A CHAINABLE CONTINUUM
NOT HOMEOMORPHIC TO AN INVERSE LIMIT ON [0, 1]
WITH ONLY ONE BONDING MAP*

BY

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1. Introduction. Mahavier has exhibited (¹) a chainable continuum not homeomorphic to an inverse limit on [0, 1] with only one bonding map. Here we present such a continuum which is not homeomorphic to Mahavier's example and is of a simpler nature than his example.

2. Definitions, notation and a theorem. If each term of the sequence \( f_1, f_2, f_3, \ldots \) maps \([0, 1]\) onto \([0, 1]\), then the inverse limit of the sequence \( f_1, f_2, f_3, \ldots \), denoted by invlim([0, 1], \( f_i \)), is the subspace of the Cartesian product \( \prod_{i=1}^{\infty} [0, 1]_i \) to which the number sequence \( x_1, x_2, x_3, \ldots \) belongs only in case \( f_n(x_{n+1}) = x_n \) for each positive integer \( n \). If \( f_1, f_2, f_3, \ldots \) is a constant sequence, say \( g \) is \( f_i \) for each positive integer \( i \), then invlim([0, 1], \( g \)) denotes invlim([0, 1], \( f_i \)). If \( x_1, x_2, x_3, \ldots \) is a constant number sequence, then \( (x_i) \) denotes the point \( (x_1, x_2, x_3, \ldots) \) in \( \prod_{i=1}^{\infty} [0, 1]_i \).

For each positive integer \( j \), \( \pi_j \) denotes the projection mapping from \( \prod_{i=1}^{\infty} [0, 1]_i \) onto the \( j \)-th factor space. By continuum we mean a non-degenerate, compact, connected metric space. The metric \( d \) on \( \prod_{i=1}^{\infty} [0, 1]_i \) is defined as

\[
\begin{align*}
  d(x, y) = \sum_{i=1}^{\infty} \frac{|\pi_i(x) - \pi_i(y)|}{2^i}
\end{align*}
\]

for each \( x \) and \( y \) belonging to \( \prod_{i=1}^{\infty} [0, 1]_i \). If \( T \) denotes an arc with non-separating points \( a \) and \( b \), we write \( T \) as \( [a, b] \).

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THEOREM. Suppose that \( k \) maps \([0, 1]\) onto \([0, 1]\) in such a way that \( A = \text{invlim}([0, 1], k) \) is the sum of two mutually exclusive connected sets \( M \) and \( N \), where \( M \) is a topological ray, \( N \) is an arc, and \( M \) is dense in \( A \). Then there exists a proper subcontinuum \([a, b]\) of \([0, 1]\) such that \( N \) is homeomorphic to \( \text{invlim}([a, b], k^2([a, b])) \) and each of the non-separating points of \( N \) corresponds to a constant number sequence in \( \text{invlim}([a, b], k^2([a, b])) \).

Proof. Since \( k \) is not the identity, the function \( h \) defined as

\[
h(x_1, x_2, x_3, \ldots) = (k(x_1), k(x_2), k(x_3), \ldots) = (k(x_1), x_1, x_2, \ldots)
\]

for each point \((x_1, x_2, x_3, \ldots)\) in \( A \) is a non-trivial homeomorphism from \( A \) onto \( A \) (op. cit.). For each positive integer \( j \) we have \( \pi_j N = \pi_j hN \), since \( h \) maps \( N \) onto \( N \). Also, by the definition of \( h \) we have \( \pi_j N = \pi_{j+1} hN \). Thus \( \pi_j N = \pi_{j+1} N \) and if \([a, b]\) denotes \( \pi_j N \), then \([a, b]\) is a proper subcontinuum of \([0, 1]\) such that \( N \) is homeomorphic to \( \text{invlim}([a, b], k^2([a, b])) \).

The two non-separating points of \( N \) must either be fixed points of \( h \) or be switched by \( h \). So we infer that any projection of a non-separating point of \( N \) is a fixed point of \( k^2 \), thus the non-separating points of \( \text{invlim}([a, b], k^2([a, b])) \) are constant number sequences.

3. Example. In this section we give an example of a chainable continuum not homeomorphic to an inverse limit on \([0, 1]\) with only one bonding map.

Let \( f \) be the mapping from \([0, 1]\) onto \([0, 1]\) defined by

\[
f(x) = \begin{cases} 
4x & \text{if } x \text{ is in } [0, 1/4], \\
-2x + 3/2 & \text{if } x \text{ is in } [1/4, 1/2], \\
x & \text{if } x \text{ is in } [1/2, 1]. 
\end{cases}
\]

Let \( g \) be the mapping from \([0, 1]\) onto \([0, 1]\) defined by

\[
g(x) = \begin{cases} 
4x & \text{if } x \text{ is in } [0, 1/4], \\
-3x + 7/4 & \text{if } x \text{ is in } [1/4, 1/3], \\
3x - 1/4 & \text{if } x \text{ is in } [1/3, 5/12], \\
-6x + 7/2 & \text{if } x \text{ is in } [5/12, 1/2], \\
x & \text{if } x \text{ is in } [1/2, 1].
\end{cases}
\]

Let \( N_1, N_2, N_3, \ldots \) denote the positive integer sequence such that \( N_1 = 2 \), and if \( n \) is a positive integer greater than 1, then \( N_n = N_{n-1} + n + 1 \). Let \( t_1, t_2, t_3, \ldots \) denote the function sequence defined as

\[
t_n = \begin{cases} 
g & \text{if } n = N_j \text{ for some positive integer } j, \\
f & \text{otherwise}. 
\end{cases}
\]

Let \( A \) denote \( \text{invlim}([0, 1], t_i) \).

The continuum \( A \) is the union of a topological ray and an arc, say \( M \) and \( N \), respectively, such that \( M \) and \( N \) are mutually exclusive and \( M \) is dense in \( A \). The arc \( N \) is \( \text{invlim}([1/2, 1], t_i | [1/2, 1]) \) with non-sepa-
rating points \((1/2)\) and \((1)\), and \(t_n|[1/2, 1]\) is the identity for each positive integer \(n\). Note here that a point \(x\) in \(A\) is a constant number sequence if and only if \(x\) belongs to \(N\) or \(\pi_1(x) = 0\). The reader will likely recognize that the continuum \(A\) is a sinusoid homeomorphic to

\[\text{\includegraphics[width=0.5\textwidth]{sinusoid.png}}\]

\[\text{\includegraphics[width=0.2\textwidth]{point.png}}\]

4. Proof. We prove here that the continuum \(A\) is not homeomorphic to an inverse limit on \([0, 1]\) using only one bonding map.

Assume that there exists a mapping \(k'\) from \([0, 1]\) onto \([0, 1]\) such that \(\text{invlim}([0, 1], k')\) is homeomorphic to \(A\). Let \(k\) denote \((k')^2\), \(A_k\) the continuum \(\text{invlim}([0, 1], k)\), and \(F\) a homeomorphism from \(A\) onto \(A_k\). By the Theorem, there exists a proper subcontinuum of \([0, 1]\), say \([a, b]\), such that \(F[N]\) is homeomorphic \(\text{invlim}([a, b], k|[a, b])\). Let \(h\) denote the non-trivial homeomorphism from \(A_k\) onto \(A_k\) defined by

\[h(x_1, x_2, x_3, \ldots) = (k(x_1), k(x_2), k(x_3), \ldots) = (k(x_1), x_1, x_2, \ldots)\]

for each point \((x_1, x_2, x_3, \ldots)\) in \(A_k\). From the Theorem we also infer that the non-separating points \(F((1/2))\) and \(F((1))\) in \(F[N]\) are constant sequences, say \((c)\) and \((d)\), respectively. The non-separating point \(F((0))\) in \(F[M]\) is a fixed point of \(h\), thus a constant sequence, say \((p)\), and \(p\) belongs to \([0, 1]\)\([a, b]\).

Let \(x_0\) be a point in \([0, 1]\) such that \(k(x_0) = c\) and such that \(x_0\) belongs to the component of \([0, 1]\)\([a, b]\) containing \(p\). That there is such a point follows from the irreducibility of \(A_k\) from \((p)\) to any point of \(F[N]\) together with the intermediate value theorem. Let \(y\) be a point in the ray \(F[M]\) such that \(\pi_i(y) = x_0\) and let \(y_1, y_2, y_3, \ldots\) denote the sequence of points in \(F[M]\) such that \(y_m = h^m(y)\) for each positive integer \(m\). We observe that, for each positive integer \(m\), if \(j\) is a positive integer not greater than \(m\), then \(\pi_j(y_m) = c\). Thus \(y_1, y_2, y_3, \ldots\) converges to \((c)\).

For each point \(x\) in the ray \(M\) there is a positive integer \(m\) such that \(\pi_m(x) < 1/2\). Let \(J\) be the function from \(M\) onto the positive integers such that if \(x\) is in \(M\), then \(J(x)\) is the least positive integer \(i\) such that \(\pi_i(x) < 1/2\).

The continuum \(A\) is ordered with respect to the following meaning of the word "precedes":

1. if each of \(u\) and \(v\) is a point on the ray \(M\), then \(u\) precedes \(v\) provided \(J(u) < J(v)\) or \(J(u) = J(v)\) and \(\pi_{J(u)}(u) \leq \pi_{J(v)}(v)\);

2. if each of \(u\) and \(v\) belongs to \(N\), then \(u\) precedes \(v\) provided \(\pi_1(u) \leq \pi_1(v)\);
(3) if one of \( u \) and \( v \) belongs to \( M \) and the other to \( N \), then \( u \) precedes \( v \) provided \( u \) belongs to \( M \).

We observe that, for points in the ray \( M \), the above-defined meaning of "precedes" is equivalent to the usual order on a ray. Since \( F \) is order-preserving, \( x \) precedes \( y \) in the ray \( F[M] \) provided \( x \) is \((p)\) or \( x \) is a separating point of the arc \((p, y)\).

It follows from the order on \( A_k \), the definition of the homeomorphism \( h \) and the convergence of the sequence \( y_1, y_2, y_3, \ldots \) to a point of \( F[N] \) that if \( i \) denotes a positive integer, then \( y_i \) precedes \( y_{i+1} \). Thus, for each positive integer \( i \), if \( C_i \) denotes the arc \([y_i, y_{i+1}]\) and \( D_i \) the arc \( F^{-1}[C_i] \), then \( h[C_i] = C_{i+1} \) and \( F^{-1}hF[D_i] = D_{i+1} \). Let \( G \) denote \( F^{-1}hF \).

We now show the following:

1. There exist positive integers \( L \) and \( Q \) such that if \( s \) is an integer greater than \( L \), then the subset \( V_s \) of \( D_s \) to which \( v \) belongs only in case \( \pi_1(v) = 1 \) contains only \( Q \) elements.

To see this we first show:

2. There exists a positive integer \( L' \) such that if \( m \) is an integer greater than \( L' \), then \( D_m \) contains a point \( u \) such that \( \pi_1(u) = 1 \).

Since \( y_1, y_2, y_3, \ldots \) converges to \((c)\), \( F^{-1}(y_1), F^{-1}(y_2), F^{-1}(y_3), \ldots \) converges to \((1/2)\). Let \( \varepsilon > 0 \) be such that \( \varepsilon < 1/8 \). Let \( W \) denote a positive integer such that if \( v \) is an integer greater than \( W \), then

\[
d(F^{-1}(y), (1/2)) < \varepsilon.
\]

Thus \( \pi_1(F^{-1}(y)) < 3/4 \), for if \( x \) belongs to \( M \) and \( \pi_1(x) \geq 3/4 \), then

\[
d(x, (1/2)) = \sum_{i=1}^{\infty} \frac{|\pi_1(x) - 1/2|}{2^i} \geq \frac{1}{8} + \sum_{i=2}^{\infty} \frac{|\pi_1(x) - 1/2|}{2^i} > \varepsilon.
\]

Assuming that statement (2) is not true, we let \( T \) denote the set to which the integer \( s \) belongs if and only if \( s > W \) and 1 is not in \( \pi_1[D_s] \). Let \( T_1, T_2, T_3, \ldots \) denote the increasing integer sequence with final set \( T \). For each of \( m \) and \( n \) denotes a positive integer, then \( \pi_n[D_{T_m}] \) is a subset of the half-open interval \([0, 3/4]\). This is a consequence of the order of the elements in \( F^{-1}(y_1), F^{-1}(y_2), F^{-1}(y_3), \ldots \) together with the fact that, given a positive integer \( j \), \( d(F^{-1}(y_{T_j}), (1/2)) < \varepsilon \) but no point \( z \) in \( D_{T_j} \) is such that \( \pi_1(z) = 1 \). Let \( Q_1, Q_2, Q_3, \ldots \) denote an infinite, increasing, positive integer sequence such that if \( j \) denotes a positive integer, then \( D_{Q_j} \) contains a point \( y \) such that \( \pi_1(y) = 1 \) and \( D_{Q_j - 1} = D_{T_n} \) for some positive integer \( n \). Let \( z_1, z_2, z_3, \ldots \) denote a sequence such that if \( m \) is a positive integer, then \( z_m \) belongs to \( D_{Q_m} \) and \( \pi_1(z_m) = 1 \). The sequence \( z_1, z_2, z_3, \ldots \) converges to \((1)\), and so does the sequence \( G^{-1}(z_1), G^{-1}(z_2), G^{-1}(z_3), \ldots \). However, if \( j \) is a positive integer, then \( G^{-1}(z_j) \) is a point of \( D_{T_m} \) for some integer \( m \), so \( \pi_n G^{-1}(z_j) \) is not greater than 3/4 for any
positive integer \( n \), and thus \( G^{-1}(z_1), G^{-1}(z_2), G^{-1}(z_3), \ldots \) cannot converge to (1). This is a contradiction from which it follows that statement (2) is true.

Let \( L' \) denote a positive integer as in statement (2) and such that \( L' > W \). For each positive integer \( i \) greater than \( L' \), let \( V_i \) denote the set to which \( v \) belongs if and only if \( v \) is a point of \( D_i \) and \( \pi_1(v) = 1 \); let \( p(i) \) denote the number of elements belonging to \( V_i \). We write \( V_i \) as \( \{v_{i1}, v_{i2}, \ldots, v_{ip(i)}\} \) and note that if \( v_{js} \) precedes \( v_{kl} \), where \( v_{js} \) and \( v_{kl} \) are points of \( \bigcup_{i > L'} V_i \), then either \( j < k \) or \( j = k \) and \( s < t \). Let \( a_1, a_2, a_3, \ldots \) denote the sequence of points in \( M \) with final set \( \bigcup_{i > L'} V_i \) and such that if \( a_j \) precedes \( a_k \), then \( j < k \). The sequence \( a_1, a_2, a_3, \ldots \) converges to (1) as do the sequences \( G(a_1), G(a_2), G(a_3), \ldots \) and \( G^{-1}(a_1), G^{-1}(a_2), G^{-1}(a_3), \ldots \). Let \( b > 0 \) be such that \( b < 1/16 \). Let \( R \) denote an integer such that if \( s \) is an integer greater than \( R \), then

\[
d(a_s, (1)) < b, \quad d(G(a_s), (1)) < b \quad \text{and} \quad d(G^{-1}(a_s), (1)) < b.
\]

Thus we have

\[
\pi_1(a_s) > 7/8, \quad \pi_1(G(a_s)) > 7/8 \quad \text{and} \quad \pi_1(G^{-1}(a_s)) > 7/8,
\]

for if \( z \) belongs to \( M \) and \( d(z, (1)) < b \), then

\[
\frac{|\pi_1(z) - 1|}{2} < b < \frac{1}{16},
\]

so \( \pi_1 z > 7/8 \).

Assume now that statement (1) is not true. Let \( m_1, m_2, m_3, \ldots \) denote an increasing, positive integer sequence such that \( m_1 > L' \) and either \( p(m_j) < p(m_j + 1) \) for each positive integer \( j \) or \( p(m_j) > p(m_j + 1) \) for each positive integer \( j \).

Suppose first that \( p(m_j) < p(m_j + 1) \) for each positive integer \( j \). Let \( K \) denote a positive integer such that whenever \( u \) denotes a positive integer greater than \( K \) and \( i \) denotes a positive integer not greater than \( p(m_u) \), then \( v_{mu} \) is \( a_s \) for some positive integer \( s \) greater than \( R \). Now let \( u \) denote a positive integer greater than \( K \); there exist points \( x(u) \) and \( y(u) \) in \( V_{m_u + 1} \) such that \( x(u) \) precedes \( y(u) \) and such that each point \( x \) in the arc \( G^{-1}([x(u), y(u)]) \) has the property that

\[
\pi_1(x) \geq \min \{\pi_1(G^{-1}(x(u))), \pi_1(G^{-1}(y(u)))\}.
\]

Thus \( \pi_1 z > 7/8 \). Let \( a_{m_u + 1} \) denote the arc \([x(u), y(u)]\) and let \( a_{m_1} \) denote \( G^{-1}([x(u), y(u)]) \); i.e., \( a_{m_u} \) is the arc \([G^{-1}(x(u)), G^{-1}(y(u))] \). Let \( w_1, w_2, w_3, \ldots \) denote a sequence of points such that \( w_1 \) belongs to \( a_{m_j} \) for each positive integer \( j \). The sequence \( w_1, w_2, w_3, \ldots \) has the sequential limit point (1) and \( G(w_j) \) belongs to \( a_{m_j + 1} \) for each positive integer \( j \).
For each positive integer $j$, $\alpha_{n_j+1}$ contains a point, say $d_j$, such that $\pi_1(d_j) = \frac{3}{4}$, and so the sequence $d_1, d_2, d_3, \ldots$ converges to $\frac{3}{4}$. The sequence $G^{-1}(d_1), G^{-1}(d_2), G^{-1}(d_3), \ldots$ converges to $(1)$ but $G^{-1}(\frac{3}{4}) \neq (1)$, a contradiction from which it follows that $p(m_j) > p(m_j + 1)$ for each positive integer $j$. A similar argument contradicts $p(m_j) > p(m_j + 1)$ for each positive integer $j$ from which it follows that statement $(1)$ is true.

Let $L$ and $Q$ denote positive integers which satisfy statement $(1)$. There exists a positive integer greater than $L$, say $L_0$, such that if $s$ denotes a positive integer greater than $L_0$ and $i$ denotes the least positive integer $i$ such that $N_i \leq J(z)$ for each point $z$ in $D_s$, then $N_{i+1} - N_i \geq 3Q$.

There exists an increasing, positive integer sequence, say $s_1, s_2, s_3, \ldots$, such that $s_i > L_0$ and if $i$ is a positive integer, then $D_{s_i}$ contains an arc, say $\gamma_{s_i}$, with non-separating points in $V_{s_i}$, and such that if $z$ is a point of $\gamma_{s_i}$, then $J(z) = N_i$ for some positive integer $i$. So $\pi_1(z) \geq \frac{3}{4}$, and if $z_1, z_2, z_3, \ldots$ denotes a sequence of points such that $z_n$ is a point of $\gamma_{s_n}$ for each positive integer $n$, then any limit point of $\{z_1, z_2, z_3, \ldots\}$ is a point of the arc $N$ and has the first projection not less than $\frac{3}{4}$. Let $(q)$ denote $G((\frac{3}{4})); q$ is a number such that $1/2 < q < 1$.

If $i$ is a positive integer, then $G(\gamma_{s_i})$ is a subset of $D_{s_{i+1}}$, call it $\gamma_{s_{i+1}}$. Since $s_i > L_0$, $D_{s_{i+1}}$ does not contain a point $u$ such that $J(u) = N_i$ for any positive integer $i$. For each positive integer $i$, let $x(s_i)$ and $y(s_i)$ denote the non-separating points of $\gamma_{s_i}$ with $x(s_i)$ preceding $y(s_i)$. It follows from an argument similar to that in proving statement $(1)$ that there do not exist more than finitely many integers $m$ such that if $z$ is a point of the arc $G[[x(s_m), y(s_m)]]$, then

$$\pi_1(z) \geq \min \{ \pi_1 G[x(s_m)], \pi_1 G[y(s_m)] \}.$$ 

So there is a positive integer $T$ such that if $n$ is a positive integer greater than $T$, then $JG(x(s_n)) \neq JG(y(s_n))$. Let $B$ denote a positive integer such that if $n$ is a positive integer greater than $B$, then $\gamma_{s_n+1}$ contains a point $z$ with $\pi_1(z) = 1/2$. Let $h_1, h_2, h_3, \ldots$ denote a sequence of points in $M$ such that if $i$ is a positive integer, then $h_i$ belongs to $\gamma_{s_{(B+1)+1}}$ and $\pi_1(h_i) = 1/2$. The sequence $h_1, h_2, h_3, \ldots$ converges to $(1/2)$ and so does $G^{-1}(h_1), G^{-1}(h_2), G^{-1}(h_3), \ldots$ But the sequence $G^{-1}(h_1), G^{-1}(h_2), G^{-1}(h_3), \ldots$ converges to a point with the first projection not less than $3/4$. Thus $G$ is not a homeomorphism, a contradiction from which it follows that the continuum $A$ is not an inverse limit on $[0, 1]$ with only one bonding map.

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