

**STRUCTURE AND EXTREMAL PROBLEMS
FOR CLASSES OF FUNCTIONS ANALYTIC IN AN ANNULUS**

BY

A. E. LIVINGSTON (NEWARK, DELAWARE)
AND J. A. PFALTZGRAFF (CHAPEL HILL, NORTH CAROLINA)

Introduction. In this paper* we study classes of functions that are analytic and either typically real or have a positive real part in the annulus $A_q = \{z: q < |z| < 1\}$. We develop characterizations of each class in terms of Herglotz-type Stieltjes integral representations and in terms of Carathéodory-Toeplitz-type semi-definite quadratic forms (Theorems 1.1, 3.1, 4.1, 4.2). The latter type of characterization is a powerful tool for the solving of quite general extremal problems and enables us to extend to A_q some recent work of Atzmon [1] for classes in the unit disk $\Delta = \{z: |z| < 1\}$ that are related to

$$\mathcal{P} = \left\{ P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n : \text{analytic and } \operatorname{Re} P(z) > 0 \text{ in } \Delta \right\}.$$

1. The class $\tilde{\mathcal{P}}_q$. In this section we give three characterizations of the special class

$$\tilde{\mathcal{P}}_q = \{f \in \mathcal{P}_q : \operatorname{Re} f(z) \equiv 1 \text{ on } |z| = q\},$$

where

$$\mathcal{P}_q = \{f(z) : \text{analytic and } \operatorname{Re} f(z) \geq 0 \text{ in } A_q\}.$$

This subclass is important, since it will be used to give a complete characterization of the structure of the full class \mathcal{P}_q (Theorem 3.1). Komatu [4] obtained the Herglotz-type integral representation (1.2) by means of the Villat representation formula for functions analytic in A_q and the Helly selection theorems. Our approach is different from that of Komatu

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[4] and Nishimiya [6]. By means of elementary manipulations with Laurent series and the Carathéodory quadratic inequalities for \mathcal{P} (see [3], p. 148) we establish a "quadratic inequalities" characterization of $\tilde{\mathcal{P}}_q$. This approach avoids appeal to Villat's formula and properties of elliptic functions and shows clearly the explicit connection between the classes $\tilde{\mathcal{P}}_q$ in the annulus and \mathcal{P} in the unit disk. There is also an advantage that our approach leads directly to useful characterizations of coefficient regions of variability and to the solutions of general extremal problems for $\tilde{\mathcal{P}}_q$ and \mathcal{P}_q (Sections 2 and 3).

The function

(1.1)

$$\begin{aligned}\Phi^*(z) &= 1 + \sum'_{k=-\infty}^{\infty} \frac{2}{1-q^{2k}} z^k = \frac{1+z}{1-z} + 2 \sum'_{k=-\infty}^{\infty} \frac{q^{2k}}{1-q^{2k}} (z^k - z^{-k}) \\ &= 1 + 2 \sum'_{k=-\infty}^{\infty} \frac{1}{1-q^{2k}} \left[z^k - \left(\frac{q^2}{z} \right)^k \right]\end{aligned}$$

is the basic kernel function for the class $\tilde{\mathcal{P}}_q$ and plays a role analogous to that of $(1+z)/(1-z)$ in \mathcal{P} . It is clear from the three representations (1.1) that $\Phi^*(z) \in \tilde{\mathcal{P}}_q$. We also note that every function in $\tilde{\mathcal{P}}_q$ is analytic in the larger annulus $q^2 < |z| < 1$ by the reflection principle. In (1.1) and henceforth, the symbol $\sum'_{k=-\infty}^{\infty}$ means summation over all non-zero integers k .

THEOREM 1.1. *The following statements are equivalent:*

(a) $f(z) = 1 + \sum'_{k=-\infty}^{\infty} a_k z^k \in \tilde{\mathcal{P}}_q$.

(b) $a_{-k} = -q^{2k} \bar{a}_k$, $k = \pm 1, \pm 2, \dots$, and

$$P(z) = 1 + \sum_{k=1}^{\infty} (1-q^{2k}) a_k z^k \in \mathcal{P}.$$

(c) If $p_k = a_k(1-q^{2k})$, $k = \pm 1, \pm 2, \dots$, $p_0 = 2$, then

$$p_{-k} = \bar{p}_k \quad \text{and} \quad \sum_{j,k=0}^N p_{j-k} \lambda_k \bar{\lambda}_j \geq 0$$

for every choice of complex numbers $\lambda_0, \lambda_1, \dots, \lambda_N$ and $N = 0, 1, \dots$

(d) There is a unique probability measure $d\mu(t)$ on $-\pi \leq t \leq \pi$ such that

$$(1.2) \quad f(z) = \int_{-\pi}^{\pi} \Phi^*(ze^{-it}) d\mu(t), \quad q < |z| < 1.$$

Proof. Statements (b) and (c) are equivalent by the classical theorem of Carathéodory for functions with positive real part in the unit disk (see [3], p. 148). (Note that $p_{-k} = \bar{p}_k$ if and only if $a_{-k} = -q^{2k}\bar{a}_k$.) Clearly, (d) implies (a), since $\Phi^*(z)$ belongs to $\tilde{\mathcal{P}}_q$. To prove (c) \Rightarrow (d), suppose that

$$P(z) = 1 + \sum_{k=1}^{\infty} (1 - q^{2k}) a_k z^k$$

belongs to \mathcal{P} . The Herglotz theorem for \mathcal{P} implies the existence of a unique probability measure $d\mu(t)$ on $[-\pi, \pi]$ such that

$$(1 - q^{2k}) a_k = 2 \int_{-\pi}^{\pi} e^{-ikt} d\mu(t), \quad k = \pm 1, \pm 2, \dots$$

Hence

$$\begin{aligned} f(z) &= 1 + \sum_{k=-\infty}^{\infty} a_k z^k = 1 + \sum_{k=-\infty}^{\infty} 2 \int_{-\pi}^{\pi} e^{-ikt} \frac{z^k}{1 - q^{2k}} d\mu(t) \\ &= \int_{-\pi}^{\pi} \left(1 + \sum_{k=-\infty}^{\infty} \frac{2}{1 - q^{2k}} (ze^{-it})^k \right) d\mu(t) = \int_{-\pi}^{\pi} \Phi^*(ze^{-it}) d\mu(t). \end{aligned}$$

To prove that (a) implies (b) we first consider $f \in \tilde{\mathcal{P}}_q$ such that $f(z)$ is analytic in $q < |z| \leq 1$, hence in $q^2 < |z| \leq 1$. The function

$$g(z) = f(z) - \sum_{k=1}^{\infty} a_{-k} z^{-k} + \sum_{k=1}^{\infty} \bar{a}_{-k} (zq^{-2})^k = 1 + \sum_{k=1}^{\infty} (a_k + q^{-2k} \bar{a}_{-k}) z^k$$

is analytic and satisfies the condition $\operatorname{Re} g(z) = \operatorname{Re} f(z) \equiv 1$ in $|z| \leq q$, since $\operatorname{Re} g = \operatorname{Re} f$ on $|z| = q$. Since $g(0) = 1$, we have

$$g(z) - 1 = \sum_{k=1}^{\infty} (a_k + q^{-2k} \bar{a}_{-k}) z^k = 0$$

identically in $|z| < q$ and, consequently, $a_{-k} = -q^{2k} \bar{a}_k$ for $k = \pm 1, \pm 2, \dots$. The function

$$P(z) = 1 + \sum_{k=1}^{\infty} (1 - q^{2k}) a_k z^k$$

is analytic in $|z| \leq 1$ and

$$P(z) = f(z) + \sum_{k=1}^{\infty} q^{2k} (\bar{a}_k z^{-k} - a_k z^k).$$

Hence $\operatorname{Re} P(z) = \operatorname{Re} f(z) \geq 0$ on $|z| = 1$ and, therefore, $\operatorname{Re} P(z) > 0$ in $|z| < 1$.

Finally, we remove the restriction that $f(z)$ be analytic on $|z| = 1$ by an approximation argument. If

$$f(z) = 1 + \sum_{k=-\infty}^{\infty} a_k z^k \in \tilde{\mathcal{P}}_q,$$

then

$$f_r(z) = f(rz) = 1 + \sum_{k=-\infty}^{\infty} a_k r^k z^k$$

belongs to $\tilde{\mathcal{P}}_{q'}$ for all $r < 1$ sufficiently near 1, $q' = q/r$, and is analytic on $|z| = 1$. Hence the corresponding function

$$P_r(z) = 1 + \sum_{k=1}^{\infty} (1 - (q/r)^{2k}) a_k r^k z^k$$

belongs to \mathcal{P} . The Carathéodory inequalities

$$\sum_{j,k=0}^N a_{j-k} r^{j-k} (1 - (q/r)^{2(j-k)}) \lambda_k \bar{\lambda}_j \geq 0, \quad N = 1, 2, \dots,$$

hold for all r near 1 ($r < 1$), and hence for $r = 1$. This completes the proof of Theorem 1.1.

The inequalities $|p_n| \leq 2$ ($n = 1, 2, \dots$) for the Taylor coefficients of a function in \mathcal{P} and Theorem 1.1 yield immediately the sharp bounds

$$|a_k| \leq 2 |1 - q^{2k}|^{-1}, \quad k = \pm 1, \pm 2, \dots,$$

for the coefficients of a function belonging to $\tilde{\mathcal{P}}_q$ (see [6]). Leutwiler and Schober [5] have shown that the Taylor coefficients of a function belonging to \mathcal{P} satisfy

$$\left| p_{2n} - \frac{1}{2} p_n^2 \right| \leq 2 - \frac{1}{2} |p_n|^2, \quad n = 1, 2, \dots$$

With Theorem 1.1 we translate this to the following

COROLLARY 1.1. *If*

$$f(z) = 1 + \sum_{k=-\infty}^{\infty} a_k z^k \in \tilde{\mathcal{P}}_q,$$

then

$$\left| (1 - q^{4k}) a_{2k} - \frac{1}{2} (1 - q^{2k})^2 a_k^2 \right| \leq 2 - \frac{1}{2} (1 - q^{2k})^2 |a_k|^2, \quad k = \pm 1, \pm 2, \dots$$

This defines the precise disk of values for a_{2k} corresponding to each preassigned a_k . Clearly, Theorem 1.1 enables one to translate from \mathcal{P} to $\tilde{\mathcal{P}}_q$ a variety of coefficient inequalities.

2. Coefficient regions and extremal problems for $\tilde{\mathcal{P}}_q$. For a function

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \in \mathcal{H}(A_q),$$

where $\mathcal{H}(A_q)$ is the set of functions analytic on A_q with the topology of local uniform convergence, we let

$$T_{-m}^n(f) = (c_{-m}, c_{-m+1}, \dots, c_{-1}, c_1, \dots, c_n)$$

viewed as a point in \mathbb{C}^{m+n} . The set

$$\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q) = \{T_{-n}^n(f) : f \in \tilde{\mathcal{P}}_q\}$$

is a subset of \mathbb{C}^{2n} that we call the *n-th coefficient region* of the class $\tilde{\mathcal{P}}_q$. In this section we exploit the connection between $\tilde{\mathcal{P}}_q$ and \mathcal{P} (Theorem 1.1) to give a description of the coefficient region $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$ in terms of Toeplitz determinant inequalities. Our description includes the explicit identification of the functions in $\tilde{\mathcal{P}}_q$ that correspond to the boundary points of $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$.

In [6] (Theorem 1) Nishimiya gave a description of coefficient regions of $\tilde{\mathcal{P}}_q$ (denoted by \mathcal{R}_q therein) consisting of points $T_{-m}^n(f)$. However, Nishimiya's result contains more free parameters than necessary. Examination of his argument ([6], p. 29) reveals that his restriction $0 < p \leq n+m$ on the number of parameters can easily be reduced to $0 < p \leq \max\{m, n\}$ by means of Rolle's theorem. Furthermore, the relations $a_{-k} = -q^{2k} \bar{a}_k$ ($k = \pm 1, \pm 2, \dots$), satisfied by the coefficients of a function in $\tilde{\mathcal{P}}_q$ which we have established in Theorem 1.1, show that the symmetric coefficient regions $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$ considered here are completely general.

Given complex numbers c_0, c_1, \dots, c_m we let $D_m(c_0, c_1, \dots, c_m)$ denote the determinant of the $(m+1) \times (m+1)$ Toeplitz matrix (c_{j-k}) ($0 \leq j, k \leq m$) with the usual convention $c_{-j} = \bar{c}_j$. Corresponding to a function

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}$$

we write $T_n(P) = (p_1, \dots, p_n)$, and let $\mathcal{K}_n(\mathcal{P}) = \{T_n(P) : P \in \mathcal{P}\}$ denote the *n-th coefficient body* or region of variability for the class \mathcal{P} . For convenient reference we state the following

THEOREM OF CARATHÉODORY AND TOEPLITZ ([3], p. 152). *It follows that*

$$\mathcal{K}_n(\mathcal{P}) = \{(p_1, p_2, \dots, p_n) \in \mathbb{C}^n : D_k(2, p_1, \dots, p_k) \geq 0, k = 1, 2, \dots, n\}.$$

Furthermore, $(p_1, \dots, p_n) \in \text{bdry } \mathcal{K}_n(\mathcal{P})$ if and only if

$$D_j(2, p_1, \dots, p_j) = 0 \quad \text{for some } j \in \{1, 2, \dots, n\},$$

and for each such boundary point of $\mathcal{K}_n(\mathcal{P})$ the function $P(z) \in \mathcal{P}$ such that $T_n(P) = (p_1, \dots, p_n)$ is uniquely determined and of the form

$$(2.1) \quad P(z) = \sum_{k=1}^n \lambda_k \frac{1+z\varepsilon_k}{1-z\varepsilon_k}, \quad |\varepsilon_k| = 1, \quad 0 \leq \lambda_k, \quad \sum_{k=1}^n \lambda_k = 1.$$

We now prove our generalization of the Carathéodory-Toeplitz theorem for the class $\tilde{\mathcal{P}}_q$. Independently of this result it is easy to see that the coefficient region $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$ is a convex compact subset of \mathbb{C}^{2n} , since $\tilde{\mathcal{P}}_q$ is a convex compact subset of $\mathcal{H}(A_q)$.

THEOREM 2.1. *Let a_k ($k = \pm 1, \pm 2, \dots, \pm n$) be $2n$ complex numbers. Necessary and sufficient conditions for the existence of a function $f \in \tilde{\mathcal{P}}_q$ such that $T_{-n}^n(f) = (a_{-n}, \dots, a_{-1}, a_1, \dots, a_n)$ are*

$$(1) \quad a_{-k} = -q^{2k} \bar{a}_k \text{ for } k = 1, 2, \dots, n,$$

$$(2) \quad D_k(2, p_1, \dots, p_k) \geq 0 \text{ for } k = 1, 2, \dots, n, \text{ where}$$

$$(2.2) \quad p_k = (1 - q^{2k}) a_k, \quad k = 1, 2, \dots, n.$$

Furthermore, if (1) and (2) hold and if $D_j(2, p_1, \dots, p_j) = 0$ for some $j \in \{1, 2, \dots, n\}$, then the function $f \in \tilde{\mathcal{P}}_q$ such that $T_{-n}^n(f) = (a_{-n}, \dots, a_n)$ is uniquely determined and of the form

$$(2.3) \quad f(z) = \sum_{k=1}^n \lambda_k \Phi^*(z\varepsilon_k), \quad |\varepsilon_k| = 1, \quad 0 \leq \lambda_k, \quad \sum_{k=1}^n \lambda_k = 1.$$

Proof. The result follows from the equivalence of parts (a) and (b) in Theorem 1.1, and from the Carathéodory-Toeplitz theorem for the class \mathcal{P} . Indeed, the necessity follows from the implication (a) \Rightarrow (b) and the observation that condition (1) yields $p_{-k} = \bar{p}_k$ for the numbers p_k defined in (2.2). For if

$$(a_{-n}, \dots, a_n) = T_{-n}^n(f) \quad \text{for some } f(z) = 1 + \sum_{k=-\infty}^{\infty} a_k z^k \in \tilde{\mathcal{P}}_q,$$

then (1) must hold and

$$P(z) = 1 + \sum_{k=1}^{\infty} (1 - q^{2k}) a_k z^k \in \mathcal{P}$$

by Theorem 1.1. Hence condition (2) follows from the Carathéodory-Toeplitz theorem.

To prove the sufficiency suppose that a_k ($k = \pm 1, \pm 2, \dots, \pm n$) satisfy (1) and (2) and let p_k ($k = 1, 2, \dots, n$) be the corresponding numbers defined by (2.2). By the Carathéodory-Toeplitz theorem there exists a function

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

such that $T_n(P) = (p_1, \dots, p_n)$. This gives an infinite sequence $\{p_k\}$ with which we extend the given sequence $\{a_k: k = \pm 1, \pm 2, \dots, \pm n\}$ by defining $a_k = p_k(1 - q^{2k})^{-1}$ and $a_{-k} = -q^{2k}\bar{a}_k$ for $k = 1, 2, \dots$. The implication (b) \Rightarrow (a) in Theorem 1.1 shows that the function

$$f(z) = 1 + \sum_{k=-\infty}^{\infty} a_k z^k$$

belongs to $\tilde{\mathcal{P}}_q$.

Finally, suppose that (1) and (2) hold and $D_j(2, p_1, \dots, p_j) = 0$ for some $j \in \{1, 2, \dots, n\}$. The corresponding uniquely determined function $P(z) \in \mathcal{P}$ is of the form (2.1) and has coefficients

$$p_k = \sum_{j=1}^n \lambda_j 2\varepsilon_j^k.$$

Thus

$$a_k = p_k/(1 - q^{2k}) = \sum_{j=1}^n \lambda_j 2\varepsilon_j^k/(1 - q^{2k})$$

and the uniquely determined corresponding function $f \in \tilde{\mathcal{P}}_q$ is of the form

$$f(z) = 1 + \sum_{k=-\infty}^{\infty} a_k z^k = \sum_{j=1}^n \lambda_j \left\{ 1 + \sum_{k=-\infty}^{\infty} 2(\varepsilon_j z)^k/(1 - q^{2k}) \right\} = \sum_{j=1}^n \lambda_j \Phi^*(z\varepsilon_j).$$

One frequently encounters extremal problems on a compact family \mathcal{F} of the type: maximize $\operatorname{Re} F(T_m^n(f))$, $f \in \mathcal{F}$, where $F = F(w_1, \dots, w_{n-m+1})$ is a complex analytic function of $n - m + 1$ variables on a region containing the coefficient region $\mathcal{K}_m^n(\mathcal{F})$. By the maximum principle such a functional achieves its maximum only on the boundary of $\mathcal{K}_m^n(\mathcal{F})$. In the next theorem we consider extremal problems defined by a general type of functional with this property.

THEOREM 2.2. *Let $G = G(w_1, \dots, w_{2n})$ be a real-valued function of $2n$ complex variables that is defined and continuous on $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$ and has the property that it achieves its maximum over $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$ only on the boundary. Let $J(f) = G(T_{-n}^m(f))$, $f \in \tilde{\mathcal{P}}_q$. Then the functional J is defined and continuous*

on $\tilde{\mathcal{P}}_q$. Furthermore, the solution to the extremal problem

$$(2.4) \quad \max\{J(f): f \in \tilde{\mathcal{P}}_q\}$$

in $\tilde{\mathcal{P}}_q$ must be a function of the form (2.3).

Proof. The map

$$\psi(a_{-n}, \dots, a_{-1}, a_1, \dots, a_n) = (p_1, \dots, p_n),$$

where $p_k = (1 - q^{2k})a_k$, $k = 1, 2, \dots, n$, defines a homeomorphism of $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$ onto $\mathcal{K}_n^n(\mathcal{P})$ by Theorems 1.1 and 2.1. If (p_1, \dots, p_n) is a point in $\mathcal{K}_n^n(\mathcal{P})$, then

$$\psi^{-1}(p_1, \dots, p_n) = (a_{-n}, \dots, a_n) \in \mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q),$$

where $a_k = p_k/(1 - q^{2k})$ and $a_{-k} = -q^{2k}\bar{a}_k$, $k = 1, 2, \dots, n$. By means of this homeomorphism we can transplant J to the functional $J_1(P) = G(\psi^{-1}(T_n(P)))$, $P(z) \in \mathcal{P}$. Then J_1 is defined and continuous on $\mathcal{K}_n^n(\mathcal{P})$ and achieves its maximum over this set only on the boundary. By a recent theorem of Atzmon [1] the solution to the extremal problem $\max\{J_1(P): P \in \mathcal{P}\}$ must be a function of the form (2.1). Hence the solution to (2.4) must be a function in $\tilde{\mathcal{P}}_q$ of the form (2.3).

3. The class \mathcal{P}_q . In this section we generalize to the class \mathcal{P}_q the results of Sections 1 and 2. We recall the definition

$$\mathcal{P}_q = \left\{ f(z) = 1 + \sum_{k=-\infty}^{\infty} c_k z^k \in \mathcal{H}(A_q): \operatorname{Re} f(z) \geq 0, z \in A_q \right\}.$$

THEOREM 3.1. *The following four statements are equivalent:*

- (a) $f(z) = 1 + \sum_{k=-\infty}^{\infty} c_k z^k \in \mathcal{P}_q$.
- (b) For each r in $q \leq r \leq 1$

$$P_r(z) = 1 + \sum_{k=1}^{\infty} (c_k r^k + \bar{c}_{-k} r^{-k}) z^k \in \mathcal{P}$$

or, equivalently,

$$(3.1) \quad \sum_{j,k=0}^N (c_{j-k} r^{j-k} + \bar{c}_{k-j} r^{k-j}) \lambda_k \bar{\lambda}_j \geq 0$$

for every choice of complex numbers $\lambda_0, \lambda_1, \dots, \lambda_N$ and $N = 0, 1, \dots$

(c) There exist unique probability measures $d\mu$ and $d\nu$ on $[-\pi, \pi]$ such that

$$(3.2) \quad c_k = \frac{2}{1 - q^{2k}} \int_{-\pi}^{\pi} e^{-ikt} d\mu(t) - \frac{2q^k}{1 - q^{2k}} \int_{-\pi}^{\pi} e^{-ikt} d\nu(t), \quad k = \pm 1, \pm 2, \dots,$$

or, equivalently,

$$(3.3) \quad f(z) = \int_{-\pi}^{\pi} \Phi^*(ze^{-it}) d\mu(t) + \int_{-\pi}^{\pi} \Phi^*(q/ze^{-it}) d\nu(t) - 1.$$

(d) *There exist unique functions $F, G \in \tilde{\mathcal{P}}_q$ such that*

$$(3.4) \quad f(z) = F(z) + G(q/z) - 1, \quad z \in A_q.$$

Proof. The equivalence of (c) and (d) and the implication (d) \Rightarrow (a) follow immediately from the properties of the subclass $\tilde{\mathcal{P}}_q$. To prove that (a) implies (b) we first consider

$$f(z) = 1 + \sum_{k=-\infty}^{\infty} c_k z^k \in \mathcal{P}_q$$

such that $f(z)$ is analytic in $q < |z| \leq 1$. We let

$$(3.5) \quad P(z) = f(z) - \sum_{k=1}^{\infty} c_{-k} z^{-k} + \sum_{k=1}^{\infty} \bar{c}_{-k} z^k = 1 + \sum_{k=1}^{\infty} (c_k + \bar{c}_{-k}) z^k.$$

Then $P(z)$ is analytic and $\operatorname{Re} P(z) \geq 0$ in $|z| \leq 1$, since $\operatorname{Re} P(z) = \operatorname{Re} f(z)$ on $|z| = 1$. Letting $p_k = c_k + \bar{c}_{-k}$ ($k = 0, \pm 1, \dots$), we have $p_{-k} = \bar{p}_k$, $p_0 = 2$, and the Carathéodory inequalities

$$(3.6) \quad \sum_{j,k=0}^N (c_{j-k} + \bar{c}_{k-j}) \lambda_k \bar{\lambda}_j \geq 0, \quad N = 0, 1, \dots,$$

hold. We now drop the restriction that $f(z)$ be analytic on $|z| = 1$, and consider for arbitrary $f \in \mathcal{P}_q$ the function

$$f_r(z) = f(rz) = 1 + \sum_{k=-\infty}^{\infty} c_k r^k z^k.$$

An application of the preceding reasoning for f_r in $q/r < |z| \leq 1$ shows that (3.6) with c_m replaced by $c_m r^m$ holds for each fixed N , arbitrary complex λ_k ($k = 0, 1, \dots, N$) and $q < r < 1$. Thus (3.1) holds for $q < r < 1$ and, by continuity (for each fixed choice $\lambda_0, \lambda_1, \dots, \lambda_N$), also for $q \leq r \leq 1$. It is also clear now that the $P_r(z)$ belong to \mathcal{P} .

To prove that (b) implies (c) we consider the function

$$P_1(z) = 1 + \sum_{k=1}^{\infty} (c_k + \bar{c}_{-k}) z^k \in \mathcal{P}$$

and apply the Herglotz theorem for \mathcal{P} . Thus there is a probability measure

$d\mu$ on $[-\pi, \pi]$ such that

$$(3.7) \quad c_k + \bar{c}_{-k} = 2 \int_{-\pi}^{\pi} e^{-ikt} d\mu(t), \quad k = \pm 1, \pm 2, \dots$$

Similarly, corresponding to $P_q(z)$, there is a probability measure $d\nu$ such that

$$(3.8) \quad c_k q^k + \bar{c}_{-k} q^{-k} = 2 \int_{-\pi}^{\pi} e^{-ikt} d\nu(t), \quad k = \pm 1, \pm 2, \dots$$

Formula (3.2) follows immediately from (3.7) and (3.8). The representation formula (3.3) follows from (3.2) in the obvious way by putting the integral representations (3.2) into the Laurent series

$$f(z) = 1 + \sum'_{k=-\infty}^{\infty} c_k z^k,$$

interchanging the order of summation and integration, and summing the resulting series in the integrals. We omit further details which are similar to those in the proof of Theorem 1.1.

Finally, we prove that the measures in (3.2) are uniquely determined. Suppose that there are probability measures $d\mu, d\eta, d\nu, d\tau$ on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} e^{-ikt} d\mu - q^k \int_{-\pi}^{\pi} e^{ikt} d\nu = \frac{(1 - q^{2k})c_k}{2} = \int_{-\pi}^{\pi} e^{-ikt} d\eta - q^k \int_{-\pi}^{\pi} e^{-ikt} d\tau.$$

An easy computation yields

$$\int_{-\pi}^{\pi} e^{-ikt} d(\nu - \tau)(t) = 0, \quad k = \pm 1, \pm 2, \dots,$$

and, therefore, $d(\nu - \tau)$ is the zero measure. It then follows that $d(\mu - \eta)$ is also the zero measure, and the proof of the theorem is complete.

Remark. Parts (c) and (d) of Theorem 3.1 appear in [4] and [6] where they were established by other methods.

Our next theorem characterizes the n -th coefficient region $\mathcal{K}_{-n}^n(\mathcal{P}_q)$ of \mathcal{P}_q and generalizes Theorem 2.1 to this larger class.

THEOREM 3.2. *Let c_k ($k = \pm 1, \pm 2, \dots, \pm n$) be $2n$ complex numbers, and define the two related sets $\{A_k\}, \{B_k\}$ by*

$$(3.9) \quad A_k = c_k + \bar{c}_{-k}, \quad B_k = q^k c_k + q^{-k} \bar{c}_{-k}, \quad k = 1, 2, \dots, n.$$

Necessary and sufficient conditions for the existence of a function $f(z) \in \mathcal{P}_q$ such that $T_{-n}^n(f) = (c_{-n}, \dots, c_n)$ are

$$(3.10) \quad D_k(2, A_1, \dots, A_k) \geq 0, \quad D_k(2, B_1, \dots, B_k) \geq 0, \\ k = 1, 2, \dots, n.$$

Furthermore, if inequalities (3.10) hold and if, for some pair of integers $j, k \in \{1, 2, \dots, n\}$,

$$D_j(2, A_1, \dots, A_j) = 0 = D_k(2, B_1, \dots, B_k),$$

then the function $f \in \mathcal{P}_q$ such that $T_{-n}^n(f) = (c_{-n}, \dots, c_n)$ is uniquely determined and of the form

$$(3.11) \quad f(z) = \sum_{k=1}^n \mu_k \Phi^*(z\varepsilon_k) + \sum_{k=1}^n \nu_k \Phi^*(q/z\zeta_k) - 1,$$

where

$$|\varepsilon_k| = |\zeta_k| = 1, \quad 0 \leq \mu_k, \nu_k \quad \text{and} \quad \sum_{k=1}^n \mu_k = \sum_{k=1}^n \nu_k = 1.$$

Proof. If $(c_{-n}, \dots, c_n) = T_{-n}^n(f)$ for the function

$$f(z) = 1 + \sum'_{k=-\infty}^{\infty} c_k z^k \in \mathcal{P}_q,$$

then by Theorem 3.1, (3.2), we have

$$(1 - q^{2k})c_k = A_k - q^k B_k,$$

where

$$(3.12) \quad A_k = 2 \int_{-\pi}^{\pi} e^{-ikt} d\mu(t), \quad B_k = 2 \int_{-\pi}^{\pi} e^{-ikt} d\nu(t), \quad k = 1, 2, \dots$$

Thus we have $A_k = c_k + \bar{c}_{-k}$, $B_k = q^k c_k + q^{-k} \bar{c}_{-k}$ for all $k = 1, 2, \dots$ and the functions

$$(3.13) \quad A(z) = 1 + \sum_{k=1}^{\infty} A_k z^k, \quad B(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$$

both belong to \mathcal{P} . The necessity of inequalities (3.10) follows from the Carathéodory-Toeplitz theorem. For the sufficiency we assume that (3.10) holds and that the three sets of numbers $\{A_k\}$, $\{B_k\}$ ($k = 1, 2, \dots, n$) and $\{c_k: k = \pm 1, \pm 2, \dots, \pm n\}$ are related by (3.9). The Carathéodory-Toeplitz theorem guarantees the existence of functions $A(z)$ and $B(z)$ of the form (3.13) with coefficients (3.12) belonging to \mathcal{P} and satisfying $T_n(A(z)) = (A_1, \dots, A_n)$ and $T_n(B(z)) = (B_1, \dots, B_n)$. The coefficient formulas (3.12) enable us to extend the given finite sequence of $2n$ numbers c_k to an infinite sequence ($k = \pm 1, \pm 2, \dots$) in the obvious way that yields

$$f(z) = 1 + \sum'_{k=-\infty}^{\infty} c_k z^k \in \mathcal{P}_q \quad \text{with} \quad T_{-n}^n(f) = (c_{-n}, \dots, c_n).$$

The proof that the boundary points of $\mathcal{K}_{-n}^n(\mathcal{P}_q)$ correspond to unique functions of the form (3.11) is quite similar to the proof of the corresponding result about the functions (2.3) for $\mathcal{K}_{-n}^n(\tilde{\mathcal{P}}_q)$ and we omit the details.

THEOREM 3.3. *Let $G = G(w_1, \dots, w_n)$ be a real-valued function of $2n$ complex variables that is defined and continuous on $\mathcal{K}_{-n}^n(\mathcal{P}_q)$ and achieves its maximum over $\mathcal{K}_{-n}^n(\mathcal{P}_q)$ only on the boundary. Let $J(f) = G(T_{-n}^n(f))$, $f \in \mathcal{P}_q$. Then J is a continuous functional on \mathcal{P}_q , and the solution to the extremal problem*

$$(3.14) \quad \max\{J(f): f \in \mathcal{P}_q\}$$

in \mathcal{P}_q must be a function of the form (3.11).

Proof. The map

$$\varphi(c_{-n}, \dots, c_n) = (A_1, \dots, A_n) + (B_1, \dots, B_n),$$

where $A_k = c_k + \bar{c}_{-k}$ and $B_k = q^k c_k + q^{-k} \bar{c}_{-k}$, defines a homeomorphism of the compact convex set $\mathcal{K}_{-n}^n(\mathcal{P}_q)$ onto the algebraic sum

$$(3.15) \quad \mathcal{K}_n(\mathcal{P}) + \mathcal{K}_n(\mathcal{P}) = \{T_n(A) + T_n(B) \in C^n: A(z), B(z) \in \mathcal{P}\}$$

by Theorems 3.1 and 3.2. The inverse mapping is

$$\psi^{-1}[(A_1, \dots, A_n) + (B_1, \dots, B_n)] = (c_{-n}, \dots, c_n),$$

where $c_k = (A_k - q^k B_k)/(1 - q^{2k})$ and $A_{-k} = \bar{A}_k$, $B_{-k} = \bar{B}_k$, $k = \pm 1, \pm 2, \dots, \pm n$. Hence $J_1 = G \circ \psi^{-1}$ is a continuous function that achieves its maximum over the set (3.15) only on the boundary. By Theorem 3.2 a point is on the boundary of the set (3.15) only if there is a pair of integers $j, k \in \{1, 2, \dots, n\}$ such that

$$D_j(2, A_1, \dots, A_j) = 0 = D_k(2, B_1, \dots, B_k).$$

Thus a solution to (3.14) in \mathcal{P}_q must be of the form (3.11).

The next theorem characterizes the extreme points of $\tilde{\mathcal{P}}_q$ and \mathcal{P}_q . We let $\text{Ext}(S)$ denote the set of extreme points of a set S .

THEOREM 3.4. *We have*

$$(3.16) \quad \text{Ext}(\tilde{\mathcal{P}}_q) = \{\Phi^*(z\eta): \eta \in C, |\eta| = 1\},$$

$$(3.17) \quad \text{Ext}(\mathcal{P}_q) = \{\Phi^*(z\eta) + \Phi^*(q/z\zeta) - 1: |\eta| = 1 = |\zeta|\}.$$

Proof. Statement (3.16) is an immediate consequence of (1.2), a result of Brickman et al. [2], and the uniqueness of the measure in the integral representation (1.2). The integral representation (3.3) shows that \mathcal{P}_q is the algebraic sum of the two function classes $\tilde{\mathcal{P}}_q$ and $\{F(q/z) - 1: F \in \tilde{\mathcal{P}}_q\}$. It follows easily from (3.16) that $\text{Ext}(\mathcal{P}_q)$ is contained in the right-hand

set of (3.17). To establish the reverse containment, assume that

$$\Phi^*(z\eta) + \Phi^*(q/z\zeta) - 1 = \lambda f_1(z) + (1-\lambda)f_2(z)$$

for some choice of $\eta, \zeta \in C$, $|\eta| = 1 = |\zeta|$, $0 < \lambda < 1$ and functions

$$f_j(z) = \int_{-\pi}^{\pi} \Phi^*(ze^{-it}) d\mu_j + \int_{-\pi}^{\pi} \Phi^*(q/ze^{-it}) d\nu_j - 1, \quad j = 1, 2,$$

belonging to \mathcal{P}_q . It follows that

$$\Phi^*(z\eta) + \Phi^*(q/z\zeta) = \int_{-\pi}^{\pi} \Phi^*(ze^{-it}) d\mu_\lambda + \int_{-\pi}^{\pi} \Phi^*(q/ze^{-it}) d\nu_\lambda,$$

where

$$d\mu_\lambda = \lambda d\mu_1 + (1-\lambda)d\mu_2, \quad d\nu_\lambda = \lambda d\nu_1 + (1-\lambda)d\nu_2.$$

The uniqueness of measures in the representation (Theorem 3.1 (c)) implies that $d\mu_\lambda$ must be a point mass concentrated at t_1 ($\eta = \exp[-it_1]$) and $d\nu_\lambda$ has its mass concentrated at t_2 ($\zeta = \exp[-it_2]$). Therefore, $d\mu_1 = d\mu_2$, $d\nu_1 = d\nu_2$ and $f_1(z) \equiv f_2(z)$. This completes the proof of the theorem.

4. Typically real functions in A_q . In this section we consider functions

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

that are analytic and satisfy the condition $\operatorname{Im} f(z) \operatorname{Im} z > 0$ in the annulus A_q . Clearly, such a function must be real on the segments of the real axis in A_q , and hence $f(z) = \overline{f(\bar{z})}$ throughout A_q by the reflection principle. It follows that all of the Laurent coefficients c_n ($n = 0, \pm 1, \pm 2, \dots$) are real, and there will be no loss of generality in assuming $c_0 = 0$. We define the following two classes of typically real functions:

(4.1)

$$TR_q = \left\{ f(z) = \sum_{n=-\infty}^{\infty} c_n z^n : \text{analytic and } \operatorname{Im} f(z) \operatorname{Im} z > 0, z \in A_q \right\},$$

(4.2)

$$\tilde{TR}_q = \{ f(z) \in TR_q \text{ and } \operatorname{Im} f(z) \equiv 0 \text{ on } |z| = q \}.$$

The subclass (4.2) plays an important role relative to the full class (4.1) similar to that of $\tilde{\mathcal{P}}_q$ relative to \mathcal{P}_q . Clearly, the functions in \tilde{TR}_q are analytic at least in the annulus $q^2 < |z| < 1$. Let

$$(4.3) \quad TR = \left\{ f(z) = \sum_{n=1}^{\infty} a_n z^n : \text{analytic and } \operatorname{Im} f(z) \operatorname{Im} z > 0, z \in \Delta \right\}$$

be the class of typically real functions in the unit disk.

For typically real functions in A_q the analogue of the kernel function $\Phi^*(ze^{it})$ in $\tilde{\mathcal{P}}_q$ is

$$(4.4) \quad \Psi^*(z, t) = \frac{1}{4i \sin t} \{ \Phi^*(ze^{it}) - \Phi^*(ze^{-it}) \} = \sum_{n=-\infty}^{\infty}' \frac{1}{(1-q^{2n})} \frac{\sin nt}{\sin t} z^n,$$

where $0 < t < \pi$ and $z \in A_q$.

LEMMA 4.1. *For each $t \in (0, \pi)$ the kernel function (4.4) belongs to the class \tilde{TR}_q .*

Proof. Fix $t \in (0, \pi)$ and note that $\operatorname{Im} \Psi^*(z, t) = 0$ on $|z| = q$, since $\operatorname{Re} \Phi^*(w) = 1$ on $|w| = q$. We observe that $\operatorname{Re} \Phi^*(e^{i(\theta-a)}) = 0$ if $\theta \neq a$, since

$$\Phi^*(e^{i(\theta-a)}) = i \operatorname{ctn} \left(\frac{\theta-a}{2} \right) + 4i \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin n(\theta-a)$$

by (1.1). Furthermore, $\operatorname{Re} \Phi^*(ze^{it}) = 0$ for every z on the semicircle $C^+ = \{|z| = 1, \operatorname{Im} z \geq 0\}$ and $\operatorname{Re} \Phi^*(ze^{-it}) = 0$ for every $z \in C^+$ except $z = e^{it}$. Thus $\operatorname{Im} \Psi^*(z, t) = 0$ on the boundary of the semiannulus $A_q^+ = \{z \in A^q : \operatorname{Im} z > 0\}$ (Ψ^* has real coefficients) except at the point $z = e^{it}$. However, $\operatorname{Im} \Psi^*(z, t)$ tends to ∞ over positive values as $z \rightarrow e^{it}$ ($z \in A_q^+$), since

$$\begin{aligned} \operatorname{Im} \Psi^*(z, t) = & -\frac{1}{4 \sin t} \operatorname{Re} \Phi^*(ze^{-it}) + \frac{1}{4 \sin t} \frac{1-|z|^2}{|1-ze^{-it}|^2} + \\ & + \frac{1}{2 \sin t} \operatorname{Re} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \{(ze^{-it})^n - (ze^{-it})^{-n}\}. \end{aligned}$$

By the minimum principle for harmonic functions, we have $\operatorname{Im} \Psi^*(z, t) > 0$ in A_q^+ . Finally, $\operatorname{Im} \Psi^*(\bar{z}, t) = -\operatorname{Im} \Psi^*(z, t)$ since Ψ^* has real coefficients, and it follows that $\Psi^*(z, t) \in \tilde{TR}_q$.

THEOREM 4.1. *The following statements are equivalent:*

(a) $f(z) = \sum_{n=-\infty}^{\infty}' c_n z^n \in TR_q$.

(b) *There exist unique probability measures $d\mu, d\nu$ on $[0, \pi]$ such that*

$$(4.5) \quad c_k = \frac{c_1 - c_{-1}}{1 - q^{2k}} \int_0^\pi \frac{\sin kt}{\sin t} d\mu(t) - \frac{(c_1 q - c_{-1} q^{-1}) q^k}{1 - q^{2k}} \int_0^\pi \frac{\sin kt}{\sin t} d\nu(t),$$

$$k = \pm 1, \pm 2, \dots,$$

or, equivalently,

$$(4.6) \quad f(z) = (c_1 - c_{-1}) \int_0^\pi \Psi^*(z, t) d\mu(t) - (c_1 q - c_{-1} q^{-1}) \int_0^\pi \Psi^*(q/z, t) d\nu(t).$$

(c) *There exist unique functions $g, h \in \tilde{TR}_q$ such that*

$$(4.7) \quad f(z) = g(z) - h(q/z), \quad z \in A_q.$$

Proof. To prove that (a) implies (b) we first consider

$$f(z) = \sum'_{n=-\infty}^{\infty} c_n z^n \in TR_q$$

such that $f(z)$ is analytic in $q \leq |z| \leq 1$. The function

$$G(z) = f(z) - \sum_{n=1}^{\infty} c_{-n} (z^{-n} + z^n) = \sum_{n=1}^{\infty} (c_n - c_{-n}) z^n$$

is analytic in $|z| \leq 1$ and $\operatorname{Im} G(z) \operatorname{Im} z > 0$ on $|z| = 1$, since f has real coefficients and, therefore, $\operatorname{Im} G(z) = \operatorname{Im} f(z)$ on $|z| = 1$. It follows that

$$\operatorname{Re} \left\{ \frac{1-z^2}{z} G(z) \right\} = 2 \operatorname{Im} G(z) \operatorname{Im} z > 0$$

on $|z| = 1$, and then, by the minimum principle, $\operatorname{Re}\{(1-z^2)G(z)/z\} > 0$ for all $z \in A$. Hence $G(z)$ belongs to TR (see [8], p. 14), and necessarily $G'(0) = c_1 - c_{-1} > 0$. There exists a unique probability measure $d\mu$ on $[0, \pi]$ such that

$$\frac{G(z)}{c_1 - c_{-1}} = \int_0^\pi \frac{z}{1 - 2z \cos t + z^2} d\mu(t) = \sum_{k=1}^{\infty} \int_0^\pi \frac{\sin kt}{\sin t} d\mu(t) z^k$$

(see [8], p. 14) and, therefore,

$$(4.8) \quad c_k - c_{-k} = (c_1 - c_{-1}) \int_0^\pi \frac{\sin kt}{\sin t} d\mu(t), \quad k = 1, 2, \dots$$

In a similar fashion we find that there is a unique probability measure $d\nu$ on $[0, \pi]$ such that

$$(4.9) \quad c_k q^k - c_{-k} q^{-k} = (c_1 q - c_{-1} q^{-1}) \int_0^\pi \frac{\sin kt}{\sin t} d\nu(t), \quad k = 1, 2, \dots,$$

since

$$-f(q/z) = - \sum_{n=1}^{\infty} (c_{-n} q^{-n} z^n + c_n q^n z^{-n})$$

is typically real and analytic in the closed annulus $q \leq |z| \leq 1$. It is easily verified that (4.8) and (4.9) are equivalent to (4.5). Finally, the restriction

that f be analytic in $q \leq |z| \leq 1$ is removed by considering $f_r(z) = f(rz)$ ($q < r < 1$, $f \in TR_q$) which is analytic and typically real in $q/r \leq |z| \leq 1$. One obtains two sequences of probability measures $d\mu_r$, $d\nu_r$ on $[0, \pi]$, and by means of the Helly selection principle the general result is established. The equivalence of (4.5) and (4.6) is easily verified by manipulations of Laurent series and formula (4.4).

One obtains (c) from (b) by noting that $d\mu$, $d\nu$ in (4.6) are probability measures and $\Psi^*(w, t) \in \tilde{TR}_q$. The special properties of the functions $g, h \in \tilde{TR}_q$ in (4.7) yield (a) as a consequence of (b), and the proof is complete.

Remarks. There is an additional characterization of TR_q in terms of quadratic inequalities. We have omitted this result, since the complicated form of the inequalities diminishes the utility of such a condition. The equivalence of (a) and (b) in Theorem 4.1 has previously been proved by Nishimiya [7] via the Villat generalization of the Poisson formula for the annulus. Our proof involves only elementary computations with Laurent series and illuminates the connection between the classes of typically real functions in the annulus and in the unit disk. Part (c) of Theorem 4.1 seems to be new and indicates the significance of the important subclass \tilde{TR}_q .

THEOREM 4.2. *The following statements are equivalent:*

$$(a) f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \in \tilde{TR}_q.$$

$$(b) c_{-k} = q^{2k} c_k, \quad k = \pm 1, \pm 2, \dots, \text{ and}$$

$$(4.10) \quad F(z) = \sum_{k=1}^{\infty} \frac{c_k(1-q^{2k})}{c_1(1-q^2)} z^k \in TR.$$

(c) *There exists a unique probability measure $d\mu$ on $[0, \pi]$ such that*

$$(4.11) \quad c_k = \frac{(1-q^2)c_1}{1-q^{2k}} \int_0^\pi \frac{\sin kt}{\sin t} d\mu(t), \quad k = \pm 1, \pm 2, \dots,$$

or, equivalently,

$$(4.12) \quad f(z) = c_1(1-q^2) \int_0^\pi \Psi^*(z, t) d\mu(t).$$

(d) *If $p_k = \{(1-q^{2k+2})c_{k+1} - (1-q^{2k-2})c_{k-1}\}/(c_1(1-q^2))$, then $p_{-k} = \bar{p}_k$, $\text{Im } p_k = 0$, $k = \pm 1, \pm 2, \dots$, and*

$$(4.13) \quad \sum_{j,k=0}^N p_{j-k} \lambda_k \bar{\lambda}_j \geq 0$$

for every choice of complex numbers $\lambda_0, \lambda_1, \dots, \lambda_N$ and $N = 0, 1, \dots$

Proof. The equivalence of (b), (c) and (d) is established by straightforward computations, and appeal to the connection between the classes TR and \mathcal{P} . We omit the details. The properties of Ψ^* in (4.12) yield that (c) implies (a).

To prove that (a) implies (b) we first assume that $f \in \tilde{TR}_q$ and f is analytic in $q < |z| \leq 1$, hence in $q^2 < |z| \leq 1$. The function

$$H(z) = f(z) - \sum_{k=1}^{\infty} c_{-k}(z^{-k} + (z/q^2)^k) = \sum_{k=1}^{\infty} (c_k - q^{-2k} c_{-k}) z^k$$

is analytic and identically zero in $|z| \leq q$, since $\operatorname{Im} H(z) = \operatorname{Im} f(z) \equiv 0$ on $|z| = q$ (the coefficients of f are real). Thus $c_{-k} = q^{2k} c_k$ and a computation on the circle $|z| = q^2$ shows that

$$\operatorname{Im} \sum_{k=1}^{\infty} (c_k q^{2k} e^{ik\theta} + c_{-k} e^{-ik\theta}) = 0, \quad 0 \leq \theta \leq 2\pi.$$

The function

$$G(z) = f(z) - \sum_{k=1}^{\infty} (c_{-k} z^{-k} + c_k q^{2k} z^k) = \sum_{k=1}^{\infty} (1 - q^{2k}) c_k z^k$$

is analytic in $|z| \leq 1$ and $\operatorname{Im} G(z) = \operatorname{Im} f(z)$ on $|z| = 1$. Since $\operatorname{Im} f(z) \operatorname{Im} z > 0$ on $|z| = 1$, it follows that $\operatorname{Re}\{(1 - z^2)G(z)/z\} > 0$ in $|z| < 1$ by the minimum principle. Therefore, $F(z)$ in (4.10) belongs to TR when $f \in \tilde{TR}_q$ is analytic in $q < |z| \leq 1$. The restriction of analyticity on $|z| = 1$ is removed by the familiar approximation argument with functions $f_r(z) = f(rz)$, $q < r < 1$, $q/r < |z| \leq 1$.

Remark. In view of the preceding result there is no loss of generality in assuming that $c_1(1 - q^2) = 1$. Thus, in order to simplify notation in subsequent results, we introduce the normalized class

$$\tilde{TR}'_q = \left\{ f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \in \tilde{TR}_q : (1 - q^2)c_1 = 1 \right\}.$$

We now turn to the question of coefficient regions and general extremal problems for typically real functions in A_q . Here the coefficient points $T_{-n}^m(f)$ and coefficient regions lie in real Euclidean spaces R^m (appropriate m), since all Laurent coefficients are real. We let \mathcal{P}_R denote the subclass of functions

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}$$

with all p_k real ($k = 1, 2, \dots$). The Carathéodory-Toeplitz theorem (see Section 2) and the observation that whenever $P(z) \in \mathcal{P}$ the function

$(P(z) + \overline{P(\bar{z})})/2$ belongs to \mathcal{P}_R yield

(4.14)

$$\mathcal{K}_n(\mathcal{P}_R) = \{(p_1, \dots, p_n) \in R^n: D_k(2, p_1, \dots, p_k) \geq 0, k = 1, 2, \dots, n\};$$

$(p_1, \dots, p_n) \in \text{bdry } \mathcal{K}_n(\mathcal{P}_R)$ if and only if $D_j(2, p_1, \dots, p_j) = 0$ for some $j \in \{1, 2, \dots, n\}$, and for each such boundary point the function $P(z) \in \mathcal{P}_R$ such that $T_n(P) = (p_1, \dots, p_n)$ is uniquely determined and of the form

(4.15)

$$P(z) = \sum_{k=1}^n \lambda_k \frac{1-z^2}{1-2z \cos \theta_k + z^2}, \quad \theta_k \in [0, \pi], \quad 0 \leq \lambda_k, \quad \sum_{k=1}^n \lambda_k = 1.$$

THEOREM 4.3. *Let c_k for $k = \pm 1, \pm 2, \dots, \pm n$ ($c_1 = 1/(1-q^2)$) be $2n$ real numbers. Necessary and sufficient conditions for the existence of a function $f \in \widetilde{TR}'_q$ such that $T_{-n}^n(f) = (c_{-n}, \dots, c_{-1}, c_1, \dots, c_n)$ are*

$$(4.16) \quad c_{-k} = q^{2k} c_k, \quad k = 1, 2, \dots, n,$$

$$(4.17) \quad D_k(2, p_1, \dots, p_k) \geq 0, \quad p_k = (1 - q^{2k+2})c_{k+1} - (1 - q^{2k-2})c_{k-1}, \\ k = 1, 2, \dots, n-1,$$

where $c_0 = 0$. Furthermore, if (4.16) and (4.17) hold, and if $D_j(2, p_1, \dots, p_j) = 0$ for some $j \in \{1, 2, \dots, n\}$, then the function $f \in \widetilde{TR}'_q$ such that $T_{-n}^n(f) = (c_{-n}, \dots, c_n)$ is uniquely determined and of the form

$$(4.18) \quad f(z) = \sum_{k=1}^{n-1} \lambda_k \Psi^*(z, \theta_k), \quad \theta_k \in [0, \pi], \quad 0 \leq \lambda_k, \quad \sum_{k=1}^{n-1} \lambda_k = 1.$$

Proof. The necessity of the conditions and (4.18) follow directly from Theorem 4.2 and the Carathéodory-Toeplitz results (4.14) and (4.15). Conversely, if c_k for $k = \pm 1, \pm 2, \dots, \pm n$ ($c_1 = 1/(1-q^2)$) satisfy the hypothesis of the theorem, and p_k ($k = 1, 2, \dots, n-1$) are the corresponding numbers in (4.17), then (4.14) implies the existence of a function

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad \text{in } \mathcal{P}_R$$

such that $T_{n-1}(P) = (p_1, \dots, p_n)$. Then

$$F(z) = \frac{z}{1-z^2} P(z) = z + \sum_{k=1}^{\infty} a_k z^k$$

belongs to TR and $p_1 = a_2$, $p_2 = a_2 - 1$, $p_k = a_{k+1} - a_{k-1}$, $k = 3, 4, \dots$. We extend the given sequence $\{c_k\}$ ($k = \pm 1, \pm 2, \dots, \pm n$) by defining

$c_k = a_k/(1 - q^{2k})$ and $c_{-k} = q^{2k}c_k$ for $k = 1, 2, \dots$. Hence

$$F(z) = z + \sum_{k=1}^{\infty} c_k(1 - q^{2k})z^k$$

belongs to TR and, by Theorem 4.2, the function

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$$

is in \tilde{TR}'_q with $T_{-n}^n(f) = (c_{-n}, \dots, c_n)$.

THEOREM 4.4. *Let $G = G(x_1, \dots, x_{2n})$ be a real-valued function of $2n$ real variables that is defined and continuous on $\mathcal{X}_{-n}^n(\tilde{TR}'_q)$ and has the property that it achieves its maximum over $\mathcal{X}_{-n}^n(\tilde{TR}'_q)$ only on the boundary. Let $J(f) = G(T_{-n}^n(f))$, $f \in \tilde{TR}'_q$. Then J is a continuous functional on \tilde{TR}'_q and the solution to the extremal problem*

$$\max \{J(f) : f \in \tilde{TR}'_q\}$$

must be a function of the form (4.18).

Proof. The map $\varphi(c_{-n}, \dots, c_n) = (p_1, \dots, p_{n-1})$ with p_k as determined in (4.17) defines a homeomorphism of $\mathcal{X}_{-n}^n(\tilde{TR}'_q)$ onto $\mathcal{X}_{n-1}(\mathcal{P}_R)$ by Theorems 4.2 and 4.3. If $(p_1, \dots, p_{n-1}) \in \mathcal{X}_{n-1}(\mathcal{P}_R)$, then

$$\varphi^{-1}(p_1, \dots, p_{n-1}) = (c_{-n}, \dots, c_n) \in \mathcal{X}_{-n}^n(\tilde{TR}'_q),$$

where

$$c_k = \begin{cases} \frac{p_1 + p_3 + \dots + p_{k-1}}{1 - q^{2k}} & \text{for } k \text{ even,} \\ \frac{1 + p_2 + \dots + p_{k-1}}{1 - q^{2k}} & \text{for } k \text{ odd,} \end{cases}$$

and

$$c_{-k} = q^{2k}c_k, \quad k = 1, 2, \dots, n.$$

By means of this homeomorphism we can transplant J to the functional

$$J_1(P) = G(\varphi^{-1}(T_{n-1}(P))), \quad P \in \mathcal{P}_R.$$

The rest of the proof is essentially the same as that of Theorem 2.2 and we omit the details.

If

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \in TR_q,$$

then (4.6) shows that c_1 and c_{-1} are parameters that must be fixed in order to specify a compact family for the study of extremal problems. We let $TR_q(c_{-1}, c_1)$ denote the set of functions in TR_q with fixed c_{-1}, c_1 . It is easy to verify that (4.7) yields the following representation for $f \in TR_q(c_{-1}, c_1)$:

$$(4.19) \quad f(z) = (c_1 - c_{-1})h(z) - (qc_1 - q^{-1}c_{-1})g(q/z), \quad h, g \in \tilde{TR}'_q.$$

It is also convenient to introduce the coefficient region

$$\begin{aligned} \mathcal{K}_{-n}^n(\tilde{TR}'_q)^* &= \{T_{-n}^n(f(q/z)): f \in \tilde{TR}'_q\} \\ &= \{(q^n c_n, \dots, qc_1, c_{-1}/q, \dots, c_{-n}/q^n): (c_{-n}, \dots, c_n) \in \mathcal{K}_{-n}^n(\tilde{TR}'_q)\}. \end{aligned}$$

By an obvious correspondence, $\mathcal{K}_{-n}^n(\tilde{TR}'_q)$ and $\mathcal{K}_{-n}^n(\tilde{TR}'_q)^*$ are homeomorphic and each boundary point of $\mathcal{K}_{-n}^n(\tilde{TR}'_q)^*$ corresponds to a unique function $f(q/z)$, where f is of the form (4.18). As a consequence of these remarks and the representation (4.19) we get

THEOREM 4.5. *We have*

$$\mathcal{K}_{-n}^n(TR_q(c_{-1}, c_1)) = (c_1 - c_{-1})\mathcal{K}_{-n}^n(\tilde{TR}'_q) - (c_1 q - c_{-1}/q)\mathcal{K}_{-n}^n(\tilde{TR}'_q)^*.$$

Furthermore, a boundary point of $\mathcal{K}_{-n}^n(TR_q(c_{-1}, c_1))$ is associated with a unique function of the form

$$(4.20) \quad f(z) = (c_1 - c_{-1}) \sum_{k=1}^{n-1} \lambda_k \Psi^*(z, \theta_k) - (c_1 q - c_{-1}/q) \sum_{k=1}^{n-1} \gamma_k \Psi^*(q/z, \varphi_k),$$

$$\theta_k, \varphi_k \in [0, \pi], \lambda_k, \gamma_k \geq 0 \text{ and } \sum_{k=1}^{n-1} \lambda_k = 1 = \sum_{k=1}^{n-1} \gamma_k.$$

By this result we obtain immediately the following

THEOREM 4.6. *Let $G = G(x_1, \dots, x_{2n})$ be a real-valued function that is continuous on $\mathcal{K}_{-n}^n(TR_q(c_{-1}, c_1))$ and achieves its maximum over this set only on the boundary. If $J(f) = G(T_{-n}^n(f))$, $f \in TR_q(c_{-1}, c_1)$, then the solution to the extremal problem*

$$\max\{J(f): f \in TR_q(c_{-1}, c_1)\}$$

must be a function of the form (4.20).

Finally, we mention the following

THEOREM 4.7. *We have*

$$\text{Ext}(\tilde{TR}'_q) = \{\Psi^*(z, \theta): \theta \in [0, \pi]\}$$

and

$$\begin{aligned} \text{Ext}(TR_q(c_{-1}, c_1)) \\ = \{(c_1 - c_{-1})\Psi^*(z, \theta) - (c_1 q - c_{-1}/q)\Psi^*(z, \varphi): \theta, \varphi \in [0, \pi]\}. \end{aligned}$$

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UNIVERSITY OF DELAWARE
NEWARK, DELAWARE

UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA

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