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On algebraic curves (of low genus) defined by Kleinian groups *

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Abstract. Let S be a Riemann surface of finite type uniformized by a finitely generated Kleinian group Γ. Using the group Γ, one should be able to reconstruct the function theory of S. For example, one should be able to construct from Γ two meromorphic functions on the compactification $\bar{S}$ of S that generate the field of meromorphic functions on $\bar{S}$. Explicit algorithms for such constructions are, however, not known.

This paper is a contribution towards the solution of this general problem. We treat the simplest cases: tori, punctured tori, and (hyperelliptic) surfaces of genus two. Our work on punctured tori complements the extensive investigations of Keen [5].

The key tool in our approach is Poincaré series. Such series provide the intermediate step between the group theory of the Kleinian group Γ and the function theory on the surfaces represented by Γ. The role of Poincaré series in this situation is analogous to the role of the Riemann theta function in studying function theory on a surface by means of its period matrix.

1. Teichmüller spaces and Bers fiber spaces. Let G be a finitely generated Kleinian group. A quasi-conformal self-map w of $\hat{C} = C \cup \{\infty\}$ is a *deformation* of G if wGw^{-1} is again a Kleinian group. A deformation w is *trivial* if there exists a Möbius transformation A such that $w \circ g \circ w^{-1} = A \circ g \circ A^{-1}$ for all $g \in G$. The *deformation space* $T(G)$ is the space of deformations of G modulo trivial deformations. The deformation space is known to be a complex manifold [2], [11], [6]. If G is a non-elementary, then we can pick three distinct points $\alpha_1, \alpha_2, \alpha_3$ in the limit set Λ of G. Every deformation w of G is equivalent to a normalized w (that is, a w that fixes α_1, α_2, α_3) and

$$T(G) \cong \left\{\begin{matrix}\text{restrictions to } \Lambda \text{ of normalized} \\ \text{deformations } w \text{ of } G\end{matrix}\right\}. \tag{1.1}$$

Let Δ be an invariant union of components of the region of discontinuity Ω of G. The space $T(G, \Delta)$ consists of the image in $T(G)$ of

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those deformations w that are conformal off Δ. It is a submanifold (see, for example, [6]) of $T(G)$. Sullivan [13] has shown that $T(G) = T(G, \Omega)$.

If Δ is connected and simply connected and Δ/G is of type (p, n), then $T(G, \Delta)$ is a model for the Teichmüller space $T(p, n)$. See [6] and the literature quoted there. We proceed to describe various specific models for Teichmüller spaces. Assume that $2p-2+n > 0$.

Bers model. $T(p, n) = T(G, \Delta)$, where G is a finitely generated quasi-Fuchsian group of type (p, n) and Δ is an invariant component of G. In particular, G can be taken to be Fuchsian and Δ, a disk in $\hat{C}$. See [1], [9].

Maskit model. $T(p, n) = T(G)$, where G is a finitely generated non-elementary function group with a simply connected invariant component Δ such that Δ/G is of type (p, n) and $(\Omega-\Delta)/G$ is a finite union of thrice punctured spheres. See [12], [9].

Earle model. $T(p, n) = T(G)$, where G is a $\boldsymbol{Z}_2$-extension of a quasi-Fuchsian group Γ of type (p, n) by an element γ that permutes the invariant components of Γ. See [4], [9].

If G is a finitely generated quasi-Fuchsian group of type (p, n), then $T(G)$ is the space of quasi-Fuchsian groups of type (p, n), and

$$T(G) \cong T(p, n) \times T(p, n),$$

and the real points in $T(G)$ are a real analytic model for $T(p, n)$. See [8].

Assume now that G is non-elementary and Δ is an invariant union of components of G. Let us normalize all deformations w at three points in Δ. The fiber $\pi^{-1}([w])$ of the *Bers fiber space*

$$\pi\colon\ F(G, \Delta) \to T(G, \Delta)$$

over the equivalence class $[w] \in T(G, \Delta)$ is the open subset $w(\Delta)$ of $\hat{C}$. The set $w(\Delta)$ is invariant under the Kleinian group wGw^{-1}.

One of the interesting problems in the theory of moduli is to obtain explicit maps between the various models of $T(p, n)$. For the case $(p, n) = (1, 1)$, we will show how to construct the classical modulus for the torus from the various models.

2. The classical uniformization of tori. Let G be a discrete rank two parabolic group with generators A, B, where

$$A(z) = z+1, \qquad B(z) = z+i, \qquad z \in \hat{C}.$$

(We will write $G = \langle A, B \rangle$.) For every deformation w of G and every Möbius transformation C, $C \circ w$ is equivalent to w. Hence we may assume that w is normalized; that is, w fixes $0, 1, \infty$. A normalized deformation w conjugates $G = G_i$ onto $G_{w(i)} = \langle A, B_{w(i)} \rangle$, where

$$B_a(z) = z+a.$$

A normalized deformation w is equivalent to a Möbius transformation if and only if $w(i) = i$. It is easy to see that for every normalized deformation w, we have $\operatorname{Im} w(i) > 0$. For $\varepsilon \in C$, $0 \leqslant |\varepsilon| < 1$,

$$w(z) = \frac{z+\varepsilon\bar{z}}{1+\varepsilon}, \quad z \in \hat{C}, \tag{2.1}$$

is a normalized deformation of G with

$$w(i) = i\frac{1-\varepsilon}{1+\varepsilon}.$$

Hence,

$$T(G) \cong \{\tau \in C;\ \operatorname{Im}\tau > 0\} = T(1, 0).$$

The curves in the fundamental group $\pi_1(C/G)$ corresponding to A and B form a canonical homology basis on the torus C/G. We conclude that

$$w(i) = \tau = \int_B dz \quad \text{on } C/wGw^{-1},$$

where dz is, of course, the normalized abelian differential of the first kind dual to the canonical homology basis. We also remark that the maps of the form (2.1) are the extremal or Teichmüller maps for G, and that a normalized Teichmüller map w takes the lattice points of G_i onto the lattice points of $G_{w(i)}$; that is, w preserves the origins of the tori. Similarly, w maps points of order two (half-lattice point) to points of order two. We call τ the *classical modulus* of the marked torus $C/G_{w(i)}$ (the marking is provided by the homology classes of the curves represented by A and $B_{w(i)}$).

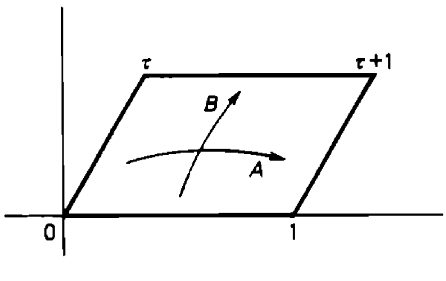

Fig. 1. Euclidean model for punctured torus

3. Quasi-Fuchsian groups representing punctured tori (the Bers model). For $\tau \in U = \{z \in C;\ \operatorname{Im} z > 0\}$, G_τ denotes the group generated by

$$z \mapsto z+1, \quad z \mapsto z+\tau.$$

We view G_τ as acting on

$$C_\tau = \{z \in C;\ z \neq n+m\tau \text{ for all } n, m \in Z\}.$$

We let

$$\pi = \pi_\tau : U \to C_\tau$$

be a holomorphic universal covering map, and we let $\Gamma = \Gamma_\tau$ be the Fuchsian model for the action of G_τ on C_τ; that is,

$$\Gamma_\tau = \{\gamma \in \mathrm{PSL}(2, \mathbf{R});\ \pi \circ \gamma = g \circ \pi \text{ for some } g \in G_\tau\}.$$

Choose A, $B \in \Gamma_\tau$ such that

$$\pi(Az) = \pi(z) + 1, \quad \pi(Bz) = \pi(z) + \tau \quad \text{for all } z \in U, \tag{3.1}$$

A, B generate Γ_τ, and

$$P = B^{-1} \circ A^{-1} \circ B \circ A \tag{3.2}$$

is parabolic. Furthermore, the elements A and B of $\pi_1(S)$ correspond to a canonical homology basis in $H_1(\bar{S})$, $S = U/\Gamma_\tau = C_\tau/G_\tau$, $\bar{S}$ = one point compactification of S. We normalize Γ_τ so that $P(z) = z + 1$ (replace π by $\pi \circ \gamma$ and Γ by $\gamma \Gamma \gamma^{-1}$ with $\gamma \in \mathrm{PSL}(2, \mathbf{R})$) and so that

$$\lim_{z \to i\infty} \pi(z) = 0 \tag{3.3}$$

(replace π by $g \circ \pi$ with $g \in G_\tau$). The normalization of P and (3.3) imply that $U(1)/\langle P \rangle$ is mapped bijectively by π onto a punctured neighbourhood of the origin. Here, for $c > 0$,

$$U(c) = \{z \in \mathbf{C};\ \mathrm{Im}\, z > c\}.$$

We lift the fundamental domain for G_τ given in Fig. 1 to a fundamental domain for Γ_τ as shown in Fig. 2. The lattice points 0, 1, $1+\tau$, τ for G_τ correspond to the boundary points ∞, a, 0, b, with $a < 0$, and $b > 0$. (Without loss of generality, we may take 0 to be one of these boundary points.)

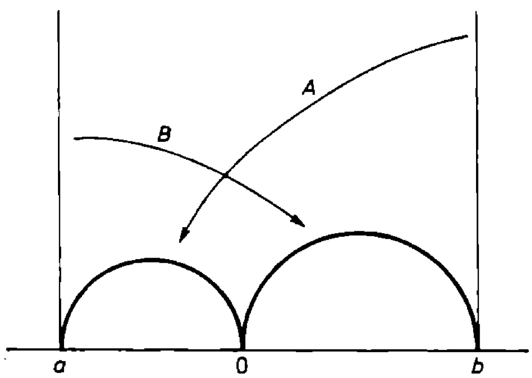

Fig. 2. Non-euclidean model for punctured torus

Lemma 3.1. *Let* $\varphi(z) = \pi'(z)^2$, $z \in U$. *Then* φ *is a cusp form for* Γ *of weight* (-4) *defined on* U.

Proof. From (3.1) we see that $\varphi(\gamma z)\gamma'(z)^2 = \varphi(z)$, all $z \in U$, all $\gamma \in \Gamma$. As a consequence of (3.3) and the equation $\varphi(z+1) = \varphi(z)$, all $z \in U$, we see that

$$\lim_{z \to i\infty} \varphi(z) = 0.$$

Hence φ is a cusp form.

LEMMA 3.2. *Let Γ be a torsion free function group with a simply connected invariant component Δ such that Δ/Γ is of type* (1,1). *Let $\varphi \neq 0$ be a cusp form for Γ of weight (-4) defined on Δ. Let f be a solution of*

$$(f')^2 = \varphi. \tag{3.4}$$

Then f is a covering map of the complement D of a rank 2 lattice G in C such that $\Delta/\Gamma \cong D/G$.

Proof. Assume first that $\Gamma = \Gamma_\tau$, $\tau \in U$, as constructed above and $\Delta = U$. There exist in this case, a non-zero constant c and a constant b such that $f = c\pi + b$.

For the general case, let ϱ: $U \to \Delta$ be a Riemann map and $\Gamma_0 = \varrho^{-1}\Gamma\varrho$. Then Γ_0 is a torsion free Fuchsian group of type (1,1), and by replacing ϱ by $\varrho \circ g$ with $g \in \mathrm{PSL}(2, \mathbf{R})$ if necessary, we may assume that $\Gamma_0 = \Gamma_\tau$, as above. The cusp form φ is the square of η, the lift to Δ of a non-trivial abelian differential of the first kind on Δ/Γ. Further, without loss of generality

$$f' = \eta.$$

Now $\tilde{\varphi} = (\varphi \circ \varrho)\varrho'^2$, $\tilde{\eta} = (\eta \circ \varrho)\varrho'$, $\tilde{f} = f \circ \varrho$ are the corresponding forms and functions for Γ_0 (defined on U). The chain rule and the special case treated at the beginning of this proof yield the conclusion of the lemma.

Let Γ be a torsion free quasi-Fuchsian group representing two surfaces of type (1,1). Let Ω be its region of discontinuity. Assume that $\Gamma = \langle A, B \rangle$ with A, B loxodromic with P defined by (3.2) parabolic. A stratification for Γ (see [8]) consists of the fixed points $x_1, \ldots, x_5$ of

$$P,\ A \circ P \circ A^{-1},\ P \circ A \circ P \circ A^{-1} P^{-1},\ B \circ A \circ P \circ A^{-1} B^{-1},\ B \circ P \circ B^{-1},$$

respectively. Note that

$$x_2 = Ax_1, \quad x_3 = Px_2, \quad x_4 = Bx_2, \quad x_5 = Bx_1.$$

Conjugating Γ by an element of $\mathrm{PSL}(2, \mathbf{C})$ we may assume

$$x_1 = \infty, \quad x_2 = 0, \quad x_3 = 1. \tag{3.5}$$

Write $x_4 = x$, $x_5 = y$.

The fact that $\{0, 1, \infty, x, y\}$ form a stratification for the (conjugated) group Γ means that if we view $T(\Gamma)$ as given by (1.1), then

$$T(\Gamma) \ni w \mapsto (w(x), w(y)) \in \mathbf{C}^2$$

is a biholomorphic mapping of $T(\Gamma)$ onto a domain in $\mathbf{C}^2$.

We now form the two Poincaré series (cusp forms of weight (-4))

$$(3.6)\qquad \varphi(z)=\sum_{\gamma\in\Gamma}\frac{\gamma'(z)^2}{\gamma z(\gamma z-1)(\gamma z-x)},\qquad \psi(z)=\sum_{\gamma\in\Gamma}\frac{\gamma'(z)^2}{\gamma z(\gamma z-1)(\gamma z-y)},\qquad z\in\Omega.$$

The cusp forms φ, ψ are a basis for the space of quadratic differentials (cusp forms of weight (-4)) for Γ. See [9].

The two functions φ and ψ can be regarded as holomorphic functions on $F(\Gamma,\Omega)$ by replacing x and y by $w(x)$, $w(y)$ with $w\in T(\Gamma)$. These two functions should yield all the function theory on the two punctured tori represented by Γ. However, we do not know the answers to even some of the simplest questions.

PROBLEM. *Let Δ be a component of Ω. Does φ vanish identically on Δ?*

For certain groups we can answer the above question in the following

THEOREM 1. *Let Γ be a torsion free Fuchsian group of type* (1,1) *acting on the upper half-plane U (thus $\Gamma\subset \mathrm{PSL}(2,\mathbf{R})$). Normalize Γ as above (including* (3.5)). *The classical modulus on the torus U/Γ (the marking is provided by the homology classes of the curves corresponding to the generators A and B) is given by*

$$\tau=\frac{\eta(Bz)-\eta(z)}{\eta(Az)-\eta(z)},$$

where $z\in U$ is arbitrary and η is a holomorphic function on U defined by

$$(\eta')^2=\varphi.$$

Proof. We must only verify that φ does not vanish identically on U (hence it never vanishes on U). This is a consequence of the linear independence of φ and ψ (over $\mathbf{C}$) and the symmetry

$$\overline{\varphi(z)}=\varphi(\bar z),\qquad \overline{\psi(z)}=\psi(\bar z),\qquad \text{all } z\in U.$$

4. Earle coordinates for punctured tori. We start with the symmetric torus $\mathbf{C}/G_i$ and the corresponding punctured torus U/Γ_i. The torus $\mathbf{C}/G_i$ has an anti-conformal involution $j(z)=i\bar z$. This involution lifts to a mapping J of U that fixes the geodesic between $b=0$ and ∞ (Fig. 2). It follows that

$$J(z)=-\bar z,$$

and that

$$J\circ A\circ J=B,\qquad J\circ B\circ J=A.$$

It follows also that

$$J\circ P\circ J=P^{-1}.$$

Let Γ be the group obtained by adjoining to Γ_i the conformal map

E: $z \mapsto -z$. Then $[\Gamma\colon \Gamma_i] = 2$ and Γ is a Kleinian group representing the single Riemann surface U/Γ_i. (Note that the stabilizer of U in Γ is precisely Γ_i.) A stratification for Γ consists of:

$$\begin{aligned} x_1 &= \text{fixed point of } P, \\ x_2 &= \text{the fixed point of } E \neq x_1, \\ x_3 &= Px_1, \quad x_4 = Ax_1. \end{aligned}$$

We can normalize Γ by (3.5). The function φ defined by (3.6) can again be viewed as a holomorphic function on $F(\Gamma, \Omega)$. It yields, again, the classical modulus of the torus.

Remark. The construction of stratifications in [9], § 6.6, is incomplete in Case VI with $n = 1$ (the case treated above). When S is the punctured torus, the loops obtained by deforming w to lie on either side of the puncture are homotopic. A special argument is required in this case.

5. Maskit coordinates for punctured tori. We start with a torsion free triangle group; that is, a free group Γ_1 generated by two parabolic elements A and B with $A \circ B$ also parabolic (it is easy to write down a formula for A and B).

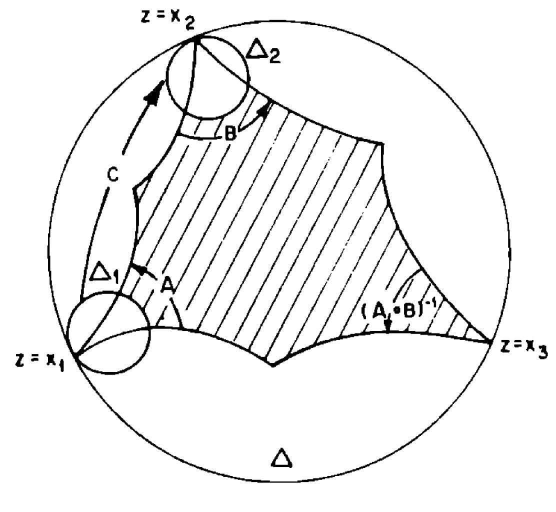

Fig. 3. An HNN extension of a triangle group

Let Δ be one of the invariant components of Γ_1. Let x_1, x_2, x_3 be the fixed points of A, B and $A \circ B$, respectively. (These three points stratify Γ_1.) Choose horocycles Δ_1 and Δ_2 for A and B in Δ. Without loss of generality these horocycles are precisely invariant under their stabilizers in Γ_1. Choose a Möbius transformation C mapping the interior of Δ_1 onto the exterior of Δ_2 with

$$C \circ A \circ C^{-1} = B.$$

For Δ_1 and Δ_2 of sufficiently small diameter, the group Γ generated by A, B and C is Kleinian, in fact an HNN extension of Γ_1. It represents a punctured torus on its invariant component, and two thrice punctured spheres. It is stratified by x_1, x_2, x_3 and $x_4 = C(x_2)$.

Again, we may normalize so that $x_1 = \infty$, $x_2 = 0$, $x_3 = 1$. The Poincaré series φ defined by (3.6) once again gives a formula for the classical modulus of the punctured torus represented by Γ (as well as the deformations of Γ).

6. Schottky groups of genus 2. Let D be a region in $\mathbf{C} \cup \{\infty\}$ bounded by four disjoint circles C_1, C_1', C_2, C_2'.

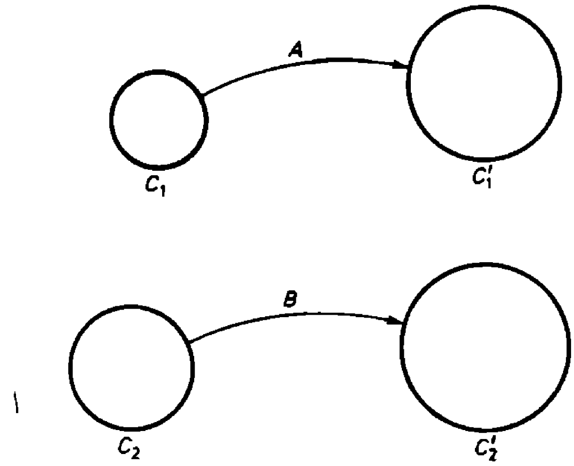

Fig. 4. A fundamental domain for a Schottky group of genus 2

We orient the circles so that the complement of D consists of the union of the interiors of the four circles. We choose Möbius transformations A and B that map the interior of C_1 (respectively, C_2) onto the exterior of C_1' (respectively, C_2'). Let Γ be the group generated by A and B. It is a free group on 2 loxodromic generators with Ω/Γ a surface of genus 2. By the retrosection theorem (see, for example, Bers [3]) every surface of genus 2 can be represented by a Schottky group – except that we may *not* insist that C_1, C_2, C_1', C_2' be circles, but only closed Jordan curves. Further, by a result of Maskit [10], every free, purely loxodromic Kleinian group on two generators is a Schottky group of genus 2.

Every Riemann surface of genus 2 is hyperelliptic and hence can be represented as a two sheeted cover of the sphere. We proceed to describe the construction of this cover from the group $\Gamma = \langle A, B \rangle$.

We use relative Poincaré series rather than Poincaré series – which could also have been used.

Let $C \in \Gamma$ be loxodromic with fixed points α, β. Let

$$g(z) = \frac{(\alpha-\beta)^2}{(z-\alpha)^2(z-\beta)^2}, \quad z \in \mathbf{C} \cup \{\infty\},$$

and define

$$\varphi_C(z) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} g(\gamma z)\gamma'(z)^2, \qquad z \in \Omega,$$

where $\Gamma_0 = \langle C \rangle$. It is shown in [7], that φ_C is a holomorphic automorphic 2-form for Γ. Furthermore, the three functions φ_A, φ_B, and $\varphi_{A \circ B}$ are linearly independent and form a basis for the space of cusp forms for Γ of weight (-4). The surface Ω/Γ has six Weierstrass points. Let $z_1, z_2, \ldots, z_6$ be lifts of the six Weierstrass points to Ω. For $1 \leqslant j < k \leqslant 6$, the automorphic form

$$\varphi_{jk}(z) = \det \begin{bmatrix} \varphi_A(z) & \varphi_B(z) & \varphi_{A \circ B}(z) \\ \varphi_A(z_j) & \varphi_B(z_j) & \varphi_{A \circ B}(z_j) \\ \varphi_A(z_k) & \varphi_B(z_k) & \varphi_{A \circ B}(z_k) \end{bmatrix}$$

vanishes at z_j and z_k. Since each of these is a Weierstrass point, φ_{jk} vanishes to order 2 at z_j and z_k, and hence has no other zeros (except, of course, at points equivalent to z_j and z_k under Γ). It follows that

$$f(z) = \frac{\varphi_{12}(z)}{\varphi_{13}(z)}, \qquad z \in \Omega,$$

defines a function of degree two on Ω/Γ with a double zero at z_2 and a double pole at z_3.

It remains to locate the Weierstrass points on Ω/Γ in terms of the group Γ. It is a trivial exercise to show that there exists a Möbius transformation E of order 2, with

$$E \circ A \circ E = A^{-1}, \qquad E \circ B \circ E = B^{-1}.$$

It follows that the fixed points of E, $E \circ A$, $E \circ B$ are in Ω and these six fixed points are lifts of the Weierstrass points of Ω/Γ.

THEOREM 2. *Let Γ be a Schottky group on the two free generators A, B. Then Ω/Γ is conformally equivalent to the algebraic curve*

$$w^2 = z(z-e_1)(z-e_4)(z-e_5)(z-e_6); \qquad \text{where } e_j = f(z_j).$$

It should be noted that the e_j, $j = 1, 4, 5, 6$, can be computed (by transcendental formulae) from the group Γ. The algorithm for the computation was developed above.

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