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Characterization of quasi-analytic functions of several variables by means of rational approximation

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1. Introduction. Let F be a compact set in the space C^n of n complex variables. Let $\mathscr{C}(F)$ denote the Banach algebra of all complex continuous functions on F with the norm

$$\|f\|_F = \sup_{z \in F} |f(z)| \quad \text{for } f \in \mathscr{C}(F).$$

Let $\mathscr{P}_\nu$ denote the set of all polynomials in $z = (z_1, \ldots, z_n) \in C^n$ of degree $\leqslant \nu$ and let $\mathscr{Q}_\nu$ denote the set of all rational functions r such that $r = P/Q$, where $P, Q \in \mathscr{P}_\nu$ and the polynomials P, Q are assumed to be mutually prime.

Given a function $f \in \mathscr{C}(F)$, we write

$$E_\nu(f, F) = \inf_{P \in \mathscr{P}_\nu} \|f - P\|_F$$

and

$$R_\nu(f, F) = \inf_{r \in \mathscr{Q}_\nu} \|f - r\|_F.$$

Let $\mathscr{B}(F)$ and $\mathscr{R}(F)$ denote the sets of all functions $f \in \mathscr{C}(F)$ such that

$$\liminf_{\nu \to \infty} \sqrt[\nu]{E_\nu(f, F)} < 1$$

and

$$\liminf_{\nu \to \infty} \sqrt[\nu]{R_\nu(f, F)} < 1,$$

respectively. It is obvious that $\mathscr{B}(F) \subset \mathscr{R}(F)$.

Now let $n = 1$ and $F = [-1, 1]$. In this case the following identity principle for functions of the class $\mathscr{B}(F)$ has been proved by Bernstein (see [1], p. 286):

(I) *If $f \in \mathscr{B}(F)$ and $f(x) = 0$ for $x \in [a, \beta] \subset F$, then $f(x) = 0$ for $x \in F$.*

This classical result admits generalizations and can also be proved for functions $f \in \mathscr{B}(F)$ in the case where F is a sufficiently "good" compact set in C^n (see [6]). Owing to property (I) the functions of the class $\mathscr{B}(F)$ are called *quasi-analytic* in the sense of Bernstein.

Recently Gončar [3] has announced (without proof) an identity principle for functions $f \in \mathscr{R}(F)$, $F = [-1, 1]$, analogous to (I)(1).

The main purpose of this paper is to give a characterization of functions $f \in \mathscr{B}(F)$ by means of rational approximation. To formulate our result we use the extremal function $\Phi(z, F)$ of a compact set F in C^n introduced by Siciak [8]

$$\Phi(z, F) = \sup_{\nu \geq 1}\{\sup\{|P(z)|^{1/\nu}: P \in \mathscr{P}_\nu, \|P\|_F \leq 1\}\}, \qquad z \in C^n.$$

Given any compact set F in C^n, we shall denote throughout by $\hat{F}$ the polynomially convex envelope of F. Then our result reads as follows.

THEOREM 1. *Let F be a compact set in C^n such that the extremal function $\Phi(z, F)$ is locally bounded in C^n and let f be a function of the algebra $\mathscr{C}(F)$. Then f is the restriction to F of a function $\tilde{f} \in \mathscr{B}(\hat{F})$ if and only if for a sequence $\nu_k \to \infty$ of positive integers there exist rational functions $r_{\nu_k} \in \mathscr{Q}_{\nu_k}$ and an open neighbourhood Ω of the set $\hat{F}$ such that*

$$\text{(i)} \qquad \limsup_{k \to \infty} \sqrt[\nu_k]{\|f - r_{\nu_k}\|_F} < 1$$

and

(ii) *the functions r_{ν_k} are holomorphic in Ω for $k \geq 1$.*

Remark. Recently Szabados [10] has proved the following weaker version of Gončar's identity principle:

Let f be a function continuous in $F = [-1, 1]$. Assume that there exist rational functions $r_{\nu_k} \in \mathscr{Q}_{\nu_k}$, $k \geq 1$, and a neighbourhood Ω (in C) of F satisfying (i) and (ii) of Theorem 1. Then $f(x) = 0$ for $x \in [\alpha, \beta] \subset F$ $\Rightarrow f(x) = 0$ for $x \in F$.

Since here $F = \hat{F}$ and $\Phi(z, F) = |z + \sqrt{z^2 - 1}|$, it follows from Theorem 1 that the result of Szabados is in fact a consequence of Bernstein's theorem (I).

Let F be a compact set in C^n and let $\mathscr{R}_0(F)$ denote the subclass of the class $\mathscr{R}(F)$ consisting of all functions f such that

$$\liminf_{\nu \to \infty} \sqrt[\nu]{R_\nu(f, F)} = 0.$$

In Section 3 we give an identity principle for functions $f \in \mathscr{R}_0(F)$ which generalizes an identity principle for these functions given by Gončar [2] in the case where $F \subset [0, 1]$ and $\operatorname{mes}(F) > 0$(1).

2. Proof of Theorem 1. We shall need two lemmas.

LEMMA 1. *Let $F = F_1 \times \ldots \times F_n$, where F_j is a compact set in the complex z_j-plane, $j = 1, \ldots, n$. Let $\mathscr{F}$ be a family of polynomials P in $z = (z_1, \ldots, z_n)$ such that*

(1) See: Note added in proof, p. 156.

(i) *there exists an open neighbourhood Ω of the set F such that $P(z) \neq 0$ for $z \in \Omega$, $P \in \mathscr{F}$.*

Then there exists a constant $\delta > 0$ such that

$$|P(z)| \geqslant \left(\frac{\delta}{|F|+\delta}\right)^{n \deg P} \|P\|_F \quad \text{for } z \in F,\ P \in \mathscr{F},$$

where $|F| = \operatorname{diam} F$ [2].

Proof. Choose a constant $\delta > 0$ such that $\{z \in C^n: \operatorname{dist}(z, F) < \delta\} \subset \Omega$. Take a polynomial $P \in \mathscr{F}$ and fix a point $a = (a_1, \ldots, a_n) \in F$ in such a way that $\|P\|_F = |P(a)|$. Write

$$p(z_1) = P(z_1, a_2, \ldots, a_n).$$

The polynomial p may be rewritten in the form

$$p(z_1) = \xi(z_1 - a_1) \ldots (z_1 - a_l),$$

where $0 \leqslant l \leqslant \deg P$, ξ and a_j ($j = 1, \ldots, l$) are complex numbers depending on the point a and the polynomial P. It follows from assumption (i) that $p(a_1) = P(a) \neq 0$. Hence

$$\left|\frac{p(z_1)}{p(a_1)}\right| = \left|\frac{z_1 - a_1}{a_1 - a_1}\right| \ldots \left|\frac{z_1 - a_l}{a_1 - a_l}\right|. \tag{1}$$

If $z_1 \in F_1$, then $|z_1 - a_1| \leqslant |F_1| \leqslant |F|$. On the other hand, by assumption (i), $|z_1 - a_j| \geqslant \delta$ for $z_1 \in F_1$, $j = 1, \ldots, l$. Hence

$$\left|\frac{z_1 - a_j}{a_1 - a_j}\right| \geqslant \frac{|z_1 - a_j|}{|z_1 - a_1| + |z_1 - a_j|} \geqslant \frac{\delta}{|F| + \delta} \tag{2}$$

for $z_1 \in F_1$, $j = 1, \ldots, l$. By (1) and (2) we obtain

$$\left|\frac{p(z_1)}{p(a_1)}\right| \geqslant \left(\frac{\delta}{|F|+\delta}\right)^l \quad \text{for } z_1 \in F_1$$

and hence

$$|P(z_1, a_2, \ldots, a_n)| \geqslant \left(\frac{\delta}{|F|+\delta}\right)^{\deg P} \|P\|_F \quad \text{for } z_1 \in F_1. \tag{3}$$

Because of (3), to complete the proof of the lemma it suffices to apply induction with respect to n.

Further on we shall often use the following properties of the extremal function Φ defined in Section 1 (see [8]):

(a) $|P(z)| \leqslant \|P\|_F[\Phi(z, F)]^{\deg P}$, $z \in C^n$, for every polynomial P.

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(2) The constant $\left(\frac{\delta}{|F|+\delta}\right)^n$ may be replaced by $\frac{\delta^n}{(|F_1| + \delta) \ldots (|F_n| + \delta)}$, where $|F_j| = \operatorname{diam} F_j$, $j = 1, \ldots, n$.

(b) If a compact set F contains the Cartesian product $G_1 \times \ldots \times G_n$, where G_j is a compact set in the z_j-plane with positive transfinite diameter $d(G_j)$, $j = 1, \ldots, n$, then the function $\Phi(z, F)$ is locally bounded in C^n.

LEMMA 2. *Let Ω be an open set in C^n and let $\mathcal{F}$ be a family of polynomials P such that*

(i) $$P(z) \neq 0 \quad \text{for } z \in \Omega,\ P \in \mathcal{F}.$$

Then for every compact set F, $F \subset \Omega$, there exist a constant θ, $0 < \theta < 1$, and a compact set G such that $F \subset \operatorname{int} G \subset G \subset \Omega$ and

$$|P(z)| \geqslant \theta^{\deg P} \|P\|_G \quad \text{for } z \in G,\ P \in \mathcal{F}.$$

Proof. Fix a compact set F, $F \subset \Omega$. Let Δ be a component of the set Ω such that $F_\Delta = F \cap \Delta \neq \emptyset$. In virtue of the Borel–Lebesgue theorem, there exists a finite system of closed polydisks $T_1, \ldots, T_m$ such that

(1) $$T_j \subset \Delta \quad \text{for } j = 1, \ldots, m$$

and

(2) $$F_\Delta \subset \operatorname{int} \bigcup_{j=1}^{m} T_j.$$

Since the set Δ is open and connected, the polydisks T_j may be chosen in such a way that

(3) $$T_j \cap T_{j+1} \neq \emptyset \quad \text{for } j = 1, \ldots, m,$$

where $T_{m+1} = T_1$. Take a polynomial $P \in \mathcal{F}$. Because of assumption (i) and (1), we may apply Lemma 1 for every T_j. So there exist constants τ_j, $0 < \tau_j < 1$, $j = 1, \ldots, m$, independent of P and such that

(4) $$|P(z)| \geqslant \tau_j^{\deg P} \|P\|_{T_j} \quad \text{for } z \in T_j,\ j = 1, \ldots, m.$$

Set $G_\Delta = \bigcup_{j=1}^{m} T_j$. Because of (3), we may assume that $\|P\|_{G_\Delta} = \|P\|_{T_1}$. Then, by (3) and (4), we obtain

$$\|P\|_{T_2} \geqslant \tau_1^{\deg P} \|P\|_{G_\Delta}.$$

Hence and again by (4) we get

$$|P(z)| \geqslant (\tau_1 \cdot \tau_2)^{\deg P} \|P\|_{G_\Delta} \quad \text{for } z \in T_1 \cup T_2.$$

Repeating, if necessary, our procedure we come to the inequality

(5) $$|P(z)| \geqslant (\tau_1 \ldots \tau_m)^{\deg P} \|P\|_{G_\Delta} \quad \text{for } z \in G_\Delta.$$

Now let Δ_k be the k-th component of the set Ω, $k = 1, 2, \ldots$ Write $F^k = F \cap \Delta_k$ and note that there exists an index r such that $F^k \neq \emptyset$ for $k \leqslant r$ and $F^k = \emptyset$ for $k > r$. In virtue of (5), for every $k = 1, \ldots, r$ we can find a compact set G^k being the sum of polydisks and a constant

$\eta_k \in (0, 1)$ such that $F^k \subset \operatorname{int} G^k \subset G^k \subset \Omega$ and

$$(6) \qquad |P(z)| \geqslant \eta_k^{\deg P} \|P\|_{G^k} \quad \text{for } z \in G^k,\ P \in \mathscr{F}.$$

Set $G = \bigcup_{k=1}^{r} G^k$. It follows from property (b) of the extremal function Φ that the functions $\Phi(z, G^k)$, $k = 1, \ldots, r$, are locally bounded in C^n. Hence

$$(7) \qquad H_k = \sup_{z \in G} \Phi(z, G^k) < +\infty, \quad k = 1, \ldots, r.$$

Write $H = \max\{H_1, \ldots, H_r\}$. By (7) and by property (a) of Φ, we obtain

$$(8) \qquad \|P\|_{G^k} \geqslant (1/H)^{\deg P} \|P\|_G, \quad P \in \mathscr{F}.$$

From (6) and (8) we get

$$|P(z)| \geqslant \theta^{\deg P} \|P\|_G \quad \text{for } z \in G,\ P \in \mathscr{F},$$

where $\theta = \min_{1 \leqslant k \leqslant r} \{\eta_k\}/H$. This completes the proof of the lemma.

Now we are able to prove Theorem 1. The proof of the necessity of conditions (i) and (ii) follows immediately from the definition of the class $\mathscr{B}(F)$. It remains to prove the sufficiency.

Suppose that $r_{\nu_k} = P_{\nu_k}/Q_{\nu_k}$, where $P_{\nu_k}, Q_{\nu_k} \in \mathscr{P}_{\nu_k}$ and P_{ν_k}, Q_{ν_k} are mutually prime, $k = 1, 2, \ldots$, is a sequence of rational functions satisfying assumptions (i) and (ii) of Theorem 1. We may assume that

$$(1) \qquad \|Q_{\nu_k}\|_{\hat{F}} = 1 \quad \text{for } k \geqslant 1.$$

By virtue of (ii), $Q_{\nu_k}(z) \neq 0$ for $z \in \Omega$, $\hat{F} \subset \Omega$. Hence applying Lemma 2 we can find a compact set G, $\hat{F} \subset \operatorname{int} G \subset G \subset \Omega$, and a constant $\theta \in (0, 1)$ independent of k and such that

$$(2) \qquad |Q_{\nu_k}(z)| \geqslant \theta^{\nu_k} \quad \text{for } z \in G,\ k \geqslant 1.$$

On the other hand, it follows from (i) and (1) that there exists a constant M such that

$$\|P_{\nu_k}\|_{\hat{F}} \leqslant M \quad \text{for } k \geqslant 1.$$

Hence, since the extremal function $\Phi(z, F)$ is assumed to be locally bounded in C^n, applying property (a) of Φ we obtain

$$(3) \qquad \|P_{\nu_k}\|_G \leqslant M H^{\nu_k}, \quad \geqslant 1,$$

where H is a constant independent of k. By (2) and (3) we get

$$\|r_{\nu_k}\|_G \leqslant A^{\nu_k}, \quad k \geqslant 1,$$

A being a constant independent of k. Hence, since the functions r_{ν_k} are holomorphic in the common neighbourhood int G of the set $\hat{F}$, by Lemma 1 in [7] for every $k \geqslant 1$ we can find polynomials $W_\nu^k \in \mathscr{P}_\nu$, $\nu \geqslant 1$, and constants M_1, τ, $0 < \tau < 1$, independent of k and ν and such that

(4) $$\|r_{\nu_k} - W_\nu^k\|_{\hat{F}} \leqslant M_1 A^{\nu_k}\tau^\nu, \quad \nu \geqslant 1,\ k \geqslant 1.$$

Take a positive integer l so large that $A\tau^l \leqslant \tau$. Then, by (4), we get

(5) $$\|r_{\nu_k} - W_{l\cdot\nu_k}^k\|_{\hat{F}} \leqslant M_1 \tau^{\nu_k}, \quad k \geqslant 1.$$

By (i), (5) and the triangle inequality we obtain

$$\|f - W_{l\cdot\nu_k}^k\|_F \leqslant M_2 \eta^{\nu_k}, \quad k \geqslant 1,$$

M_2 and η being constants independent of k, $0 < \eta < 1$. This implies that $f \in \mathscr{B}(F)$. Hence, in virtue of Lemma 3 in [7], there exists a function $\tilde{f} \in \mathscr{B}(\hat{F})$ such that $\tilde{f}|_F = f$. The proof is completed.

Let $\mathscr{S}$ denote the set of all increasing sequences of positive integers. Given a sequence $\{\nu_k\} \in \mathscr{S}$, we denote by $[\{\nu_k\}]$ the set of all sequences $\{\mu_k\} \in \mathscr{S}$ such that

$$1/M < \nu_k/\mu_k < M, \quad k \geqslant 1,$$

for a constant M independent of k. For a fixed sequence $\{\nu_k\} \in \mathscr{S}$ we denote by $\mathscr{B}(F, [\{\nu_k\}])$ and $\mathscr{R}(F, [\{\nu_k\}])$ the sets of functions $f \in \mathscr{C}(F)$ satisfying the requirements

$$\limsup_{k\to\infty} \sqrt[\mu_k]{E_{\mu_k}(f, F)} < 1 \quad \text{for an } \{\mu_k\} \in [\{\nu_k\}],$$

and

$$\limsup_{k\to\infty} \sqrt[\mu_k]{R_{\mu_k}(f, F)} < 1 \quad \text{for an } \{\mu_k\} \in [\{\nu_k\}],$$

respectively. One can check that $\mathscr{R}(F, [\{\nu_k\}])$ is a ring with respect to the ordinary point-wise addition and multiplication of functions and $\mathscr{B}(F, [\{\nu_k\}])$ is a subring of the ring $\mathscr{R}(F, [\{\nu_k\}])$.

It is seen from the proof that Theorem 1 gives the following characterization of the subring $\mathscr{B}(F, [\{\nu_k\}])$ in the ring $\mathscr{R}(F, [\{\nu_k\}])$:

THEOREM 2. *Let F satisfy the assumptions of Theorem* 1 *and let $f \in \mathscr{R}(F, [\{\nu_k\}])$. Then $f \in \mathscr{B}(F, [\{\nu_k\}])$ if and only if there exist rational functions $r_{\mu_k} \in \mathscr{Q}_{\mu_k}$, where $\{\mu_k\} \in [\{\nu_k\}]$, and an open set Ω, $\hat{F} \subset \Omega$, satisfying* (i) *and* (ii) *of Theorem* 1.

3. The identity principle for functions of the class $\mathscr{R}_0(F)$. Let F be a fixed compact set in C^n. Given any set G in C^n, we write

$$\varphi_F(G) = \sup\{1/\sup_{z\in F}\Phi(z, K)\colon K \text{ is a compact subset of } G\},$$

where $\Phi(z, K)$ is the extremal function of K. We propose to call the set-function φ_F a *Φ-capacity of the set G with respect to F.*

Now assume that K is a compact set in the space C of one complex variable. Then it is known [9] that

$$\Phi(z, K) = +\infty \quad \text{for } z \in C\setminus K \Leftrightarrow \Phi(\mathring{z}, K) = +\infty$$
$$\text{at a point } \mathring{z} \in C\setminus K \Leftrightarrow d(K) = 0,$$

where $d(K)$ denotes the transfinite diameter of K. Hence we obtain the following characterization of polar sets G with respect to the Φ-capacity in terms of the logarithmic capacity $c_l(G)$ (comp. [5]):

For every Borel set G in C, $c_l(G) = 0$ if and only if $\varphi_F(G) = 0$ for any compact set $F \subset C$ such that $F \cap (C\setminus G) \neq \emptyset$.

THEOREM 3. *Let F be a compact set in C^n and let f be a function of the class $\mathscr{R}_0(F)$. If f vanishes on a closed subset I of F such that $\varphi_F(I) > 0$, then $\varphi_I(\{z \in F\colon f(z) \neq 0\}) = 0$.*

Proof. Since $f \in \mathscr{R}_0(F)$, there exist polynomials $P_{\nu_k}, Q_{\nu_k} \in \mathscr{P}_{\nu_k}$, where $\{\nu_k\} \in \mathscr{S}$, and a sequence $\{\varepsilon_k\}$ of positive numbers, $\lim\limits_{k\to\infty} \varepsilon_k = 0$, such that

$$\left\| f - \frac{P_{\nu_k}}{Q_{\nu_k}} \right\|_F \leqslant \varepsilon_k^{\nu_k}, \quad k \geqslant 1. \tag{1}$$

Since $f(z) = 0$ for $z \in I$, we have

$$\left\| \frac{P_{\nu_k}}{Q_{\nu_k}} \right\|_I \leqslant \varepsilon_k^{\nu_k}, \quad k \geqslant 1. \tag{2}$$

We may assume that

$$\|Q_{\nu_k}\|_I = 1, \quad k \geqslant 1. \tag{3}$$

Suppose that $f(z) \neq 0$ for $z \in G \subset F$, $\varphi_I(G) > 0$. Then, by the definition of $\varphi_I(G)$, there exists a compact set $K \subset G$ such that $\varphi_I(K) > 0$. Now for every $k = 1, 2, \ldots$ there must exist a point $z^{(k)} \in K$ such that

$$|Q_{\nu_k}(z^{(k)})| \geqslant [\varphi_I(K)]^{\nu_k}, \quad k \geqslant 1. \tag{4}$$

Indeed, if $|Q_{\nu_k}(z)| < [\varphi_I(K)]^{\nu_k}$ for $z \in K$, then, by property (a) of the extremal function Φ, we would have $|Q_{\nu_k}(z)| < 1$ for $z \in I$, which is impossible because of (3).

On the other hand, by (2) and (3), applying property (a) of Φ we obtain

$$|P_{\nu_k}(z)| \leqslant [\varepsilon_k/\varphi_K(I)]^{\nu_k} \quad \text{for } z \in K,\ k \geqslant 1, \tag{5}$$

where $\varphi_K(I) \geqslant \varphi_F(I) > 0$. Choosing, if necessary, a subsequence of the sequence $\{z^{(k)}\}$, we may assume that $\lim\limits_{k\to\infty} z^{(k)} = a \in K$. Then, by (1), (4) and (5), we get

$$|f(a)| = \lim_{k\to\infty}|f(z^{(k)})| \leqslant \lim_{k\to\infty}\left|f(z^{(k)}) - \frac{P_{\nu_k}(z^{(k)})}{Q_{\nu_k}(z^{(k)})}\right| + \lim_{k\to\infty}\left|\frac{P_{\nu_k}(z^{(k)})}{Q_{\nu_k}(z^{(k)})}\right|$$
$$\leqslant \lim_{k\to\infty}\varepsilon_k^{\nu_k} + \lim_{k\to\infty}[\varepsilon_k/\varphi_K(I)\cdot\varphi_I(K)]^{\nu_k} = 0.$$

We have got a contradiction, which completes the proof.

COROLLARY. *If* $f\in\mathscr{R}_0(F)$ *and* f *vanishes on a subset* I *of* F *such that* $\varphi_F(I) > 0$, *then* $f(z) = 0$ *at each point* $z\in F$ *such that*

$$\varphi_I\big(F\cap B(z,\varepsilon)\big) > 0 \quad \textit{for every } \varepsilon > 0,$$

where $B(z,\varepsilon)$ *denotes the ball with centre* z *and radius* ε.

In particular, $f(z) = 0$ at each point $z = (z_1, \ldots, z_n)\in F$ such that

$$d\big(K_j\cap B(z_j,\varepsilon)\big) > 0 \quad \text{for every } \varepsilon > 0,\ j = 1, \ldots, n.$$

Remark. Theorem 3 generalizes a certain result of Gončar (Theorem 1 in [2]) equivalent to the following:

Let F be a closed subset of the line-segment $[0,1]$ and let $f\in\mathscr{R}_0(F)$.

If $f(x) = 0$ for $x\in I\subset F$, $\operatorname{mes}(I) > 0$, then f vanishes almost everywhere in F.

In the case where F is a compact subset of C, another proof of Theorem 3 can be obtained by using an important result of Gončar dealing with a connection of rational functions with the modulus of a plane condenser (see [4], Theorem 1).

Note added in proof. Recently Gončar [11] has proved that every function $f\in\mathscr{R}(F)$, where $F = [0,1]$, vanishing on a subset I of F with $d(I) > 0$ vanishes identically on F. This result includes Theorem 3 in the case $n = 1$ and $F = [0,1]$.

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