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ON AN INTEGRAL INEQUALITY CONNECTED WITH HARDY'S INEQUALITY (III)

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Let $f = f(t)$ be a real function defined in the interval $\langle -1, +1\rangle$, absolutely continuous, and satisfying conditions $f(-1) = f(+1) = 0$ and $\int_{-1}^{+1} p\dot{f}^2 dt < \infty$, where $p = (1-t^2)^{-1/2}$ and the dot means differentiation with respect to t. As shown in [2], for such a function, there holds the integral identity

$$(1) \qquad \int_{-1}^{+1} p\dot{f}^2 dt - 2\int_{-1}^{+1} p^5 f^2 dt = \int_{-}^{+1} p^{-3}\dot{h}^2 dt,$$

where $h = p^2 f$. This identity implies, in turn, the inequality

$$(2) \qquad \int_{-1}^{+1} p^5 f^2 dt \leqslant \frac{1}{2}\int_{-1}^{+1} p\dot{f}^2 dt$$

in which the coefficient $1/2$ cannot be diminished. Equality in (2) appears for functions $f = \mathrm{const}\cdot f_0$ and $f_0 \equiv p^{-2}$ only.

In what follows we assume the following orthogonality condition for f:

$$(3) \qquad \int_{-1}^{+1} p^5 f f_0 dt \equiv \int_{-1}^{+1} p^3 f dt = 0.$$

THEOREM 1. *Under the assumptions listed above we have the integral inequality*

$$(4) \qquad \int_{-1}^{+1} p^5 f^2 dt \leqslant \frac{4}{9}\int_{-1}^{+1} p\dot{f}^2 dt,$$

where the coefficient $4/9$ *cannot be diminished. Equality in* (4) *occurs for* $f \equiv 0$ *only.*

Theorem 1 follows immediately from the theorem in [3], but we present here an elementary proof of it based on a certain modification of Hardy's inequality Let us recall that inequality. The well-known Hardy Theorem says (cf. [1]) that if a sequence $\{\beta_k\}_{k=1}^{\infty}$ of real numbers satisfies the condition

$$0 < \sum_{k=1}^{\infty} \beta_k^2 < \infty,$$

then the inequality

$$\sum_{k=1}^{\infty} k^{-2}(\beta_1+\ldots+\beta_k)^2 < 4\sum_{k=1}^{\infty} \beta_k^2 \tag{5}$$

is valid; the coefficient 4 cannot be diminished.

Our proof of Theorem 1 will be based on the following

LEMMA. *If a sequence $\{a_k\}_{k=0}^{\infty}$ of real numbers satisfies the condition*

$$0 < \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty} a_k^2 < \infty,$$

then the inequality

$$a_0^2 + \sum_{k=1}^{\infty}\left(k+\frac{1}{2}\right)^{-2}\left(\frac{1}{2}a_0+a_1+\ldots+a_k\right)^2 < 4\left(\frac{1}{2}a_0^2+\sum_{k=1}^{\infty} a_k^2\right) \tag{6}$$

is valid; the coefficient 4 cannot be diminished.

Proof. For any real function $f \in L^2(0,\infty)$, $f \not\equiv 0$, we have the integral inequality (see [1])

$$\int_0^{\infty} t^{-2}F^2 dt < 4\int_0^{\infty} f^2 dt, \tag{7}$$

where $F(t) = \int_0^t f(\tau)d\tau$. Setting $f(t) = \frac{1}{3}a_0$ for $0 \leqslant t < 3/2$ and $f(t) = a_k$ for $k+1/2 \leqslant t < k+3/2$, where $k = 1, 2, \ldots$, we obtain

$$\int_0^{\infty} f^2 dt = \frac{1}{6}a_0^2 + \sum_{k=1}^{\infty} a_k^2$$

and

$$\int_0^{\infty} t^{-2}F^2 dt \geqslant \frac{1}{6}a_0^2 + \sum_{k=1}^{\infty}\left(k+\frac{1}{2}\right)^{-2}\left(\frac{1}{2}a_0+a_1+\ldots+a_k\right)^2,$$

whence, by virtue of (7), there follows (6).

Now, setting $a_k = k^{-1/2}$ for $k = 1, 2, \ldots, K$ and $a_k = 0$ for $k = 0$ and $k > K$, we find first

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{K} k^{-1}$$

and

$$(k+\tfrac{1}{2})^{-2}(\tfrac{1}{2}a_0+a_1+\ldots+a_k)^2 > 4k^{-1}(1-\varepsilon_k)$$

for $k = 1, 2, \ldots, K$, where $\varepsilon_k = (\frac{1}{2}+k)^{-2}(\frac{1}{4}+2k^{3/2}) > 0$. Then we obtain

$$(8)\qquad a_0^2+\sum_{k=1}^{\infty}\left(k+\frac{1}{2}\right)^{-2}\left(\frac{1}{2}a_0+a_1+\ldots+a_k\right)^2$$

$$\geqslant \sum_{k=1}^{K}\left(k+\frac{1}{2}\right)^{-2}\left(\frac{1}{2}a_0+a_1+\ldots+a_k\right)^2 > 4\sum_{k=1}^{K}(1-\eta_K)k^{-1}$$

$$= 4(1-\eta_K)\left(\frac{1}{2}a_0^2+\sum_{k=1}^{\infty}a_k^2\right)$$

where

$$\eta_K = \Big(\sum_{k=1}^{K}\varepsilon_k k^{-1}\Big)\Big(\sum_{k=1}^{K}k^{-1}\Big)^{-1}.$$

Finally, if we note that $\eta_K > 0$ and $\eta_K \to 0$ for $K\to\infty$, inequality (8) shows that the coefficient 4 in (6) cannot be diminished.

Proof of Theorem 1. Let $T_k = T_k(t)$, $k = 0, 1, \ldots$, denote Chebyshev polynomials for the interval $\langle -1, +1\rangle$. If $h = p^2 f$ the Parseval identity gives the formulae

$$(9)\quad I_1 \equiv \frac{2}{\pi}\int_{-1}^{+1} ph^2dt = 2a_0^2+\sum_{k=1}^{\infty}a_k^2,\qquad I_2 \equiv \frac{2}{\pi}\int_{-1}^{+1} p^{-3}\dot{h}^2dt = 2b_0^2+\sum_{k=1}^{\infty}b_k^2,$$

where

$$\pi a_0 = \int_{-1}^{+1} phT_0dt,\qquad \pi b_0 = \int_{-1}^{+1} p^{-1}\dot{h}T_0dt,$$

and, for $k = 1, 2, \ldots$,

$$\pi a_k = 2\int_{-}^{+1} phT_kdt,\qquad \pi b_k = 2\int_{-1}^{+1} p^{-1}\dot{h}T_kdt.$$

Performing integration by parts, we obtain

$$(10)\qquad a_1 = 2b_0\quad\text{and}\quad (k+1)a_{k+1}-(k-1)a_{k-1} = 2b_k\qquad\text{for } k = 1, 2, \ldots$$

These equalities allow to express coefficients a_k in terms of b_k.

According to (3), we have $a_0 = 0$. Putting $b_0 = \frac{1}{2}a_0$, $b_{2k} = a_k$ and $b_{2k-1} = \beta_k$ for $k = 1, 2, \ldots$, we obtain from (10) the formulae

$$(11) \qquad a_1 = a_0, \qquad a_{2k+1} = (k+\tfrac{1}{2})^{-1}(\tfrac{1}{2}a_0+a_1+\ldots+a_k),$$
$$a_{2k} = k^{-1}(\beta_1+\ldots+\beta_k) \quad \text{for } k = 1, 2, \ldots$$

Now integrals I_1 and I_2 take the form

$$I_1 = a_0^2+\sum_{k=1}^{\infty}\left(k+\frac{1}{2}\right)^{-2}\left(\frac{1}{2}a_0+a_1+\ldots+a_k\right)^2+\sum_{k=1}^{\infty}k^{-2}(\beta_1+\ldots+\beta_k)^2,$$

$$I_2 = \frac{1}{2}a_0^2+\sum_{k=1}^{\infty}a_k^2+\sum_{k=1}^{\infty}\beta_k^2.$$

If $f \not\equiv 0$, we have, according to (5) and (6), the inequality $I_1 < 4I_2$. Of course, the coefficient 4 cannot be diminished here. Applying now identity (1), which can be rewritten in the form

$$\frac{2}{\pi}\int_{-1}^{+1} p\dot{f}^2 dt = 2I_1+I_2,$$

we complete the proof of the Theorem.

REFERENCES

[1] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, London 1951.

[2] A. Krzywicki and A. Rybarski, *On some integral inequalities involving Chebyshev weight function*, Colloquium Mathematicum 18 (1967), p. 147-150.

[3] — *On an integral inequality connected with Hardy's inequality*, Zastosowania Matematyki 10 (1969), p. 37-41.

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