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A characterization of a minimal submanifold in R^{n+p}

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Abstract. Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Euclidean space R^{n+p} with $n \geqslant 2$, $p \geqslant 1$. By a Euclidean vector field X on M we mean a differentiable mapping which assigns to each point of M a tangent vector to R^{n+p}. We formulate the definition of divergence of such vector field with respect to a moving frame on M and we obtain some conditions, stated in terms of divergence of such fields, for M to be a minimal submanifold. These generalize some results of [1], [5].

We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ in R^{n+p} such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M (and consequently, the remaining vectors $e_{n+1}, \ldots, e_{n+p}$ are normal to M). With respect to this frame field, let $W_1, \ldots, W_{n+p}$ be the field of dual frames. If W_{AB} $(A, B = 1, \ldots, n+p)$ are the connection forms, then

$$de_A = \sum_{B=1}^{n+p} W_{AB} e_B, \tag{1}$$

where de_A is the covariant differential of e_A and the multiplication in the product $W_{AB} e_B$ is in the sense of tensor product. If we restrict the forms to M, then

$$W_a = 0, \quad a = n+1, \ldots, n+p.$$

By Cartan's lemma we may write

$$W_{ai} = \sum_{j=1}^{n} h_{ij}^a W_j \quad (i = 1, \ldots, n,\ a = n+1, \ldots, n+p). \tag{2}$$

M is called a *minimal submanifold* if the mean curvature vector

$$H = \frac{1}{n} \sum_{i,a} h_{ii}^a e_a \tag{3}$$

vanishes identically, i.e. if $\sum_i h_{ii}^a = b$ for all a.

DEFINITION 1. The divergence of a Euclidean vector field X on M is defined by

$$\operatorname{div} X = \langle dX, \theta \rangle (e_1, \ldots, e_n), \tag{4}$$

where

(5) $$\Theta = \sum_{i=1}^{n} (-1)^{i-1} (W_1 \wedge \ldots \wedge W_{i-1} \wedge W_{i+1} \wedge \ldots \wedge W_n) e_i$$

and the multiplication of the 1-form coefficients in the inner product $\langle dX, \Theta \rangle$ is in the sense of exterior multiplication.

Remark. The above definition is just the well-known definition of divergence for a vector field on M [3].

We will prove

THEOREM 1. *Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Euclidean space R^{n+p} with $n \geq 2$, $p \geq 1$. Let $X = \sum_{A=1}^{n+p} X^A e_A$ be a Euclidean vector fields on M. If $\bar{X} = \sum_{i=1}^{n} X^i e_i$, then*

$$\operatorname{div} X = \operatorname{div} \bar{X} + n \sum_{a=n+1}^{n+p} X^a H^a,$$

where

(6) $$H^a = \frac{1}{n} \sum_{i=1}^{n} h^a_{ii}, \quad a = n+1, \ldots, n+p.$$

THEOREM 2. *Let M be an n-dimensional manifold immersed in an $(n+p)$-dimensional Euclidean space R^{n+p}. If X_i $(i = 1, \ldots, n)$ are n linearly independent vector fields on M, then*

$$(\operatorname{div} L) e_{n+k} = n H^{n+k} L,$$

where

$$L = X_1 \times \ldots \times X_n \times e_{n+1} \times \ldots \times e_{n+k-1} \times e_{n+k+1} \times \ldots \times e_{n+p}$$

for $k = 1, \ldots, p$.

We denote by $U_1 \times \ldots \times U_k$ the vector product of k vectors $U_1, \ldots, U_k$ of R^{k+1}, [4].

To prove Theorems 1, 2 we will need the following three lemmas.

The first lemma follows immediately from the linearity of d and Definition 1.

LEMMA 1. *If X, Y are two Euclidean vector fields on M, then*

$$\operatorname{div}(X+Y) = \operatorname{div} X + \operatorname{div} Y.$$

The following lemma generalizes relation (b) of Theorem 2.1 of [1].

LEMMA 2. *Let X be a Euclidean vector field on M. If f is a differ-*

entiable function on M, then

$$\operatorname{div}(fX) = f\operatorname{div}X + \langle \operatorname{grad} f, X\rangle,$$

where grad f *is the gradient of* f.

Proof. We have

$$\begin{aligned}
\operatorname{div}(fX) &= \langle d(fX), \Theta\rangle(e_1, \dots, e_n) \\
&= df\langle X, \Theta\rangle(e_1, \dots, e_n) + f\langle dX, \Theta\rangle(e_1, \dots, e_n) \\
&= df\langle \sum_{i=1}^{n} X^i e_i, \Theta\rangle(e_1, \dots, e_n) + df\langle \sum_{a=n+1}^{n+p} X^a e_a, \Theta\rangle(e_1, \dots, e_n) + f\operatorname{div}X \\
&= df\langle \sum_{i=1}^{n} X^i e_i, \Theta\rangle(e_1, \dots, e_n) + f\operatorname{div}X \\
&= df\sum_{i=1}^{n}(X^i(-1)^{i-1}W_1\wedge \dots \wedge W_{i-1}\wedge W_{i+1}\wedge \dots \wedge W_n)\times \\
&\qquad \times(e_i, \dots, e_n) + f\operatorname{div}X \\
&= \sum_{i=1}^{n}(X^i W_1\wedge \dots \wedge W_{i-1}\wedge df\wedge W_{i+1}\wedge \dots \wedge W_n)(e_1, \dots, e_n) + f\operatorname{div}X \\
&= \sum_{i=1}^{n} X^i df(e_i) + f\operatorname{div}X = df(\sum_{i=1}^{n} X^i e_i) + f\operatorname{div}X \\
&= df(\bar{X}) + f\operatorname{div}X = \langle \operatorname{grad} f, \bar{X}\rangle + f\operatorname{div}X = \langle \operatorname{grad} f, X\rangle + f\operatorname{div}X.
\end{aligned}$$

LEMMA 3. $\sum_{i=1}^{n} W_1\wedge \dots \wedge W_{i-1}\wedge W_{ai}\wedge W_{i+1}\wedge \dots \wedge W_n = nH^a W_1\wedge \dots \wedge W_n$.

Proof. If T is the first member, then from (2) we have

$$\begin{aligned}
T &= \sum_{i=1}^{n} W_1\wedge \dots \wedge W_{i-1}\wedge (\sum_{j=1}^{n} h_{ij}^a W_j)\wedge W_{i+1}\wedge \dots \wedge W_n \\
&= \sum_{i=1}^{n} W_1\wedge \dots \wedge W_{i-1}\wedge h_{ii}^a W_i\wedge W_{i+1}\wedge \dots \wedge W_n \\
&= (\sum_{i=1}^{n} h_{ii}^a) W_1\wedge \dots \wedge W_i\wedge \dots \wedge W_n = nH^a W_1\wedge \dots \wedge W_n.
\end{aligned}$$

Proof of Theorem 1. Since

$$X = \bar{X} + \sum_{a=n+1}^{n+p} X^a e_a,$$

we have

$$dX = d\bar{X} + \sum_{a=n+1}^{n+p}(dX^a)e_a + \sum_{a=n+1}^{n+p} X^a de_a.$$

Moreover, from Lemmas 1–3 we get

$$\begin{aligned}\operatorname{div} X &= \langle dX, \Theta\rangle(e_1, \dots, e_n) = \langle d\dot{X}, \Theta\rangle(e_1, \dots, e_n) + \sum_{a=n+1}^{n+p} X^a \langle de_a, \Theta\rangle (e_1, \dots, e_n)\\
&= \operatorname{div} \dot{X} + \sum_{a=n+1}^{n+p} X^a \langle de_a, \Theta\rangle (e_1, \dots, e_n)\\
&= \operatorname{div} \dot{X} + \sum_a X^a \langle \sum_{i=1}^{n} W_{ai} e_i, \Theta\rangle (e_1, \dots, e_n)\\
&= \operatorname{div} \dot{X} + \sum_a X^a \sum_{i=1}^{n} (-1)^{i-1} W_{ai} \wedge (W_1 \wedge \dots \wedge W_{i-1} \wedge W_{i+1} \wedge \dots \wedge W_n) \cdot\\
&\qquad \cdot (e_1, \dots, e_n)\\
&= \operatorname{div} \dot{X} + \sum_a X^a \sum_{i=1}^{n} (W_1 \wedge \dots \wedge W_{i-1} \wedge W_{ai} \wedge W_{i+1} \wedge \dots \wedge W_n)(e_1, \dots, e_n)\\
&= \operatorname{div} \dot{X} + n \sum_a X^a H^a \quad \text{(from Lemma 3).}\end{aligned}$$

The next corollary follows immediately from Theorem 1.

Corollary 1. $\operatorname{div} e_a = nH^a$, $a = n+1, \dots, n+p$.

This generalizes relation (2.9) of [1].

Proof of Theorem 2. If $X_i = \sum_j X_i^j e_j$, then (see [4])

$$X_1 \times \dots \times X_n \times e_{n+1} \times \dots \times e_{n+k-1} \times e_{n+k+1} \times \dots \times e_{n+p} = (-1)^{n+p-1}(-1)^{n+k+1} (\det X) e_{n+k},$$

where $\det X = \det (X_i^j)$.

Moreover, from Lemma 2 and Corollary 1 we obtain

$$\begin{aligned}&\operatorname{div} (X_1 \times \dots \times X_n \times e_{n+1} \times \dots \times e_{n+k-1} \times e_{n+k+1} \times \dots \times e_{n+p}) e_{n+k}\\
&\quad = (-1)^{n+p-1}(-1)^{n+k+1} \{(\det X) \operatorname{div} e_{n+k}\} e_{n+k}\\
&\quad = nH^{n+k}(-1)^{n+k+1}(-1)^{-(n+k+1)} X_1 \times \dots \times X_n \times\\
&\qquad\qquad \times e_{n+1} \times \dots \times e_{n+k-1} \times e_{n+k+1} \times \dots \times e_{n+p}\\
&\quad = nH^{n+k} X_1 \times \dots \times X_n \times e_{n+1} \times \dots \times e_{n+k-1} \times e_{n+k+1} \times \dots \times e_{n+k}.\end{aligned}$$

An immediate result, which follows from Theorem 2 and which generalizes Corollary 3.1 in [1], is the following

Corollary 2. *M is minimal if for any n linearly independent vector fields on M it holds*

$$\operatorname{div} (X_1 \times \dots \times X_n \times e_{n+1} \times \dots \times \hat{e}_{n+k} \times \dots \times e_{n+p}) = 0 \quad \textit{for all } k,$$

where $\hat{e}_{n+k}$ indicates that the vector e_{n+k} is ommited.

Let M be a compact hypersurface in R^{n+1} and r be the position vector of M. From Theorem 1, Green's theorem [2], p. 11, and the formulas of Minkowski–Hsiung [2], p. 196, we get the following

COROLLARY 3. $\int_M \operatorname{div} r\, dM = -n \operatorname{vol}(M)$, *where* dM *is the volume element of* M *and* $\operatorname{vol}(M)$ *volume of* M.

References

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