 3 and Theorem 1 in the case where $z_1 = 0$, and is trivial in the case where $z_2 = \infty$, we may assume $z_1 \neq 0$ and $z_2 \neq \infty$ without loss of generality. Let us consider the expression $|f(z_1) - f(z_2)|^Q / |z_2|^{Q^2-1}|z_1 - z_2|$. By Theorem 15 we have

$$\frac{|f(z_1) - f(z_2)|^Q}{|z_2|^{Q^2-1}|z_1 - z_2|} \leqslant \max_{\varphi_1} \left\{ |z_2|^{Qq_{21}(\varphi_1) - Q^2} \frac{\left| |z_1/z_2|^{q_{21}(\varphi_1)} e^{i\arg(z_1/z_2)} - 1 \right|^Q}{|z_1/z_2 - 1|} \times \right.$$
$$\left. \times \left| \frac{|z_1/z_2|^{q_{21}(\varphi_1)} e^{i\arg(z_1/z_2)} |z_1|^{q_{11}(\varphi_1) - q_{21}(\varphi_1)} - 1}{|z_1/z_2|^{q_{21}(\varphi_1)} e^{i\arg(z_1/z_2)} - 1} \right|^Q \right\},$$

where q_{21} and q_{11} are defined in the quoted theorem. Clearly, $|z_2|^{q^*} \leqslant 1$, $q^* = Qq_{21}(\varphi_1) - Q^2$, with equality if and only if either $|z_2| = 1$, or $q_{21}(\varphi_1) = Q$. Next we observe that

$$\left| \left|\frac{z_1}{z_2}\right|^{q_{21}(\varphi_1)} e^{i\arg(z_1/z_2)} - 1 \right|^Q \Big/ \left| \frac{z_1}{z_2} - 1 \right| \leqslant \left\{ 2 - 2\cos\arg\frac{z_1}{z_2} \right\}^{\frac{1}{2}Q - \frac{1}{2}}$$

with equality if and only if $|z_1| = |z_2| = 1$, and, finally,

$$\left| \left|\frac{z_1}{z_2}\right|^{q_{21}(\varphi_1)} e^{i\arg(z_1/z_2)} |z_1|^{q_{11}(\varphi_1) - q_{21}(\varphi_1)} - 1 \right| \leqslant \left| \left|\frac{z_1}{z_2}\right|^{q_{21}(\varphi_1)} e^{i\arg(z_1/z_2)} - 1 \right|$$

with equality if and only if either $|z_1| = 1$ (also if $|z_1| \to 0$) or $q_{11}(\varphi_1) = q_{21}(\varphi_1)$. Consequently $|f(z_1) - f(z_2)|^Q / |z_2|^{Q^2-1}|z_1 - z_2| \leqslant 2^{Q-1}$. Equality holds here if and only if $z_1 = e^{i\vartheta}$, $z_2 = -e^{i\vartheta}$ (ϑ real), f $(f \in E_Q^*)$ being arbitrary. Hence it follows, in particular, that the value obtained for c_1 is the best possible. Now, by Lemma 2, we get an analogous result for the lower bound. It remains to prove that the best possible value of δ_1 is $Q-1/Q$. This can be done with the help of the same example as in case (ii): $z_1^{(n)}=1$, $z_2^{(n)}=n$ $(n=1,2,\ldots)$ and $f(s)=|s|^Q e^{i\arg s}$ $(s\neq 0,\infty)$, $f(s)=s$ $(s=0,\infty)$. Thus the proof is completed in all cases.

Conclusions.

22. We conclude the paper with some remarks on possible ways of generalization of the results obtained.

The first way of generalization is to consider classes of Q-quasiconformal mappings connected with an equation

$$\mu^*(z)=e^{2i\arg h'(\bar z)}\overline{\mu^*\big(h(\bar z)\big)} \tag{41}$$

in the same sense as the class E_Q is connected with (2) (cf. Definition 2C), where μ^* denotes the complex dilatation of the continued function f^*, and h, $h(z)\not\equiv z$, is a fixed, arbitrarily chosen homography. Clearly, functions of any class under consideration should be defined in a domain D which is invariant under $w=h(\bar z)$, i.e. $D=h(\bar D)$ where $\bar D=\{z\colon \bar z\in D\}$. Any such (closed) domain will be called *natural* with respect to $w=h(\bar z)$. It can be proved that equation (41) corresponds to $h\big(\overline{f^*(z)}\big)=f^*\big(h(\bar z)\big)$ in the same way as (2) corresponds to $e^{i\alpha}/\overline{f^*(z)}=f^*(e^{i\alpha}/\bar z)$, i.e. to $f^*(z)=e^{-i\alpha}/\overline{f^*(e^{i\alpha}/\bar z)}$ (cf. the equivalence of Definition 2C and 2B).

The same way of generalization may also be applied to the class E_Q^*. In this case we consider classes of Q-quasi-conformal mappings connected with an equation

$$\mu(z)=e^{-2i\arg h'(z)}\mu\big(h(z)\big) \tag{42}$$

in the same sense as E_Q^* is connected with the equation $\mu(z)=e^{-2i\alpha}\mu(e^{i\alpha}z)$, where μ denotes the complex dilatation of f, h, $h(z)\not\equiv z$, is a fixed, arbitrarily chosen homography, and functions of any class under consideration are defined in a domain D which is invariant under $w=h(z)$, i.e. $D=h(D)$; any such (closed) domain will be called *natural* with respect to $w=h(z)$.

Now, by a well-known theorem on homographies (see e.g. [14], pp. 86-87), the investigation of the classes of mappings generated by all equations (41) and (42) can be reduced to the investigation of some *normalized* classes. Here we give a list of some normalized classes and the corresponding normalized (closed) natural domains D, and we present the proposed names and notation. Throughout this list a denotes a real number and n a positive integer. Mappings are always assumed to have 0 and 1 as their invariant points in the case where D is the closed unit disc, and $0, 1, \infty$ in the other listed cases.

(i) $h(\bar{z}) = e^{i\alpha}/\bar{z}$, $\alpha = 2\pi/n$, $n \neq 1$; $D = \{z\colon |z| \leqslant 1\}$ — the class $E_Q^{(n)}$ of *n-cyclic elliptic* Q-quasi-conformal mappings.

(ii) $h(\bar{z}) = e^{i\alpha}/\bar{z}$, α/π irrational; $D = \{z\colon |z| \leqslant 1\}$ — the class E_Q of *limit elliptic* Q-quasi-conformal mappings. Here the adjective "limit" is justified by an obvious relation $E_Q = \bigcap\limits_n E_Q^{(n)}$.

(iii) $h(\bar{z}) = a\bar{z}$, $1 < a < +\infty$; $D = \{z\colon 0 \leqslant \arg z \leqslant \pi\}$ — the class $H_Q^{(a)}$ of *a-discrete hyperbolic* Q-quasi-conformal mappings.

(iv) $h(\bar{z}) = a\bar{z}$, $a \to 1$; $D = \{z\colon 0 \leqslant \arg z \leqslant \pi\}$ — the class H_Q of *limit hyperbolic* Q-quasi-conformal mappings. Here "$a \to 1$" means that equation (41) should read as follows: there is a sequence of real numbers a_k, $a_k \to 1$ as $k \to +\infty$, such that $\mu^*(z) = \overline{\mu^*(a_k\bar{z})}$ for $z \in D$ and $k = 1, 2, \ldots$ Clearly $H_Q = \bigcap\limits_a H_Q^{(a)}$.

(v) $h(\bar{z}) = \bar{z}+1$; $D = \{z\colon 0 \leqslant \arg z \leqslant \pi\}$ — the class P_Q of *normalized parabolic* Q-quasi-conformal mappings.

(vi) $h(z) = e^{i\alpha}z$, $\alpha = 2\pi/n$, $n \neq 1$; $D = \mathcal{E}$ (the closed plane) — the class $E_Q^{*(n)}$ of *n-cyclic continued elliptic* Q-quasi-conformal mappings.

(vii) $h(z) = e^{i\alpha}z$, α/π irrational; $D = \mathcal{E}$ — the class E_Q^* of *limit continued elliptic* Q-quasi-conformal mappings. Clearly $E_Q^* = \bigcap\limits_n E_Q^{*(n)}$.

(viii) $h(z) = az$, $1 < a < +\infty$; $D = \mathcal{E}$ — the class $H_Q^{*(a)}$ of *a-discrete continued hyperbolic* Q-quasi-conformal mappings.

(ix) $h(z) = az$, $a \to 1$; $D = \mathcal{E}$ — the class H_Q^* of *limit continued hyperbolic* Q-quasi-conformal mappings. Here "$a \to 1$" means that equation (42) should read as follows: there is a sequence of real numbers a_k, $a_k \to 1$ as $k \to +\infty$, such that $\mu(z) = \mu(a_k z)$ for $z \in \mathcal{E}$ and $k = 1, 2, \ldots$ Clearly $H_Q^* = \bigcap\limits_a H_Q^{*(a)}$.

(x) $h(z) = z+1$; $D = \mathcal{E}$ — the class P_Q^* of *normalized continued parabolic* Q-quasi-conformal mappings.

In this place I should like to express my thanks to Dr J. Chądzyński for his remarks concerning this classification.

23. It is particularly interesting to consider the classes H_Q and H_Q^*, which are "dual" to E_Q and E_Q^*, respectively, have a clear geometric interpretation and may be widely applied in physics. Here duality is understood in the sense that we obtain some analogues of Definitions 2A and 4A by replacing in them the condition $f(z) = e^{i\arg z}f(|z|)$ by $f(z) = |z|f(e^{i\arg z})$. Since the same domain, namely the closed plane $\mathcal{E}$, is a natural domain for E_Q^* and H_Q^* as well, it is natural to consider the classes $E_Q^{(1)} = E_Q^* \circ H_Q^*$, $E_Q^{(n)} = E_Q^{(1)} \circ E_Q^{(n-1)}$ $(n = 2, 3, \ldots)$ or $H_Q^{(1)} = H_Q^* \circ E_Q^*$, $H_Q^{(n)} = H_Q^{(1)} \circ H_Q^{(n-1)}$ $(n = 2, 3, \ldots)$, where $A \circ B$ denotes the class of all compositions $f \circ g$, i.e. $w = f(g(z))$; $f \in A$, $g \in B$. These classes have also a clear interpretation, and the methods given for E_Q^* can be transferred to them.

24. The second way of generalization of the results obtained is to consider various classes of quasi-conformal mappings which are solutions of Beltrami differential equations with separated variables, e.g. mappings with the complex dilatation of one of the forms shown in Section 3. The classes E_Q, E_Q^*, H_Q, H_Q^* may serve here as examples. Two of these classes, E_Q and E_Q^*, consist of mappings f which transform concentric circles $\{z: |z| = r\}$ ($0 \leqslant r \leqslant 1$ in the case of E_Q and $0 < r < +\infty$ in the case of E_Q^*) onto concentric circles $\{w: |w| = |f(r)|\}$; the other two consist of mappings which transform concentric rays $\{z: \arg z = \vartheta\}$ ($0 \leqslant \vartheta \leqslant \pi$ in the case of H_Q and $-\pi < \vartheta \leqslant \pi$ in the case of H_Q^*) onto concentric rays $\{w: \arg w = \arg f(e^{i\vartheta})\}$. It is natural to replace here the family of concentric circles or of concentric rays by another family of curves, e.g. by a family of logarithmic spirals. In this way we may try to investigate most of the important plane quasi-conformal mappings. This is an analogue of the situation in the theory of partial differential equations, where a great number of results have been obtained with the help of equations with separated variables.

It is also natural to look for some analogues of the classes E_Q, E_Q^*, H_Q, H_Q^* etc. in three and higher dimensions, and to apply them to various three-dimensional physical problems.

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