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Four different unknown functions satisfying the triangle mean value property for harmonic polynomials, II

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Abstract. If f_j: $C \to R$ satisfy the quasi-triangle mean value property $f_0(x+y)+f_1(x+\theta y)+$ $+f_2(x+\theta^2 y) = 3f_3(x)$ for all $x, y \in C$, $\theta = \exp(2\pi i/3)$, then there exist generalized quadratic polynomials such that $f_j(x) = Q_j^0+Q_j^1(x)+Q_j^2(x)$ for all $x \in C$. In addition if f_j are bounded on a set of positive Lebesgue measure, then f_j are given by harmonic polynomials of degree $\leqslant 2$.

3. Reduction to generalized quadratic polynomials. In the previous note (*) we found the continuous solution of equation (M). In this note it is our purpose to prove the following two extensions of a theorem proved in (*).

One of them is:

THEOREM 1. *Let R be the set of all real numbers and let C be the set of all complex numbers. If f_j ($j = 0, 1, 2, 3$): $C \to R$ satisfy the quasi-triangle mean value property*

$$\text{(M)} \qquad \sum_{j=0}^{2} f_j(x+\theta^j y) = 3f_3(x)$$

for all $x, y \in C$, where θ is a number $\exp(2\pi i/3)$, then there exist generalized quadratic polynomials (¹) *such that*

$$\text{(3.1)} \qquad f_j(x) = Q_j^0+Q_j^1(x)+Q_j^2(x)$$

for all $x \in C$ and for each $j = 0, 1, 2, 3$, where

(i) *Q_j^0 are real constants,*

(ii) *Q_j^1: $C \to R$ are additive functions and*

(iii) *Q_j^2: $C \to R$ are symmetric bi-additive functions, i.e., $Q_j^2(x) = Q_{j,2}(x, x)$ and $Q_{j,2}(x, y)$: $C \times C \to R$ are symmetric bi-additive.*

The other is:

THEOREM 2. *Let μ denote Lebesgue measure on $R \times R$ and let $\Omega \subset R \times R$ be a measurable subset with $\mu(\Omega) > 0$. Let the functions $f_j(x) \equiv u_j(x_1, x_2)$:*

(*) Ann. Polon. Math. 33 (1977), 219–221.

(¹) See [1], [4].

$\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$, $j=0, 1, 2, 3$, *be solutions of* (M). *Then* f_j *are given by* (3.1) *for all* $x\in\boldsymbol{C}$. *In addition, if* u_j *are bounded on* Ω, *then* u_j *are continuous everywhere and are harmonic polynomials of degree* $\leqslant 2$.

4. Proof of Theorem 1. Rewrite (M) as

$$(T_0^y+T_1^{\theta y}+T_2^{\theta^2 y}-3T_3^0)f(x)=0 \tag{4.1}$$

for all $x, y\in\boldsymbol{C}$, where the shift operator is previously defined by

$$T_j^z f(x)=f_j(x+z) \quad \text{and} \quad T_j^0 f(x)=f_j(x)$$

for each $j=0, 1, 2, 3$ and for all $x, z\in\boldsymbol{C}$.

Replace x by $x-\theta z_1$ and y by $y+z_1$ in (4.1). Then clearly

$$(T_0^{(1-\theta)z_1+y}+T_1^{\theta y}+T_2^{\theta^2 y+(\theta^2-\theta)z_1}-3T_3^{-\theta z_1})f=0 \tag{4.2}$$

for all $x, y, z_1\in\boldsymbol{C}$.

If the difference operator Δ_y is defined by

$$\Delta_y f_j(x)=(T_j^y-T_j^0)f(x) \quad \text{for all } x, y\in\boldsymbol{C},$$

then by taking a difference of (4.2) and (4.1) one obtains

$$\Delta_{(1-\theta)z_1}f_0(x+y)+\Delta_{(\theta^2-\theta)z_1}f_2(x+\theta^2 y)-3\Delta_{-\theta z_1}f_3(x)=0. \tag{4.3}$$

Similarly, by replacing x by $x-\theta^2 z_2$ and y by $y+z_2$ in (4.3), we infer

$$\Delta_{(1-\theta)z_1}f_0\big(x+y+(1-\theta^2)z_2\big)+\Delta_{(\theta^2-\theta)z_1}f_2(x+\theta^2 y)- -3\Delta_{-\theta z_1}f_3(x-\theta^2 z_2)=0 \tag{4.4}$$

for all $x, y, z_1, z_2\in\boldsymbol{C}$, which with (4.3) implies

$$\Delta_{(1-\theta)z_1}\Delta_{(1-\theta^2)z_2}f_0(x+y)-3\Delta_{-\theta z_1}\Delta_{-\theta^2 z_2}f_3(x)=0. \tag{4.5}$$

Finally, set $y=z_3$ in (4.5). Then

$$\Delta_{(1-\theta)z_1}\Delta_{(1-\theta^2)z_2}f_0(x+z_3)-3\Delta_{-\theta z_1}\Delta_{-\theta^2 z_2}f_3(x)=0$$

which with (4.5) implies the desired equation

$$\Delta_{(1-\theta)z_1}\Delta_{(1-\theta^2)z_2}\Delta_{z_3}f_0(x)=0$$

for all $x, z_1, z_2, z_3\in\boldsymbol{C}$, and therefore

$$\Delta_{z_3}^3 f_0(x)=0 \quad \text{for all } x, z_3\in\boldsymbol{C}. \tag{4.6}$$

Hence it immediately follows from the results of S. Mazur and W. Orlicz [3] with equation (4.6) that there exists a quadratic polynomial such that

$$f_0(x)=Q_0^0+Q_0^1(x)+Q_0^2(x) \quad \text{for all } x\in\boldsymbol{C}, \tag{4.7}$$

where functions Q_0^j, $j=0, 1, 2, 3$, are defined in Theorem 1.

In view of equation (M) it is clear that obvious modifications can be repeated for the terms f_1, f_2 and f_3 to obtain

$$\Delta_u^3 f_j(x) = 0 \quad \text{for each } j = 1, 2, 3 \text{ and for all } x, u \in C,$$

since $\theta^m \neq \theta^p$, $m \neq p$ for m, $p = 1, 2, 3$. Theorem 1 is proved.

5. Proof of Theorem 2. By (4.7) and the additivity of Q_0^j we obtain ([3])

$$f_0(Nx) = Q_0^0 + NQ_0^1(x) + N^2 Q_0^2(x) \quad \text{for } N = 1, 2, 3. \tag{5.1}$$

System (5.1) is clearly solved for Q_0^j, $j = 0, 1, 2$, in terms of $f_0(Nx)$, $N = 1, 2, 3$, since the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{vmatrix} \neq 0.$$

But $|f_0(x)| \equiv |u_0(x_1, x_2)|$ is bounded for all $(x_1, x_2) \in \Omega$. Hence Q_0^0, $Q_0^1(x)$ and $Q_0^2(x)$ for all x are bounded on a set of positive Lebesgue measure. Further, the identity

$$Q_{0,2}(x, y) = \tfrac{1}{4}\big(Q_{0,2}(x+y, x+y) - Q_{0,2}(x-y, x-y)\big)$$

for all $x, y \in C$ shows that $Q_{0,2}(x, y)$ is also bounded on a set of positive Lebesgue measure, since $Q_{0,2}(x+y, x+y) = Q_0^2(x+y)$ and $Q_0^2(x-y)$ are bounded. If one briefly defines $Q_{0,2}(x, y) = Q_{0,2}(x_1, x_2, x_3, x_4)$, then by the bi-additivity of $Q_{0,2}$ one readily obtains

(5.2)
$$Q_{0,2}(x_1, x_2, x_3, x_4) = a_1(x_1, x_3) + a_2(x_1, x_4) + a_3(x_2, x_3) + a_4(x_2, x_4),$$

where a_1, a_2, a_3, a_4: $\boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ are additive in the first and second variables separately. Moreover, (5.2) implies

$$a_1(x_1, x_3) = Q_{0,2}(x_1, x_2, x_3, x_4) + Q_{0,2}(x_1, -x_2, x_3, -x_4) + \\ + Q_{0,2}(x_1, -x_2, x_3, x_4) + Q_{0,2}(x_1, x_2, x_3, -x_4).$$

This shows that a_1 is bounded. Similarly, a_2, a_3, a_4 are bounded.

It now follows by well-known theorems of additive functions ([2]) that Q_0^j, $j = 1, 2, 3$, must be continuous everywhere and, by (5.1), so is f_0. Similarly, if $|u_j| < k$, $k > 0$, $j = 0, 1, 2$, for all $(x_1, x_2) \in \Omega$, then f_1, f_2 and f_3 are also continuous everywhere. Hence the theorem in (*) immediately implies that f_j, $j = 0, 1, 2, 3$, are given by harmonic polynomials of degree $\leqslant 2$. This proves Theorem 2.

6. Corollaries. As consequences of Theorem 1 and Theorem 2, we obtain the following corollaries.

([2]) See [2].

COROLLARY 1. *A function f: $C \to R$ or C satisfies the triangle mean value property*

$$\sum_{j=0}^{2} f(x+\theta^j y) = 3f(x) \quad \text{for all } x, y \in C \tag{6.1}$$

if and only if there exists a generalized quadratic polynomial such that

$$f(x) = Q^0 + Q^1(x) + Q^2(x) \quad \text{for all } x \in C, \tag{6.2}$$

where Q^0 is a real or a complex constant and an additive function Q^1: $C \to R$ or C and a symmetric bi-additive function Q^2: $C \to R$ or C must satisfy the equation

$$\sum_{j=0}^{2} [Q^1(\theta^j y) + 2Q_2(x, \theta^j y) + Q^2(\theta^j y)] = 0 \quad \text{for all } x, y \in C. \tag{6.3}$$

Proof. By Theorem 1 we obtain (6.2). Substitute (6.2) in (6.1) to obtain (6.3).

COROLLARY 2. *Let f: $C \to C$. Then the measurably bounded solution of (6.1) is given by a complex polynomial such that*

$$f(x) = a_0 + a_1 x + a_2 \bar{x} + a_3 x^2 + a_4 x\bar{x} + a_5 \bar{x}^2 \tag{6.4}$$

where a_k, $k = 0, 1, \ldots, 5$, are complex constants.

Proof. If f: $C \to C$ satisfies (6.1), then f is given by (6.2). It readily follows from a similar proof of Theorem 2 that Q^1, Q^2: $C \to C$ are also bounded on a set of positive Lebesgue measure and hence are continuous everywhere. The continuous additive function Q^1: $C \to C$ is given by $Q^1(x) = ax + b\bar{x}$ for all $x \in C$ where $\bar{x}$ denotes the conjugate of x. Hence, by the bi-additivity of Q^2, we obtain

$$Q_2(x_1, x_2) = a(x_2)x_1 + b(x_2)\bar{x}_1 \tag{6.5}$$

for all x_1, $x_2 \in C$, and a, b: $C \to C$. But by the symmetry of Q_2 we have

$$a(x_2)x_1 + b(x_2)\bar{x}_1 = a(x_1)x_2 + b(x_1)\bar{x}_2. \tag{6.6}$$

Set $x_1 = 1$ and $x_1 = i$ in (6.6) to obtain

$$a(x_2) + b(x_2) = a(1)x_2 + b(1)\bar{x}_2 \tag{6.7}$$

and $a(x_2)i - b(x_2)i = a(i)x_2 + b(i)\bar{x}_2$ which implies

$$-a(x_2) + b(x_2) = a(i)ix_2 + b(i)i\bar{x}_2. \tag{6.8}$$

If we solve equations (6.7) and (6.8) for a and b, then $b(x_2) = c_1 x_2 + c_2 \bar{x}_2$ and $a(x_2) = c_3 x_2 + c_4 \bar{x}_2$ with complex constants c_1, c_2, c_3 and c_4, which with (6.5) yield $Q^2(x) = a_3 x^2 + a_4 x\bar{x} + a_5 \bar{x}^2$. Hence we obtain (6.4). Conversely, (6.4) satisfies (6.1).

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