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The sufficient condition for linear constant time-lag systems to be normal systems

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Abstract. In this paper we consider two problems: the sufficient conditions under which a control system is a normal system and an estimation of a number of switching times for bang-bang control.

Introduction. A system of differential equations with a retarded argument is a mathematical model of many physical and technological processes. The topic treated is a study of time-optimal steering described by a linear stationary system of differential equations with retarding in state variables.

We consider here two important problems: One — given in detail in Section 2 — concerns conditions under which a control system is a normal system (cf. Definition 1.3), the other — contained in Section 3 — deals with an estimation of the number of switching times for bang-bang control.

In Section 1 we summarize briefly a number of questions from optimal control theory which will play a basic role throughout the paper. We give also some examples illustrating the main theses.

Some knowledge about the solutions and Pontriagin's maximum principle allows us to find an effective time-optimal control, but in the case where a control system is not normal, Pontriagin's maximum principle does not give any information about the time-optimal control.

Analogous problems for processes described by a linear stationary system of ordinary differential equations have already been solved (cf. [4]). However, retardings of the argument in a control process give dificulites of theoretical and calculatory nature. Eoughly speaking, the components of any solution of a linear stationary system of differential equations are analytic functions whereas the components of any solution of the same system with a retarded argument are, in general, only continuous.

In order to overcome these difficulties we apply a method which consists in changing "it every step" a linear stationary system of differential equations with a retarded argument by an equivalent (cf. Definition 2.1) linear homogeneous stationary system of ordinary differential equations.

This change is possible only when the initial function for a system with a retarded argument fulfils certain assumptions and it is made at the cost of raising in the order of the system of homogeneous ordinary differential equations.

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1. Consider the linear control process with time delay

$$\dot{x}(t) = Ax(t)+Bx(t-h)+Cu(t)+g(t), \tag{1}$$

where $0 < h < \infty$ is constant, the state vector $x(t)$ is an n-vector, the controller $u(t)$ is an r-vector, A and B are $(n \times n)$ real constant matrices, C is an $(n \times r)$ real constant matrix, and $g(t)$ is a continuous n-vector function.

The problem is to find a measurable control function $u(t)$ on $[0, t_1]$ from the given compact, but not necessarily convex, restraint set $Q \subset R^r$ which steers the system response $x(t)$ from the given continuous initial function $\varphi(t)$, $-h \leqslant t \leqslant 0$, to continuously moving closed target set $G(t) \subset R^n$ in minimum time t^* $(t^* \leqslant t_1)$.

The following theorems will be used only in the exact formulation of some problems. Therefore we omit the proofs, which can be found in [2] and [5].

DEFINITION 1.1. The obtainable set $\mathscr{X}(t_1)$ of system (1) with the initial function $\varphi(t)$ is the set of all endpoints $x(t_1)$ which can be obtained from the initial function $\varphi(t)$, $-h \leqslant t \leqslant 0$, at $t = t_1$ by the use of all measurable controllers $u(t)$ with $u(t) \in Q$ on $[0, t_1]$.

DEFINITION 1.2. A controller $u(t)$ will be called *extremal* on $[0, t_1]$ if it steers the corresponding response $x(t)$ to the boundary $\partial\mathscr{X}(t_1)$ of the obtainable set $\mathscr{X}(t_1)$ at t_1, that is, $x(t_1) \in \partial\mathscr{X}(t_1)$.

THEOREM 1.1. *Consider system* (1). *A controller* $u(t) \in Q$ *on* $[0, t_1]$ *is extremal if and only if there exists a non-trivial solution* $\eta(t)$ *of the adjoint equation*

$$\begin{aligned} \dot{\eta}(t) &= -\eta(t)A-\eta(t+h)B, & 0 \leqslant t \leqslant t_1-h, \\ \dot{\eta}(t) &= -\eta(t)A, & t_1-h \leqslant t \leqslant t_1, \end{aligned} \tag{2}$$

such that

$$\eta(t)\,Cu(t) = \max_{u \in Q} \eta(t)\,Cu$$

almost everywhere on $[0, t_1]$. *Furthermore,* $\eta(t_1)$ *is outward normal to* $\mathscr{X}(t_1)$ *at* $x(t_1)$.

DEFINITION 1.3. System (1) will be called *normal* (see also [2]) on $[0, t_1]$ if no component of $\eta(t)C$ is identically zero on any subinterval of $[0, t_1]$ for all non-zero vectors $\eta(t_1)$.

The following is a uniqueness condition for extremal controllers:

THEOREM 1.2. *Consider system (1) with a compact restraint set $Q \subset R^r$ containing more than one point. If the system is normal on $[0, t_1]$, then $\mathscr{K}(t_1)$ is strictly convex and hence it has a non-empty interior in R^n. Furthermore, if two controllers $u_1(t)$ and $u_2(t)$ steer their corresponding responses $x_1(t)$ and $x_2(t)$ to the same boundary point of $\mathscr{K}(t_1)$, i.e., $x_1(t_1) = x_2(t_1) \in \partial\mathscr{K}(t_1)$, then $u_1(t) = u_2(t)$ almost everywhere on $[0, t_1]$.*

The following theorem gives the existence of a time optimal controller:

THEOREM 1.3. *Consider system (1) with a compact controller restraint set $Q \subset R^r$ and a continuously moving closed target set $G(t) \subset R^n$. If there exists one measurable controller $u(t) \in Q$ which steers the initial function $\varphi(t)$ to the target set $G(t)$ at $t = t_1 < \infty$, there exists a time optimal controller $u^*(t) \in Q$ on $0 \leqslant t \leqslant t^*$, where $t^* \leqslant t_1$.*

The following theorem gives a necessary condition for a time optimal controller:

THEOREM 1.4 (Maximum principle). *The time optimal control $u^*(t) \in Q$ on $[0, t^*]$ of Theorem 1.3 is an extremal control, i.e., $u^*(t^*) \in \partial\mathscr{K}(t^*)$ and so, by Theorem 1.1, there exists a non-trivial adjoint solution $\eta(t)$ such that*

$$\eta(t)Cu^*(t) = \max_{u \in Q} \eta(t)Cu$$

almost everywhere on $[0, t^*]$, where $\eta(t^*)$ is an outward normal to $\mathscr{K}(t^*)$ at $x^*(t^*)$.

From Theorem 1.2 it is apparent that if the system is normal, then the time optimal control is unique. Since the optimal control is given by $\eta(t)Cu^*(t) = \max_{u \in Q} \eta(t)Cu$, if Q is a given hypercube $|u_i| \leqslant 1$, $i = 1, 2, \ldots, r$, then

$$u^*(t) = \operatorname{sgn} \eta(t)C,$$

where $\operatorname{sgn} \nu = \operatorname{sgn}(\nu_1, \nu_2, \ldots, \nu_r) = (\operatorname{sgn}\nu_1, \operatorname{sgn}\nu_2, \ldots, \operatorname{sgn}\nu_r)$, and so $u^*(t)$ is a bang-bang controller.

In the next part of the paper we will use the following terms:

DEFINITION 1.4. If c_s denotes a vector whose components are the elements of the s-th column of matrix C ($s = 1, 2, \ldots, r$), then the scalar function $\sigma_s(t) = \eta(t)c_s$ is called the *switching function*. In other words, the switching function is the scalar product of the vectors $\eta(t)$ and c_s.

DEFINITION 1.5. System (1) is called *s-normal* on $[0, t_1]$, $s = 1, 2, \ldots, r$, if for every $\eta(t_1) \neq 0$ the switching function $\sigma_s(t)$ is non-identically zero on any subinterval of $[0, t_1]$.

DEFINITION 1.6. If system (1) is s-normal on $[0, t_1]$, $s = 1, 2, \ldots, r$, then the zeros of the swithing function $\sigma_s(t)$ will be called the *swithing times* for s-th component of the controller $u(t)$.

Clearly, system (1) is normal on $[0, t_1]$ iff the system is s-normal on this interval for each $s = 1, \ldots, r$.

2. We shall now formulate the conditions for the matrices A, B and C under which system (1) is normal. Let us notice that if we put $t \to -t+t_1$ in the adjoint system (2) and denote $z(t) = \eta(-t+t_1)$, then system (2) assumes the form:

(3) $$\dot{z}(t) = z(t)A + z(t-h)B$$

with the initial conditions

(4) $$z(t) = 0, \quad t \in [-h, 0); \quad z(0) = \overset{0}{z} = \eta(t_1).$$

Let us write

$$E^s_{[0,t_1]} = \{t \in [0, t_1]\colon\ \sigma_s(t) = z(t)c_s = 0\}.$$

Since $\sigma_s(t)$ is piecewise analytical, we can formulate the following

Remark 1. System (1) is s-normal on $[0, t_1]$ iff the set $E^s_{[0,t_1]}$ is finite for each $\overset{0}{z} \neq 0$ ($t_1 < \infty$).

Naturally, if system (1) is s-normal on $[0, t_1]$, then is also normal on every $[0, t^*]$, where $t^* \leqslant t_1$.

In the case where the matrix B is a null-matrix, the system is described by a system of equations of the form

(1′) $$\dot{x}(t) = Ax(t) + Cu(t), \quad x(0) = \overset{0}{x}.$$

A n.a.s.c. for s-regularity of this system on each $[0, T]$ is

$$\operatorname{rank}(c_s, Ac_s, \ldots, A^{n-1}c_s) = n.$$

The following example can show the difficulties in formulating conditions for the matrices A, B and C which guarantee the finiteness of $E^s_{[0,t_1]}$ for every $\overset{0}{z} \neq 0$.

EXAMPLE. Consider the system

$$\dot{z}(t) = z(t)A + z(t-1)B_a,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_a = \begin{pmatrix} 0 & \dfrac{a}{e} + e \\ a & 0 \end{pmatrix}, \quad C = c_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with the initial condition

$$z(t) = (z_1(t), z_2(t)) = (0, 0), \quad t \in [-1, 0); \quad z(0) = (e, -1).$$

$z(t) = (e^{t+1}, -e^{2t})$ is the solution of this system on $[0, 1]$. However, on $[1, 2]$, the system has the form

$$\dot{z}_1(t) = z_1(t) - \frac{a}{e^2} e^{2t}, \qquad z_1(1) = e^2,$$

$$\dot{z}_2(t) = 2z_2(t) + \left(\frac{a}{e} + e\right)e^t, \quad z_2(1) = -e^2,$$

and the solution is

$$z_1(t) = \left(\frac{a}{e} + e\right)e^t - \frac{a}{e^2} e^{2t},$$

$$z_2 t = -\left(\frac{a}{e} + e\right)e^t + \frac{a}{e^2} e^{2t}.$$

Since $\operatorname{rank}(c_1, Ac_1) = 2$, for every $z(0) \neq 0$ the set $E^1_{[0,1]}$ is a finite set (at most a one-point set), whereas for $z(0) = (e, -1)$ we have $E^1_{[0,1]} = \{1\}$, $E^1_{[0,2]} = [1, 2]$. Thus we see that the set $E^1_{[0,2]}$ is an interval even though for $a = e^2/(e-1)$ the matrix B_a is non-singular and symmetric.

We shall introduce the following operation:

$$R^{kn} \ni y \to y_{r,s} = (y_{rn+1}, y_{rn+2}, \ldots, y_{sn}) \in R^{(s-r)n},$$
$$k \in N, \quad 0 \leqslant r < s \leqslant k.$$

DEFINITION 2.1. The system of differential equations with a retarded argument (3), (4) is called *equivalent* on $[a, b]$ to a system of ordinary differential equations

$$\dot{y}(t) = y(t)D, \quad y(a) = \overset{0}{y},$$

where D is a real $m \times m$ matrix ($m \geqslant n$), if $z(t) = \big(y_{m-n+1}(t), y_{m-n+2}(t), \ldots, y_m(t)\big)$ on $[a, b]$, i.e. the sequence of the n components of $y(t)$ is identical with the solution $z(t)$ on $[a, b]$.

Let us introduce matrices D_k which will play an important role in the sequel.

We set

(5) $$D_1 = A, \quad D_2 = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}, \quad D_k = \begin{pmatrix} A & B & & & 0 \\ & A & B & & \\ & & A & \ddots & \\ & & & \ddots & B \\ & & & A & B \\ 0 & & & & A \end{pmatrix}.$$

D_k is a $kn \times kn$ matrix and contains $n \times n$ blocks. On the main diagonal there are k matrices A, above $k-1$ matrices B; the other elements are equal to zero.

Consider the following system on $[(k-1)h, kh]$:

(6) $$\dot{y}^k(t) = y^k(t)D_k$$

with the initial condition

(7) $$y^k\big((k-1)h\big) = \overset{k}{y},$$

where $\overset{k}{y}$ are defined inductively by

(8) $$\overset{1}{y} = \overset{0}{z}, \quad \overset{k}{y} = \Big(\overset{k-1}{y}, y^{k-1}_{k-2,k-1}\big((k-1)h\big)\Big).$$

THEOREM 2.1. *For every $k \in N$ there exists such a matrix D_k and vector $\overset{k}{y}$ that system* (3), (4) *is equivalent on* $[(k-1)h, kh]$ *to system* (6), (7).

Proof. We shall prove this theorem by induction. Let $z^k(t)$ denote a solution of system (3), (4) on $[(k-1)h, kh]$. We assume that Theorem 2.1 is valid for $k = p$ and we shall show that on $[ph, (p+1)h]$ we have

(9) $$z^{p+1}(t) = y^{p+1}_{p,p+1}(t).$$

It is sufficient to show that $z^{p+1}(t)$ and $y^{p+1}_{p,p+1}(t)$ are solutions of the same stationary system of differential equations. Conditions (5), (6), (7) and (8) imply that $y^{p+1}_{p,p+1}(t)$ fulfils the following system:

(10) $$\dot{y}^{p+1}_{p,p+1}(t) = y^{p+1}_{p,p+1}(t)A + y^{p+1}_{p-1,p}(t)B$$

with the initial condition

(11) $$y^{p+1}_{p,p+1}(ph) = y^{p}_{p-1,p}(ph).$$

From (5), (6), (7) and (8) we have also

(12) $$y^{p+1}_{0,p}(t) = \overset{p}{y}e^{(t-ph)D_p};$$

moreover, for $k = p$ in (6) and (7), it follows that on $[(p-1)h, ph]$

$$y^p(t) = \overset{p}{y}e^{(t-ph+h)D_p};$$

and hence on $[ph, (p+1)h]$

(13) $$y^p(t-h) = \overset{p}{y}\, e^{(t-ph)D_p}.$$

Utilising (12) and (13), we get

$$y^{p+1}_{0,p}(t) = y^p(t-h);$$

consequently

(14) $$y^{p+1}_{p-1,p}(t) = y^{p}_{p-1,p}(t-h).$$

According to the induction hypothesis we have on $[ph, (p+1)h]$

(15) $$y^{p}_{p-1,p}(t-h) = z^p(t-h).$$

Finally, from (10), (11), (14) and (15) it follows that, on $[ph, (p+1)h]$, $y^{p+1}_{p,p+1}(t)$ fulfils the system

$$\dot{y}^{p+1}_{p,p+1}(t) = y^{p+1}_{p,p+1}(t)A + z^p(t-h)B \tag{16}$$

with the initial condition

$$y^{p+1}_{p,p+1}(ph) = z^p(ph). \tag{17}$$

Solving system (3), (4) by the step-method, we can see that $z^{p+1}(t)$ fulfils on $[ph, (p+1)h]$ the system

$$\dot{z}^{p+1}(t) = z^{p+1}(t)A + z^p(t-h)B \tag{18}$$

with the initial condition

$$z^{p+1}(ph) = z^p(ph). \tag{19}$$

From (16), (17), (18) and (19) it follows that $y^{p+1}_{p,p+1}(t)$ and $z^{p+1}(t)$ are, on $[ph, (p+1)h]$, solutions of the same system of equations with the same initial conditions.

This ends the proof.

We shall now prove a well-known lemma.

LEMMA 1. *A sufficient condition for the set*

$$E_{[(k-1)h, kh]} = \{t \in [(k-1)h, kh]: \sigma(t) = y^k(t)u = 0, u \in R^{kn}\}$$

to be finite for every $\overset{k}{y} \neq 0$ *is* $\operatorname{rank}(u, D_k u, \ldots, D_k^{kn-1} u) = kn$.

Proof. Assume that the set $E_{[(k-1)h, kh]}$ is infinite. By virtue of the analyticity of the function $\sigma(t)$ the contradiction of our thesis is equivalent to the assertion $E_{[(k-1)h, kh]} = [(k-1)h, kh]$, i.e.,

$$y^k(t)u = 0, \quad t \in [(k-1)h, kh].$$

By differentiating this equation $kn-1$ times and by (6) we obtain the following homogeneous system of equations:

$$y^k(t)u = 0,$$
$$y^k(t)D_k u = 0,$$
$$y^k(t)D_k^{kn-1} u = 0.$$

This system is fulfilled by $y^k((k-1)h) = \overset{k}{y} \neq 0$, and thus $\operatorname{rank}(u, D_k u, \ldots, D_k^{kn-1} u) < kn$. This completes the proof.

Let us introduce the notation

$$R^{kn} \ni c_s^k = (0, 0, \ldots, 0, c_s^*)^* \tag{20}$$

(a star means transposition).

From Theorem 2.1 and the definition of c_s^k it follows that for $t \in [(k-1)h, kh]$ we have

$$z^k(t)c_s = y_{k-1,k}^k(t)c_s = y^k(t)c_s^k. \tag{21}$$

If we use (21) in Lemma 1 and next recall Remark 1, we shall get the following lemmas:

LEMMA 2. *A sufficient condition for system* (1) *to be s-normal system on* $[(k-1)h, kh]$ *is* $\operatorname{rank}(c_s^k, D_k c_s^k, \ldots, D_k^{kn-1}c_s^k) = kn$.

LEMMA 3. *If* $u \in R^{(k+1)n}$, $v = u_{1,k+1} \in R^{kn}$, $\operatorname{rank}(u, D_{k+1}u, \ldots, D_{k+1}^{(k+1)n-1}u) = (k+1)n$, *then* $\operatorname{rank}(v, D_k v, \ldots, D_k^{kn-1}v) = kn$.

Proof. Suppose that

$$\operatorname{rank}(v, D_k v, \ldots, D_k^{kn-1}v) = q < kn. \tag{22}$$

Since D_{k+1} has on the main diagonal $k+1$ matrices A und under the diagonal-zeros, we obtain the following expressions for the last $k+1$ rows of the matrix $(u, D_{k+1}u, \ldots, D_{k+1}^{(k+1)n-1}u)$:

$$(v, D_k v, \ldots, D_k^{(k+1)n-1}v). \tag{23}$$

Since D_k is a $kn \times kn$ matrix, then from (22) and the Cayley–Hamilton Theorem it follows that

$$\operatorname{rank}(v, D_k v, \ldots, D_k^{(k+1)n-1}v) = q. \tag{24}$$

Using (23) and (24), we obtain

$$\operatorname{rank}(u, D_{k+1}u, \ldots, D_{k+1}^{(k+1)n-1}u) = n+q < (k+1)n,$$

which contradicts our assumption.

THEOREM 2.2. *A sufficient condition for system* (1) *to be s-normal on* $[0, kn]$ *is* $\operatorname{rank}(c_s^k, D_k c_s^k, \ldots, D_k^{kn-1}c_s^k) = kn$.

Remark 2. Since system (1) is normal iff it is s-normal for $s = 1, 2, \ldots, r$, we infer that system (1) is normal on $[0, kh]$ if $\operatorname{rank}(c_s^k, D_k c_s^k, \ldots, D_k^{kn-1}c_s^k) = kn$.

The condition given in Theorem 2.2 is not necessary as the following example shows:

EXAMPLE. Consider the system of equations

$$\dot{z}(t) = z(t)A + z(t-1)B$$

with the initial condition

$$z(t) = 0, \quad t \in [-1, 0); \quad z(0) = (z_1(0), z_2(0)) = \overset{0}{z} \neq 0,$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = c_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It is easy to verify that

$$\operatorname{rank}(c_1^2, D_2 c_1^2, D_2^2 c_1^2, D_2^3 c_1^2) = \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 12 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix} = 3,$$

i.e., the condition given in Theorem 2.2 is not satisfied.

Solving the system by the step-method we get

$$z_1(t) = \overset{0}{z}_1 e^t, \qquad t \in [1, 2],$$
$$z_2(t) = \overset{0}{z}_2 (t + e^2 - 1) e^{2(t-1)}, \qquad t \in [1, 2];$$

hence

$$\sigma_1(t) = \dot{z}(t) c_1 = \overset{0}{z}_1 e^t + \overset{0}{z}_2 (t + e^2 - 1) e^{2(t-1)}.$$

Since $\overset{0}{z} \neq 0$, the form of the function $\sigma_1(t)$ leads us to the conslusion that the set $E_{[1,2]}$ contains at most two elements and $E_{[0,2]}$ at most three elements. This means that the system under consideration is normal on $[0, 2]$.

Lemma 4. *Given a real $m \times m$ matrix D, if there exists a vector $u \in R^m$ such that* $\operatorname{rank}(u, Du, \ldots, D^{m-1}u) = m$, *then* $\operatorname{rank} D \geqslant m-1$.

Proof. Suppose that $\operatorname{rank} D = q < m-1$. To the matrix D corresponds a linear mapping $D: R^m \to R^m$ and a q-dimensional subspace $V \subset R^m$ such that

$$D: R^m \to V, \qquad V \subset R^m, \qquad \dim V = q < m-1,$$

and hence it immediately follows that for $k \in N$

$$D^k: R^m \to V.$$

Consequently, the vectors $Du, D^2u, \ldots, D^{m-1}u$ are contained in V. Thus we have

$$\operatorname{rank}(u, Du, \ldots, D^{m-1}u) \leqslant 1 + q < m,$$

and this fact contradicts our assumption.

Lemma 5. *For the matrix D_k given in* (5) *we have*:

a) *if the Jordan matrix Λ is similar to A, i.e., $\Lambda = Q^{-1}AQ$, then*

$$\Lambda_k = \begin{pmatrix} \Lambda & Q^{-1}BQ & & 0 \\ & \Lambda & Q^{-1}BQ & \\ & & \ddots & Q^{-1}BQ \\ 0 & & & \Lambda \end{pmatrix}$$

is similar to D_k.

b) *if* λ_i *are eigenvalues of the matrix* A *with multiplicities* l_i, $i = 1, 2, \ldots, m$, *then the* λ_i *are eigenvalues of the matrix* D_k *with multiplicities* kl_i.

c) *if* $\operatorname{rank} A = n$, *then* $\operatorname{rank} D_k = kn$,

d) *if* $\operatorname{rank} D_2 = 2n-1$, *then* $\operatorname{rank} D_k = kn-1$, $k = 2, 3, \ldots$

Proof. In order to prove proposition a) it will suffice to show that there exists a non-singular $kn \times kn$ matrix Q_k such that $Q_k^{-1} D_k Q_k = \Lambda_k$. From the assumption we know that there exists a matrix Q satisfying the equation $Q^{-1} A Q = \Lambda$. It is easy to testify that

$$Q_k = \begin{pmatrix} Q & & & 0 \\ & Q & & \\ & & \ddots & \\ 0 & & & Q \end{pmatrix}$$

is the matrix in question. Cases b) and c) follow immediately from a). Case d) we will prove by induction. It is well known that rank $D_k = kn-1$ iff the dimension of the kernel of the linear mapping D_k: $R^{kn} \to R^{kn}$ is unity. Suppose that a vector $u \in R^{(k+1)n}$ satisfies the system

$$D_{k+1} u = 0,$$

which can also be written in the form

$$D_{k+1} u = \begin{pmatrix} D_{k-1} u_{0,k-1} + C_{k-1} u_{k-1,k} \\ D_2 u_{k-1,k+1} \end{pmatrix} = 0,$$

where C_{k-1} is a $(k-1)n \times n$ matrix and $C_{k-1}^* = (0, 0, \ldots, 0, B^*)$.

In view of this equality the induction scheme is evident.

This completes the proof of Lemma 5.

Assume now that rank $A = n-1$ and write

$$d = Bv, \quad \text{where } 0 \neq v \in \operatorname{Ker} A. \tag{25}$$

Consider the following system:

$$D_2 x = \begin{pmatrix} A x_{0,1} + B x_{1,2} \\ A x_{1,2} \end{pmatrix} = 0.$$

The last of the equations of this type has the solution

$$x_{1,2} = av, \quad a \in R,$$

and hence the system of the first n equations can be written in the form

$$A x_{0,1} = -ad. \tag{26}$$

Notice that $\operatorname{rank} D_2 = 2n-1$ iff system (26) has a solution only for $a = 0$. This is the case iff $\operatorname{rank}(A, d) = n$.

Remark 3. If $\operatorname{rank} A = n-1$, then $\operatorname{rank} D_2 = 2n-1$ iff $\operatorname{rank}(A, d) = n$.

Using Lemma 4, propositions c) and d) of Lemma 5 and Remark 3, we may now state the following theorem, dealing with necessary conditions of non-singularity of $(c_s^k, D_k c_s^k, \ldots, D_k^{kn-1} c_s^k)$.

THEOREM 2.3. *Given D_k, c_s^k as above (cf. (5), (20) and (25)), the following implications hold:*

$$\operatorname{rank}(c_s^k, D_k c_s^k, \ldots, D_k^{kn-1} c_s^k) = kn \Rightarrow \operatorname{rank} D_k \geqslant kn-1$$
$$\Leftrightarrow \operatorname{rank} D_2 = 2n-1 \Leftrightarrow \operatorname{rank}(A, d) = n.$$

3. Henceforth we confine our attention to an estimation of the number of switching times for the s-th component of a control vector $u(t)$ on $[0, kh]$ when the control system is s-normal on this interval.

We first recall a well-known lemma:

LEMMA 6. *If $f_i(t)$ are polynomials with real coefficients of degrees r_i and λ_i are different real numbers $i = 1, 2, \ldots, m$, then the quasi-polynomial*

$$\sum_{i=1}^{m} f_i(t) e^{\lambda_i t} \tag{27}$$

has at most $\sum_{i=1}^{m} r_i + m - 1$ real zeros.

Proof. It is seen at once that Lemma 6 holds for $m = 1$. We shall prove the passage from $m-1$ to m. Multiplying function (27) by $e^{-\lambda_m t}$, we obtain the function

$$\sum_{i=1}^{m-1} f_i(t) e^{(\lambda_i - \lambda_m)t} + f_m(t), \tag{28}$$

whose zeros are the same as for function (27). Suppose now that the estimation given in Lemma 6 does not hold, i.e., that function (27) (and consequently (28)) has at least $\sum_{i=1}^{m} r_i + m$ zeros. Between two consecutive zeros of a differentiable function there exists at least one zero of its derivative. Hence the derivative of the r_m+1 order of function (28) has at least $\sum_{i=1}^{m} r_i + m - (r_m+1)$ real zeros. This derivative has the form

$$\sum_{i=1}^{m-1} \bar{f}_i(t) e^{(\lambda_i - \lambda_m)t}, \tag{29}$$

where $\bar{f}_i(t)$ are polynomials of degrees r_i, $i = 1, 2, \ldots, n-1$. In other words, function (29) is the same type as function (27), but it has only $m-1$ terms. In view of the induction hypothesis function (29) has at most $\sum_{i=1}^{m-1} r_i + (m-1) - 1$ zeros. Thus the supposition that Lemma 6 is true for quasi-polynomials with $m-1$ terms and false for quasi-polynomials with m terms leads to contradiction.

THEOREM 3.1. *Consider system* (1) *and suppose that all eigenvalues of A are real. We can state that:*

a) *if system* (1) *is s-normal on* $[(k-1)h, kh]$, *then the number of switching times on this interval for the s-th component of* $u(t)$ *is at most* $kn-1$;

b) *if system* (1) *is s-normal on* $[0, kh]$, *then the number of switching times on this interval is at most* $k(k+1)n/2-k$.

Proof. It will suffice to prove a) since b) follows at once from a) if we apply the formula for the arithmetical progression sum.

If all the eigenvalues λ_i of A are real with multiplicities l_i, $i=1, 2, \ldots, m$, then, by virtue of Lemma 5b), λ_i are the eigenvalues of D_k with multiplicities kl_i. Hence each component $y_j^k(t)$, $j=1,2,\ldots,kn$, of the solution $y^k(t)$ of system (6), (7) is of the form

$$\sum_{i=1}^{m} f_i(t)e^{\lambda_i t}, \tag{30}$$

where $f_i(t)$ are polynomiala with real coefficients whose degrees are $r_i \leqslant kl_i - -1$.

In view of Theorem 2.1, $y_{k-1,k}^k(t) = z^k(t)$ and components of $z^k(t)$ are functions of form (30). The switching function $\sigma_s(t) = z^k(t)c_s$ is a linear combination of these components, and hence for $t \in [(k-1)h, kh]$ we have

$$\sigma_s(t) = \sum_{i=1}^{m} \bar{f}_i(t)e^{\lambda_i t}, \tag{31}$$

where $\bar{f}_i(t)$ are polynomials of degrees $\bar{r}_i \leqslant kl_i - 1$.

In view of Lemma 6, the switching function $\sigma_s(t)$ given by (31) has at most $\sum_{i=1}^{m} \bar{r}_i + m - 1$ real zeros.

Since

$$\sum_{i=1}^{m} \bar{r}_i + m - 1 = \sum_{i=1}^{m} (\bar{r}_i + 1) - 1 \leqslant \sum_{i=1}^{m} kl_i - 1 = kn - 1,$$

h9 ; witching function $\sigma_s(t)$ possesses at most $kn-1$ real zeros on $[(k-1)h, kh]$, and this completes the proof of a).

The estimation given in Theorem 3.1 can be obtained without referring to Theorem 2.1. Namely, when we solve system (3), (4) by the step-method, we solve in fact a non-homogeneous stationary system of differential equations with a quasi-polynomial as a non-homogeneity term. Using this fact, we may prove by induction that every component of the solution $z^k(t)$ is of form (30). We proceed further as in the proof of Theorem 3.1.

Now we may show that the "smoothing effect" for solutions of a system of differential equations with a retarded argument allows us to improve the estimation given in Theorem 3.1. Let us consider a finite sequence of real numbers $-\infty < t_1 < t_2 < \ldots < t_k < \infty$ and natural numbers p_j, $j = 1, 2, \ldots, k$.

Let the function

$$(32)\qquad F(t) = \begin{cases} \sum_{i=1}^{m} f_{i,1}(t)e^{\lambda_i t}, & t \in (-\infty, t_1], \\ \sum_{i=1}^{m} f_{i,j}(t)e^{\lambda_i t}, & t \in (t_{j-1}, t_j], \\ \sum_{i=1}^{m} f_{i,k+1}(t)e^{\lambda_i t}, & t \in (t_k, \infty), \end{cases}$$

where $f_{ij}(t)$ are polynomials with real coefficients of degrees r_{ij}, resp., and λ_i are different real numbers $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k+1$, fulfil at points t_j the following conditions:

$$(33)\qquad \left.\frac{d^r \sum_{i=1}^{m} f_{i,j}(t)e^{\lambda_i t}}{dt^r}\right|_{t_j} = \left.\frac{d^r \sum_{i=1}^{m} f_{i,j+1}(t)e^{\lambda_i t}}{dt^r}\right|_{t_j},$$

for $r = 0, 1, \ldots, p_j$.

LEMMA 7. *If $F(t)$ given by* (32) *satisfies* (33), *then*

a) *if $p_j \leqslant \sum_{i=1}^{m} \min(r_{ij}, r_{ij+1}) + m - 1$, then the $F(t)$ has for $k = 2p-1$ at most*

$$\sum_{j=1}^{k+1}\Big(\sum_{i=1}^{m} r_{ij} + m - 1\Big) - \sum_{r=1}^{p} p_{2r-1}$$

real zeros; however, for $k = 2p$ it has at most

$$\sum_{j=1}^{k+1}\Big(\sum_{i=1}^{m} r_{ij} + m - 1\Big) - \sum_{r=1}^{p} p_{2r}$$

real zeros;

b) *if numbers R_{ij} satisfy the inequalities $r_{ij} \leqslant R_{ij}$, $p_j \leqslant \sum_{i=1}^{m} \min(R_{ij}, R_{ij+1}) + m - 1$, then $F(t)$ has for $k = 2p-1$ at most*

$$\sum_{j=1}^{k+1}\Big(\sum_{i=1}^{m} R_{ij} + m - 1\Big) - \sum_{r=1}^{p} p_{2r-1}$$

real zeros; however, for $k = 2p$ it has at most

$$\sum_{j=1}^{k+1}\Big(\sum_{i=1}^{m} R_{ij} + m - 1\Big) - \sum_{r=1}^{p} p_{2r}$$

real zeros.

Proof. We shall first prove cases a) and b) for $k=1$. Case a) as will be proved by induction on l, where l is the least number from $1, 2, \ldots, m$ satisfying the condition

$$p_1 \leqslant \sum_{i=1}^{l} \min(r_{i,1}, r_{i,2}) + l - 1.$$

Proof for $l=1$ (i.e., $p_1 \leqslant \min(r_{1,1}, r_{1,2})$).

The number of zeros of $F(t)$ does not change if we multiply $F(t)$ by $e^{-\lambda_1 t}$. Thus we obtain the function

$$\bar{F}(t) = \begin{cases} f_{1,1}(t) + \sum_{i=2}^{m} f_{i,1}(t) e^{(\lambda_i - \lambda_1)t}, & t \leqslant t_1, \\ f_{1,2}(t) + \sum_{i=2}^{m} f_{i,2}(t) e^{(\lambda_i - \lambda_1)t}, & t > t_1. \end{cases}$$

Suppose that $\bar{F}(t)$ possesses at least $\sum_{i=1}^{m} (r_{i,1} + r_{i,2}) + 2m - 1 - p_1$ real zeros. Since between two successive zeros of a differentiable function there is at least one zero of its derivative, then the p_1-derivative of $\bar{F}(t)$ is a piecewise quasi-polynomial and has at least $\sum_{i=1}^{m} (r_{i,1} + r_{i,2}) + 2m - 1 - 2p_1$ real zeros. Therefore, using Lemma 6, we find that it has at most $\sum_{i=1}^{m} (r_{i,1} + r_{i,2}) + 2m - 2 - 2p_1$ real zeros. This is a contradiction.

We now proceed to the passage from l to $l+1$. From the definition of the number l we have

$$\sum_{i=1}^{l} \min(r_{i,1}, r_{i,2}) + l - 1 < p_1 \leqslant \sum_{i=1}^{l+1} \min(r_{i,1}, r_{i,2}) + l,$$

and hence

$$p_1 = \sum_{l=1}^{l} \min(r_{i,1}, r_{i,2}) + l - 1 + q,$$

where $0 < q \leqslant \min(r_{l=1,1}, r_{l+1,2}) + 1$.

Multiplying $F(t)$ by $e^{-\lambda_{l+1} t}$, we obtain the function

$$\bar{F}(t) = \begin{cases} f_{l+1,1}(t) + \sum_{\substack{i=1 \\ i \neq l+1}}^{m} f_{i,1}(t) e^{(\lambda_i - \lambda_{l+1})t}, & t \leqslant t_1, \\ f_{l+1,2}(t) + \sum_{\substack{i=1 \\ i \neq l+1}}^{m} f_{i,2}(t) e^{(\lambda_i - \lambda_{l+1})t}, & t > t_1. \end{cases} \tag{34}$$

Suppose that function (34) possesses at least $\sum_{i=1}^{m}(r_{i,1}+r_{i,2})+2m-1-p_1$ zeros. Repeting the same argument as for the case $l=1$, we find that the derivative of the degree q of function (34) possesses at least $\sum_{i=1}^{m}(r_{i,1}+r_{i,2})+$ $+2m-1-p_1-q$ zeros. However, the q-th derivative of function (34) is of form (32) and satisfies condition (33), where $\bar{p}_1 = p_1-q = \sum_{i=1}^{l}\min(r_{i,1},$ $r_{i,2})+l-1$.

Therefore we may use the induction hypothesis and show that the q-th derivative of function (34) has at most

$$\sum_{i=1}^{m}(r_{i,1}+r_{i,2})+2m-2-2q-\sum_{i=1}^{l}\min(r_{i,1},r_{i,2})-l+1$$
$$=\sum_{i=1}^{m}(r_{i,1}+r_{i,2})+2m-2-p_1-q$$

real zeros. We have obtained a contradiction and this completes the proof of case a) for $k=1$.

Case b) easily follows from a). It is obvious when $p_1\leqslant\sum_{i=1}^{m}\min(r_{i,1},r_{i,2})+$ $+m-1$, However, when

$$\sum_{i=1}^{m}\min(r_{i,1},r_{i,2})+m-1<p_1\leqslant\sum_{i=1}^{m}\min(R_{i,1},R_{i,2})+m-1,$$

then, using case a) for $\bar{p}_1=\sum_{i=1}^{m}\min(r_{i,1},r_{i,2})+m-1$, we find that the function $F(t)$ possesses at most

$$\sum_{i=1}^{m}(r_{i,1}+r_{i,2})+2m-2-\sum_{i=1}^{m}\min(r_{i,1},r_{i,2})-m+1$$
$$=\sum_{i=1}^{m}\max(r_{i,1},r_{i,2})+m-1$$

real zeros. Since

$$\sum_{i=1}^{m}\max(r_{i,1},r_{i,2})+m-1+p_1-p_1$$
$$\leqslant\sum_{i=1}^{m}\max(R_{i,1}R_{i,2})+m-1+\sum_{i=1}^{m}\min(R_{i,1},R_{i,2})+m-1-p_1$$
$$=\sum_{i=1}^{m}(R_{i,1},R_{i,2})+2m-2-p_1,$$

the proof of case b) for $k=1$ is completed.

We will show how case a) for $k = 2p-1$ follows from the case of $k = 1$. Let us divide the set of real numbers into intervals:

$$R = (-\infty, t_2]\cup(t_2, t_4]\cup\ldots\cup(t_{2p-2}, \infty).$$

In every interval given above we may use the case $k = 1$, just proved. Hence we find that the function $F(t)$ has at most

$$\sum_{r=1}^{p}\Big(\sum_{i=1}^{m}(r_{i,2r-1}+r_{i,2r})+2m-2-p_{2r-1}\Big) = \sum_{j=1}^{k+1}\Big(\sum_{i=1}^{m}r_{ij}+m-1\Big)-\sum_{r=1}^{p}p_{2r-1}$$

real zeros. The case of $k = 2p$ and case b) are proved in an analogous manner.

THEOREM 3.2. *Consider the control system* (1) *in which every eigenvalue of the matrix A is real. If system* (1) *is s-normal on* $[0, kh]$, *then the number of switching times on this interval for the s-th component of the control vector is*:

a) *for an odd k at most* $\dfrac{k(k+1)n}{2}-\dfrac{(k+1)^2}{4}$,

b) *for an even k at most* $\dfrac{k(k+1)n}{2}-\dfrac{k(k+2)}{4}$.

Proof It is well known (cf. [3]) that the solution of a system of differential equations with a retarded argument smoothes itself with the increasing of t. In our case, for $t > kh$, all the components of the solution $z(t)$ of system (3), (4) are of class C^k. We may set

$$p_j = j-1. \tag{35}$$

If all the eigenvalues λ_i of A are real with multiplicities l_i, $i = 1, 2, \ldots, m$, then repeating the same arguments as in the proof of Theorem 3.1 we find that the switching function $\sigma_s(t)$ is of the form

$$\sigma_s(t) = \sum_{i=1}^{m}f_{ij}(t)e^{\lambda_i t}, \quad t\in[(j-1)h, jh], \tag{36}$$

where $f_{ij}(t)$ are polynomials of degrees

$$r_{ij}\leqslant jl_i-1 = R_{ij}, \quad j = 1, 2, \ldots, k. \tag{37}$$

Using (35) and (37) in case b) of Lemma 7 we get Theorem 3.2 at once.

LEMMA 8. *The function $F(t)$, given by* (32) *with $m = 1$, fulfilling conditions* (33) *has at most*

$$\sum_{j=1}^{k+1}r_j-\sum_{j=1}^{k}\min\big(p_j, \max(r_j, r_{j+1})\big)$$

real zeros.

Proof. Multiplying function (32) by $e^{-\lambda_1 t}$ and replacing t by $t-t_k$, we obtain the function

$$(38)\qquad F(t) = \begin{cases} f_1(t), & t\in(-\infty, t_1], \\ f_j(t), & t\in(t_{j-1}, t_j], \\ f_{k+1}(t), & t\in(t_k, \infty), \end{cases}$$

fulfilling the conditions

$$(39)\qquad \frac{d^r f_j(t)}{dt^r}\bigg|_{t_j} = \frac{d^r f_{j+1}(t)}{dt^r}\bigg|_{t_j}, \quad r = 0, 1, \ldots, p_j,$$

where $f_j(t) = a_{j,r_j}t^{r_j}+\ldots+a_{j,0}$ are polynomials of degrees $r_j, j = 1, 2, \ldots, k+1$. Obviously, it suffices to prove Lemma 8 for functions (38), (39).

For $k = 1$ conditions (39) are equivalent to the equalities

$$(40)\qquad a_{1,r} = a_{2,r}, \quad r = 0, 1, \ldots, p_1.$$

Suppose that $r_1 \leqslant r_2$ and $p_1 \leqslant \max(r_1, r_2) = r_2$. Thus, if we denote by p the number of changes in the sequence $a_{2,r}, r = 0, 1, \ldots, p_1$, then using the Descartes rule and (40), we find that function (38) has at most $(r_1-p)+(r_2-p_1+p) = r_1+r_2-p_1$ real zeros.

In the case where $r_1 > r_2$ we reason analogously. On the other hand, if $p_1 > \max(r_1, r_2)$, then from (40) it follows that the polynomials $f_1(t)$ and $f_2(t)$ are identical and Lemma 8 for $k = 1$ holds.

Remark 4. Estimating the number of real zeros of function (38), we have added the maximal possible numbers of its zeros on $(-\infty, t_1]$ and (t_1, ∞).

In the proof of the passage from $k-1$ to k we shall use the following notations:

p — the number of changes in the sequence $a_{k,r}, r = 0, 1, \ldots, p_k$,
q — the maximal number of real zeros of the polynomial

$$f_k(t) \quad \text{on} \quad (t_{k-1}, \infty).$$

For $j = k$ conditions (39) have the form

$$(41)\qquad a_{k,r} = a_{k+1,r}, \quad r = 0, 1, \ldots, p_k,$$

and hence, if $p_k > \max(r_k, r_{k+1})$, then the polynomials $f_k(t)$ and $f_{k+1}(t)$ are identical and the passage from $k-1$ to k is obvious. Assume therefore that $p_k \leqslant \max(r_k, r_{k+1})$.

Using the induction hypothesis, we find that function (38), which contains only $k-1$ points t_j, has at most

$$\sum_{j=1}^{k} r_j - \sum_{j=1}^{k-1} \min(p_j, \max(r_j, r_{j+1}))$$

real zeros.

Using Remark 4 and the definition of q, we see that on $(-\infty, t_{k-1})$ function (38) has at most

$$(42) \qquad \sum_{j=1}^{k} r_j - \sum_{j=1}^{k-1} \min\big(p_j, \max(r_j, r_{j+1})\big) - q$$

real zeros. We only need to estimate the number of real zeros of function (38) on (t_{k-1}, ∞).

We have just assumed that $p_k \leqslant \max(r_k, r_{k+1})$. Let us consider two cases: $0 \leqslant p_k \leqslant r_{k+1}$ and $r_{k+1} < p_k \leqslant r_k$.

If $0 \leqslant p_k \leqslant r_{k+1}$, then using (41) and the Descartes rule we can prove that function (38) has on (t_{k-1}, ∞) at most

$$(43) \qquad (q-p)+(r_{k+1}-p_k+p) = q+r_{k+1}-p_k$$

real zeros. On the other hand, if $r_{k+1} < p_k \leqslant r_k$, then using the same arguments we can show that function (38) has on (t_{k+1}, ∞) at most

$$(44) \qquad \big(q-(p_k-r_{k+1})-p\big)+p = q+r_{k+1}-p_k$$

real zeros. From (42) and (43) or (42) and (44) it follows at once that function (38) has at most

$$\sum_{j=1}^{k} r_j - \sum_{j=1}^{k-1} \big(p_j, \max(r_j, r_{j+1})\big) + r_{k+1} - p_k$$

real zeros. This completes the proof of Lemma 8.

THEOREM 3.3. *Consider the control system* (1) *in which all the eigenvalues of A are equal. Thus, if system* (1) *is s-normal on* $[0, kh]$, *then the number of switching times on this interval for the s-th component of the control vector $u(t)$ is at most* $\dfrac{k(k+1)n}{2} - \dfrac{k^2-k+2}{2}$.

Proof. From the proof of Theorem 3.2 we know that on $[(1-1)h, jh]$ the switching function $\sigma_s(t)$ is the form

$$\sigma_s(t) = f_j(t)e^{\lambda t}, \quad j = 1, 2, \ldots, k,$$

where $f_j(t)$ are polynomials of degrees

$$(45) \qquad r_j \leqslant jn-1.$$

Using conditions (35) and (45) we get Theorem 3.3.

EXAMPLE. It is easy to verify that the solution of the differential equation

$$\dot{z}(t) = z(t) - 12ez(t-1); \quad z(0) = 1$$

is of the form

$$z(t) = \begin{cases} e^t, & t \in [0, 1], \\ (13-12t)e^t, & t \in [1, 2], \\ (301-300t+72t^2)e^t, & t \in [2, 3], \\ (8077-8076t+2664t^2-288t^3)e^t, & t \in [3, 4]. \end{cases}$$

Since $z(1) = e$, $z(2) = -11e^2$, $z(3) = 49e^3$, $z(4) = -35e^4$, the solution $z(t)$ of our equation has on $[0, 4]$ at least three real zeros. From Theorem 3.3 for $k = 4$ it follows that $z(t)$ has on $[0, 4]$ at most three real zeros.

References

[1] R. Bellman and K. L. Cooke, *Differential-difference equations*, Academic Press, New York 1963.

[2] D. H. Chyung and E. B. Lee, *Control of linear time delay systems with essentially linear cost functionals*, Differential and dynamic systems, Edited by J. Hale and J. LaSalle, Academic Press, New York 1967.

[3] R. Gabasov i F. Kiriłlova, *Kaczestvennaja tieorija optimalnych prociessov*, Izdat. Nauka, Moskva 1971.

[4] G. Leitmann, *Wprowadzenie do teorii sterovania optymalnego*, Warszawa 1971.

[5] M. N. Oguztoreli, *A time-optimal control problem for systems described by differential-difference equations*, SIAM J. on Control 1 (1963), p. 290–310.

[6] A. Turowicz, *Geometria zer wielomianów*, PWN, Warszawa 1967.

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