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THE TIME NEEDED FOR ESTIMATION OF A FUNCTIONAL BY THE MONTE CARLO METHOD

1. Introduction. In the space $L_\infty(\Omega)$ (1), the integral equation

$$u(x)-\int_\Omega K(x,y)u(y)\mu(dy) = g(x) \quad (x\in\Omega), \tag{1.1}$$

where (Ω, Σ, μ) is a measure space, and $\mu(\Omega) < +\infty$, has been solved, by the Monte Carlo method, in papers [10] and [11].

Suppose that there exists a set $\Omega_0 \in \Sigma$ with $\Omega_0 \subset \Omega$ and $\mu(\Omega_0) > 0$ and that it satisfies the following conditions:

(A) For all $f\in L_\infty(\Omega\setminus\Omega_0)$, the series $\sum_{n=0}^{\infty} T_+^n f$ converges in $L_\infty(\Omega\setminus\Omega_0)$, where the integral operator T_+ is defined by the formula

$$[T_+f](x) = \int_{\Omega\setminus\Omega_0} |K(x,y)| f(y)\mu(dy) \quad (x\in\Omega\setminus\Omega_0).$$

(B) $K(x,y)\geqslant 0$ for $x\in\Omega\setminus\Omega_0$ $(\mathrm{mod}\,\mu)$ and $y\in\Omega_0$ $(\mathrm{mod}\,\mu)$.

(C) $K(x,y) = 0$ for $x\in\Omega_0$ $(\mathrm{mod}\,\mu)$ and $y\in\Omega$ $(\mathrm{mod}\,\mu)$.

The Monte Carlo method is also used in papers [10] and [11] to obtain an estimation of the value of the functional

$$(u,\varphi) = \int_\Omega u(x)\varphi(x)\mu(dx), \tag{1.2}$$

where $u(x)$ is the solution of equation (1.1), and $\varphi\in L_1(\Omega)$ (2).

It is well known (see [7]-[9]) that various probability models have been considered in some special cases for the solution, by the Monte Carlo method, of problems (1.1) and (1.2).

(1) $L_\infty^\Sigma(A)$ denotes the space of Σ-measurable and bounded on A $(\mathrm{mod}\,\mu)$ functions, where $A\in\Sigma$.

(2) $L_1(A)$ is the space of μ-integrable functions on A, where $A\in\Sigma$.

For solving a particular case of problems (1.1) and (1.2) (without conditions (B) and (C)), a number of probability models has also been given in [2].

In order to estimate the computational times of these models, the estimation of variances of random variables constructed in suitable models is considered in [5], [4] and [3].

We know that the mentioned variances exert a great influence on the computational times of the models, but they do not precisely define the computational time. Therefore, an exact theoretical background for choosing the optimal model and for comparing the ones already known has been resolved so far (see [2], p. 1280, and [4], p. 244).

In this paper, we shall, at first, estimate the time needed for solving the problems (1.1) and (1.2) by probability models displayed in [10] and [11]. We shall also compare these models and choose an optimal one with respect to the calculation. Moreover, we shall also consider the diminution of the time needed for solving the mentioned problems.

By the results of [10] and [11], we remark that the probability models for solving the problem (1.1) are not essentially different from the ones for solving the problem (1.2). Therefore, we shall only consider the estimation of the time needed for solving the problem (1.2).

2. Concept of the ε-scheme. Suppose that there exists a $(\Sigma\times\Sigma)$-measurable function $p(x,y)$, bounded on $\Omega\times\Omega$ and satisfying the following conditions:

$$(\mathrm{P}_1)\qquad \alpha \equiv \operatorname*{vrai\,sup}_{\mu\quad x\in\Omega\setminus\Omega_0}\Big\{\int_\Omega K(x,y)p(x,y)\mu(dy)\Big\}<1,$$

$$(\mathrm{P}_2)\qquad K(x,y)p(x,y)\geqslant 0 \quad \text{for } x\in\Omega\ (\mathrm{mod}\,\mu),\ y\in\Omega\ (\mathrm{mod}\,\mu),$$

$$(\mathrm{P}_3)\qquad p(x,y)\neq 0 \quad \text{for } (x,y)\in\Omega\times\Omega\ (\mathrm{mod}\,\mu\times\mu).$$

Then there exists a set $A^*\in\Sigma$ such that $\mu(A^*)=0$ and (see [11] or [10], Lemma (2.1))

$$(\mathrm{B}^*)\qquad K(x,y)\geqslant 0 \quad \text{for } x\in\Omega_A^*\setminus\Omega_0,\ y\in\Omega_0\ (\mathrm{mod}\,\mu),$$

where $\Omega_A^*=\Omega\setminus A^*$,

$$(\mathrm{P}_1^*)\qquad \alpha^*\equiv\sup_{x\in\Omega_A^*\setminus\Omega_0}\Big\{\int_\Omega p(x,y)K(x,y)\mu(dy)\Big\}<1,$$

$$(\mathrm{P}_2^*)\qquad p(x,y)K(x,y)\geqslant 0 \quad \text{for } x\in\Omega_A^*,\ y\in\Omega\ (\mathrm{mod}\,\mu),$$

$$(2.1)\qquad G\equiv\sup_{x\in\Omega_A^*\setminus\Omega_0}\{|g(x)|\}<+\infty.$$

Let the complete measure $\bar{\mu}$ be the extension of μ on the σ-field $\bar{\Sigma} \supset \Sigma$, let δ be a positive constant, and let

(2.2) $$\tilde{\Omega} = \Omega \cup \Omega^*, \quad \tilde{\Sigma} = \bar{\Sigma} \cup \Sigma^*, \quad \tilde{\mu}(\tilde{A}) = \bar{\mu}(\tilde{A} \cap \Omega) + \delta\delta_{\tilde{A}} \qquad (\forall \tilde{A} \in \tilde{\Sigma}),$$

where Ω^* is a set satisfying the conditions

(2.3) $$\Omega^* \neq \emptyset \quad \text{and} \quad \Omega^* \cap \Omega = \emptyset.$$

Let Σ^* and $\delta_{\tilde{A}}$ be determined by

(2.4) $$\Sigma^* = \{\tilde{A} : \tilde{A} = \bar{A} \cup \Omega^*;\ \bar{A} \in \bar{\Sigma}\}, \qquad \delta_{\tilde{A}} = \begin{cases} 1 & \text{if } \tilde{A} \cap \Omega^* \neq \emptyset, \\ 0 & \text{if } \tilde{A} \cap \Omega^* = \emptyset. \end{cases}$$

It is proved (see [10] or [11], Lemma (2.3)) that under these conditions $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ is a space with a complete measure.

Basing on the measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$, we construct two homogeneous Markov processes in the broad sense in the phase space $\tilde{\Omega}$ (see [6], p. 280-283). The transition probabilities $P_i(k, x, \tilde{A})$ $(i = 1, 2)$ corresponding to each Markov process are determined (see [10] or [11], Lemma (2.4)) (3) by the formulas

(2.5) $$P_i(k, x, \tilde{A}) = \int_{\tilde{A}} P_i(k-1, y, \tilde{A}) P_i(1, x, dy) \qquad (x \in \tilde{\Omega};\ \tilde{A} \in \tilde{\Sigma};\ k = 2, 3, \ldots),$$

where

(2.6) $$P_i(1, x, \tilde{A}) = \begin{cases} \int_{\tilde{A}} F_i(x, y) \tilde{\mu}(dy) & \text{if } x \in \Omega_A^* \setminus \Omega_0, \\ \chi_{\tilde{A}}(x) & \text{if } x \in A^* \cup \Omega_0 \cup \Omega^*, \end{cases}$$

(2.7)
$$F_i(x, y) = \begin{cases} p(x, y) K(x, y) & \text{if } (x, y) \in (\Omega_A^* \setminus \Omega_0) \times (\Omega \setminus \Omega_0), \\ p(x, y) K(x, y) + \dfrac{g_i(x) p(x, y)}{\mu(\Omega_0) g_i(y)} & \text{if } (x, y) \in (\Omega_A^* \setminus \Omega_0) \times \Omega_0, \\ h_i(x) & \text{if } (x, y) \in (\Omega_A^* \setminus \Omega_0) \times \Omega^*, \end{cases}$$

(2.8) $$h_i(x) = \frac{1}{\delta}\left[1 - \int_{\Omega} p(x, y) K(x, y) \mu(dy) - \frac{g_i(x)}{\mu(\Omega_0)} \int_{\Omega_0} \frac{p(x, y) \mu(dy)}{g_i(y)}\right],$$

(2.9)
$$g_1(x) = -\left[|g(x)| + \chi_{\Omega_0}(x)\left(\frac{2GM}{1-a^*} + \Delta\right)\right], \quad g_2(x) = g(x) - g_1(x) \qquad (x \in \Omega),$$

(2.10) $$M \equiv \sup_{(x,y) \in (\Omega_A^* \setminus \Omega_0) \times \Omega_0} \{|p(x, y)|\} < +\infty,$$

Δ is a positive constant.

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(3) In [10] and [11], each such Markov process is called the *i-th process* $(i = 1, 2)$.

Let $P(\cdot)$ be a probability measure defined on the σ-field by the formula

$$P(\tilde{A}) = \int_{\tilde{A}} \pi(x)\tilde{\mu}(dx) \qquad (\forall \tilde{A} \in \tilde{\Sigma}), \tag{2.11}$$

where $\pi(x)$ is a function satisfying the following conditions:

$$0 < \pi(x) < +\infty \quad \text{on } \Omega \ (\text{mod}\,\mu), \tag{Π_1}$$

$$\pi(x) \geqslant 0 \quad \text{on } \tilde{\Omega} \ (\text{mod}\,\tilde{\mu}), \qquad \int_{\tilde{\Omega}} \pi(x)\tilde{\mu}(dx) = 1. \tag{Π_2}$$

Suppose that $\tilde{\Omega}^{(i)}$ is the sample space of trajectories of the i-th process with the initial probability distribution $P(\cdot)$. Put

$$\Omega^{(i)} = \bigcup_{n=0}^{\infty} \Omega^{(i)}[n], \tag{2.12}$$

where $\Omega^{(i)}[n]$ is the subspace of $\tilde{\Omega}^{(i)}$ consisting of trajectories of the i-th process of the form

$$x_0 \to x_1 \to \ldots \to x_n \tag{2.13}$$
$$(x_n \in \Omega_0 \cup \Omega_1;\ x_0, x_1, \ldots, x_{n-1} \in \Omega_A^* \setminus \Omega_0;\ \Omega_1 \equiv A^* \cup \Omega^* \setminus \Omega_0),$$

i.e. $\Omega^{(i)}$ is the subspace of $\tilde{\Omega}^{(i)}$ and consists of trajectories of the i-th process of the form

$$x_0 \to x_1 \to \ldots \to x_l \quad (x_l \in \Omega_0 \cup \Omega_1;\ x_0, x_1, \ldots, x_{l-1} \in \Omega_A^* \setminus \Omega_0), \tag{2.14}$$

where l is a finite natural number.

Let $F^{(i)}(x_0, x_1, \ldots, x_l)$ be the function defined on $\Omega^{(i)}$ by the formula

(2.15)

$$F^{(i)}(x_0, x_1, \ldots, x_l) = \begin{cases} \dfrac{\varphi(x_0)g_i(x_l)}{\pi(x_0)p(x_0, x_1)\ldots p(x_{l-1}, x_l)} & \text{if } x_l \in \Omega_0;\ l \geqslant 1, \\ \dfrac{\varphi(x_0)g_i(x_0)}{\pi(x_0)} & \text{if } x_0 \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega_1. \end{cases}$$

Then (see [10] or [11], Section 3) $\eta^{(i)} = F^{(i)}(x_0, x_1, \ldots, x_l)$ is the random variable defined on the probability space $(\tilde{\Omega}^{(i)}, \tilde{\Sigma}^{(i)}, \tilde{\mu}^{(i)})$, where

$$\tilde{\Sigma}^{(i)} = \tilde{\Sigma} \times \tilde{\Sigma} \times \ldots \times \tilde{\Sigma} \times \ldots$$

and

$$\tilde{\mu}^{(i)}\{\tilde{A}_0 \times \tilde{A}_1 \times \ldots \times \tilde{A}_l\} = \int_{\tilde{A}_0} P(dx_0) \int_{\tilde{A}_1} P_i(1, x_0, dx_1) \ldots \int_{\tilde{A}_l} P_i(1, x_{l-1}, dx_l) \quad (\tilde{A}_k \in \tilde{\Sigma};\ k = 0, 1, 2, \ldots, l). \tag{2.16}$$

We know that (see [10] or [11], Theorem (3.1))

$$M\eta^{(i)} = (u^{(i)}, \varphi) \equiv \int_{\Omega} u^{(i)}(x)\varphi(x)\mu(dx), \tag{2.17}$$

where $u^{(i)}(x)$ is the solution in $L_\infty(\Omega)$ of the equation

$$u^{(i)}(x) - \int_{\Omega} K(x, y) u^{(i)}(y)\mu(dy) = g_i(x) \quad (x \in \Omega). \tag{2.18}$$

Therefore,

$$M\eta^{(1)} + M\eta^{(2)} = (u, \varphi) \equiv \int_{\Omega} u(x)\varphi(x)\mu(dx), \tag{2.19}$$

where $u(x)$ is the solution in $L_\infty(\Omega)$ of equation (1.1).

From the previous results, we deduce that the computational scheme for estimating the value of functional (1.2) consists of the following steps:

1. Sample a trajectory, having form (2.14), of the i-th process (with the initial probability distribution $P(\cdot)$ and the transition probabilities $P_i(k, x, \cdot)$).

2. Calculate by formula (2.15) the value of the random variable $\eta^{(i)}$ corresponding to this trajectory.

3. For each i-th process ($i = 1, 2$), repeat all the computations in steps $1 \div 2$ for N_i times (N_i is sufficiently great). Then we obtain the values $\eta_1^{(i)}, \eta_2^{(i)}, \ldots, \eta_{N_i}^{(i)}$, where $\eta_k^{(i)}$ ($k = 1, 2, \ldots, N_i$) is the value of the random variable $\eta^{(i)}$ corresponding to the k-th trajectory of the i-th process.

4. Estimate the values of the functionals $(u^{(i)}, \varphi)$ ($i = 1, 2$) by the following approximate formulae (see (2.17)):

$$(u^{(i)}, \varphi) \approx \frac{4}{N_i} \sum_{k=1}^{N_i} \eta_k^{(i)} \equiv \bar{\eta}^{(i)} \quad (i = 1, 2). \tag{2.20}$$

5. Estimate the value of functional (1.2) by the following approximate formula (see (2.19)):

$$(u, \varphi) \approx \bar{\eta}^{(1)} + \bar{\eta}^{(2)}. \tag{2.21}$$

Suppose that

$$(2.22)\qquad D\eta^{(i)} < +\infty \quad (i = 1, 2).$$

Then we easily infer that the number N_i of "experiments" of the i-th process (in the mentioned computational scheme) is finite and connected with the error ε of the approximate formula (2.21) by the relation (see [1], p. 11)

$$(2.23)\qquad N_i = [D\eta^{(i)}/\varepsilon_i^2]+1 \quad (i = 1, 2),$$

where $[D\eta^{(i)}/\varepsilon_i^2]$ is the integral part of the value $D\eta^{(i)}/\varepsilon_i^2$ and the constants ε_i $(i = 1, 2)$ are defined by the conditions

$$(\mathrm{E})\qquad \varepsilon_1+\varepsilon_2 = \frac{1}{3}\varepsilon \quad (\varepsilon_i > 0;\ i = 1, 2).$$

Definition 2.1. Let conditions (2.22) be satisfied and let N_i $(i = 1, 2)$ be defined by formula (2.23). Then, a computational scheme consisting of the steps $1\div 5$ for estimating the value of functional (1.2) with the error ε (4) is called an *ε-scheme*. We denote the ε-scheme by S_ε.

Let $t_j^{(i)}$ be a time to realize the j-th step $(j = 1, 2)$ of S_ε for the i-th process $(i = 1, 2)$; let C_π (or C_p, C_{g_i}, C_φ) be the mean time for calculating the value of the function $\pi(x)$ (or $p(x, y)$, $g_i(x)$, $\varphi(x)$); let C_s (or C_m, C_d) be the mean time for realizing an addition (or a multiplication, a divison); and let τ_0 (or $\tau_p^{(i)}$) be the mean time for defining an initial state x_0 (or for defining a transfer from the state x_n to a state x_{n+1}) of a trajectory having form (2.14) of the i-th process.

Then $t_j^{(i)}$ $(j = 1, 2)$ are the functions defined on $\Omega^{(i)}$ by the formulae

$$(2.24)\qquad t_1^{(i)} = t_1^{(i)}(x_0, x_1, \ldots, x_l) \equiv \tau_0+l\tau_p^{(i)},$$

$$(2.25)\qquad t_2^{(i)} = t_2^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} C_1^{(i)}+lC_2 & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega_1, \end{cases}$$

where $x_l \in \Omega_0 \cup \Omega_1$, $x_0, x_1, \ldots, x_{l-1} \in \Omega_A^* \setminus \Omega_0$, and

$$(2.26)\qquad C_1^{(i)} = C_\varphi+C_{g_i}+C_\pi+C_m+C_d, \qquad C_2 = C_p+C_m.$$

We know (see [11], formula (3.11)) that

$$(2.27)\qquad \tilde{\mu}^{(i)}\{\tilde{\Omega}^{(i)} \setminus \Omega^{(i)}\} = 0.$$

Hence $t_j^{(i)}$ $(j = 1, 2)$ are the random variables defined on the probability space $(\tilde{\Omega}^{(i)}, \tilde{\Sigma}^{(i)}, \tilde{\mu}^{(i)})$.

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(4) In the computation we do not discuss the round-off error.

Let $t_4^{(i)}$ be a time to realize the calculations for estimating $(u^{(i)}, \varphi)$ in the 4-th step of S_ε, and let $\tilde{t}_4^{(i)}$ be the function defined on $\Omega^{(i)}$ by the formula

$$(2.28)\qquad \tilde{t}_4^{(i)} = \tilde{t}_4^{(i)}(x_0, x_1, \ldots, x_l) \equiv \begin{cases} 1 & \text{if } x_l \in \Omega_0, \\ 0 & \text{if } x_l \in \Omega_1. \end{cases}$$

Then we have

$$(2.29)\qquad t_4^{(i)} = C_s \sum_{k=1}^{N_i} \tilde{t}_{4,k}^{(i)} + C_d \qquad (i = 1, 2),$$

where $\tilde{t}_{4,k}^{(i)}$ $(k = 1, 2, \ldots, N_i)$ is a value of the random variable $\tilde{t}_4^{(i)}$ corresponding to the k-th trajectory of the i-th process in S_ε. Obviously, $t_4^{(i)}$ is also the random variable defined on the probability space $(\tilde{\Omega}^{(i)}, \tilde{\Sigma}^{(i)}, \tilde{\mu}^{(i)})$.

Hence, the time t_ε needed for solving the problem (1.2) by the ε-scheme is defined by the formula

$$(2.30)\qquad t_\varepsilon = \sum_{i=1}^{2} t_i + C_s,$$

where t_i ([5]) is given by

$$(2.31)\qquad t_i = \sum_{k=1}^{N_i} (t_{1,k}^{(i)} + t_{2,k}^{(i)}) + t_4^{(i)} \qquad (i = 1, 2),$$

and $t_{j,k}^{(i)}$ is a value of the random variable $t_j^{(i)}$ corresponding to the k-th trajectory of the i-th process in S_ε.

Definition 2.2. The value of

$$(2.32)\qquad T_\varepsilon = \sum_{i=1}^{2} Mt_i + C_s$$

is called the *mean time* for solving the problem (1.2) by the ε-scheme S_ε.

We infer from (2.30) and (2.32) that T_ε expresses the time, in practice, for solving the problem (1.2) by S_ε. Therefore, instead of ε-schemes S_ε we can compare suitable values of T_ε.

3. Estimation of the mean time for solving the problem (1.2). From formulae (2.29)-(2.32) it follows

$$(3.1)\qquad T_\varepsilon = \sum_{i=1}^{2} N_i (Mt_1^{(i)} + Mt_2^{(i)} + CM\tilde{t}_4^{(i)}) + C_s + 2C_d.$$

From (2.23) and (3.1) we see that in order to estimate T_ε it is necessary to estimate $Mt_1^{(i)}$, $Mt_2^{(i)}$, $M\tilde{t}_4^{(i)}$ and $D\eta^{(i)}$.

([5]) Obviously, t_i is also the random variable defined on the probability space $(\tilde{\Omega}^{(i)}, \tilde{\Sigma}^{(i)}, \tilde{\mu}^{(i)})$.

Now we consider the following problems:

Let $T^{(p)}$, $T_0^{(p)}$ and $\tilde{T}_0^{(p)}$ be the integral operators defined as follows:

$$[T^{(p)}f](x) = \int_{\Omega\setminus\Omega_0} K(x, y)p(x, y)f(y)\mu(dy), \tag{3.2}$$

$$[T_0^{(p)}f](x) = \int_{\Omega_0} K(x, y)p(x, y)f(y)\mu(dy), \tag{3.3}$$

$$[\tilde{T}_0^{(p)}f](x) = \int_{\Omega_0} p(x, y)f(y)\mu(dy). \tag{3.4}$$

Let $\tilde{\Omega}_x^{(i)}$ be the space of trajectories of the i-th process with some initial state x $(x\in\Omega_A^*\setminus\Omega_0)$. Put

$$\Omega_x^{(i)} = \bigcup_{n=1}^{\infty} \Omega_x^{(i)}[n] \quad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2), \tag{3.5}$$

where $\Omega_x^{(i)}[n]$ is the subspace of $\tilde{\Omega}_x^{(i)}$ consisting of trajectories of the i-th process having the form

$$x \to x_1 \to \ldots \to x_n \quad (x_n\in\Omega_0\cup\Omega_1;\ x, x_1, \ldots, x_{n-1}\in\Omega_A^*\setminus\Omega_0). \tag{3.6}$$

Write $\tilde{\Sigma}_x^{(i)} = \tilde{\Sigma}\times\tilde{\Sigma}\times \ldots \times\tilde{\Sigma}\times \ldots$

Then the probability measure $\tilde{\mu}_x^{(i)}$ of $(\tilde{\Omega}_x^{(i)}, \tilde{\Sigma}_x^{(i)}, \tilde{\mu}_x^{(i)})$ is defined by the formula (see [11], formula (2.29))

$$\tilde{\mu}_x^{(i)}\{\tilde{A}_1 x \ldots x\tilde{A}_l\} = \int_{\tilde{A}_1} P_i(1, x, dx_1) \int_{\tilde{A}_2} P_i(1, x_1, dx_2) \ldots \int_{\tilde{A}_l} P_i(1, x_{l-1}, dx_l) \quad (\tilde{A}_k\in\tilde{\Sigma};\ k = 1, 2, \ldots, l). \tag{3.7}$$

LEMMA 3.1. *Under assumptions* (A), (B), (C), (P_1), (P_2), (P_3), (Π_1) *and* (Π_2), *the expected values* $Mt_1^{(i)}$ $(i = 1, 2)$ *exist and are finite, and*

$$Mt_1^{(i)} = \tau_0 + \tau_p^{(i)}(\pi, z)_{\Omega\setminus\Omega_0} \ ^{(6)} \quad (i = 1, 2), \tag{3.8}$$

where $z(x)$ *is the solution in* $L_\infty(\Omega\setminus\Omega_0)$ *of the equation*

$$z(x) - [T^{(p)}z](x) = 1 \quad (x\in\Omega\setminus\Omega_0). \tag{3.9}$$

Proof. Put

$$z^{(i)}(x) = \sum_{n=1}^{\infty} n\, \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n]\} \quad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2). \tag{3.10}$$

We know that (see [11], formula (2.43))

$$\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n]\} = P_i(n, x, \Omega_0\cup\Omega_1) - P_i(n-1, x, \Omega_0\cup\Omega_1) \quad (x\in\Omega_A^*\setminus\Omega_0;\ n = 2, 3, \ldots), \tag{3.11}$$

$$P_i(k, x, \Omega_A^*\setminus\Omega_0) \leqslant (1-q)^k \quad (x\in\Omega_A^*\setminus\Omega_0;\ k = 1, 2, \ldots), \tag{3.12}$$

[6] The symbol $(f_1, f_2)_A$ denotes the value of the integral $\int_A f_1(x)f_2(x)\mu(dx)$.

where

$$(3.13)\qquad 0 < q \equiv 1 - \sup_{x\in \Omega_A^*\setminus \Omega_0} \Big\{ \int_{\Omega\setminus\Omega_0} p(x,y)K(x,y)\mu(dy)\Big\} \leqslant 1.$$

Therefore,

$$(3.14)\qquad \mu_x^{(i)}\{\Omega_x^{(i)}[n]\} \leqslant 1 - P_i(n-1, x, \Omega_0\cup\Omega_1)$$
$$= P_i(n-1, x, \Omega_A^*\setminus\Omega_0) \leqslant (1-q)^{n-1} \qquad (x\in\Omega_A^*\setminus\Omega_0;\ n = 1, 2, \ldots).$$

Hence, from (3.10) we have

$$(3.15)\qquad z^{(i)}(x) \leqslant \sum_{n=0}^{\infty}(n+1)(1-q)^n = q^{-2} \qquad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2),$$

i.e. $z^{(i)}(x)\in L_\infty(\Omega\setminus\Omega_0)$ $(i = 1, 2)$.

It follows easily from (3.7) and (3.10) that

$$(3.16)\qquad \sum_{n=2}^{\infty} n\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n]\} = \int_{\Omega_A^*\setminus\Omega_0} z^{(i)}(x_1)P_i(1, x, dx_1) +$$
$$+ \int_{\Omega_A^*\setminus\Omega_0} \sum_{n=1}^{\infty} \tilde{\mu}_{x_1}^{(i)}\{\Omega_{x_1}^{(i)}[n]\}P_i(1, x, dx_1) \qquad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2).$$

We know that (see [11], formula (2.45))

$$(3.17)\qquad \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}\} = \sum_{n=1}^{\infty} \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n]\} = 1 \qquad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2).$$

Therefore, from (3.16) we deduce

$$(3.18)\qquad \sum_{n=2}^{\infty} n\,\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n]\} = \int_{\Omega_A^*\setminus\Omega_0} z^{(i)}(x_1)P_i(1, x, dx_1) + P_i(1, x, \Omega_A^*\setminus\Omega_0)$$
$$(x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2).$$

From (3.7) we have

$$(3.19)\qquad \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[1]\} = \int_{\Omega_0\cup\Omega_1} P_i(1, x, dx_1) = P_i(1, x, \Omega_0\cup\Omega_1)$$
$$(x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2).$$

From (3.10), (3.18) and (3.19) it follows

$$(3.20)\qquad z^{(i)}(x) - \int_{\Omega_A^*\setminus\Omega_0} z^{(i)}(x_1)P_i(1, x, dx_1) = P_i(1, x, \tilde{\Omega}) \equiv 1 \qquad (x\in\Omega_A^*\setminus\Omega_0).$$

It is known that (see (2.6))

$$(3.21)\qquad \frac{dP_i(1, x, \cdot)}{d\tilde{\mu}}(\cdot) = F_i(x, \cdot), \qquad P_i(1, x, \cdot) \ll \tilde{\mu} \qquad (x\in\Omega_A^*\setminus\Omega_0).$$

Hence, from (3.20) and (2.7) we obtain

$$z^{(i)}(x) - \int_{\Omega\setminus\Omega_0} K(x, x_1)p(x, x_1)z^{(i)}(x_1)\mu(dx_1) = 1 \quad (x \in \Omega\setminus\Omega_0 \ (\mathrm{mod}\ \mu)),$$

i.e. $z^{(i)}(x)$ $(i = 1, 2)$ are solutions in $L_\infty(\Omega\setminus\Omega_0)$ of equation (3.9). On the other hand, by condition (P_1) we get

$$(3.22)\quad a_p \equiv \|T^{(p)}\| = \underset{\mu}{\mathrm{vrai}} \sup_{x\in\Omega\setminus\Omega_0} \Big\{ \int_{\Omega\setminus\Omega_0} K(x, y)p(x, y)\mu(dy)\Big\} \leqslant a < 1.$$

Therefore, the solution $z(x)$ of equation (3.9) exists and is unique in a B-space $L_\infty(\Omega\setminus\Omega_0)$. It follows that

$$(3.23)\qquad z^{(1)}(x) = z^{(2)}(x) = z(x) \quad (x\in\Omega\setminus\Omega_0 \ (\mathrm{mod}\,\mu)).$$

We have (see (2.24)) $t_1^{(i)} = t_1^{(i)}(x_0, x_1, \dots, x_l) > 0$; hence $Mt_1^{(i)}$ exists and is defined by the formula (see (2.27) and (2.24))

$$(3.24)\quad Mt_1^{(i)} \equiv \int_{\tilde\Omega^{(i)}} t_1^{(i)} d\tilde\mu^{(i)} = \int_{\Omega^{(i)}} t_1^{(i)} d\tilde\mu^{(i)} = \tau_0 \sum_{n=0}^{\infty} \int_{\Omega^{(i)}[n]} d\tilde\mu^{(i)} + \tau_p^{(i)} \sum_{n=1}^{\infty} \int_{\Omega^{(i)}[n]} n d\tilde\mu^{(i)}.$$

From (2.16), (3.7), (3.10) and (2.11) we deduce

$$(3.25)\quad \sum_{n=1}^{\infty} \int_{\Omega^{(i)}[n]} n d\tilde\mu^{(i)} = \sum_{n=1}^{\infty} \int_{\Omega_A^*\setminus\Omega_0} n\tilde\mu_x^{(i)}\{\Omega_x^{(i)}[n]\}P(dx) = \int_{\Omega_A^*\setminus\Omega_0} z^{(i)}(x)P(dx)$$
$$= \int_{\Omega_A^*\setminus\Omega_0} z(x)\pi(x)\mu(dx) = (\pi, z)_{\Omega_A^*\setminus\Omega_0} = (\pi, z)_{\Omega\setminus\Omega_0}.$$

We know (see [11], formula (3.10))

$$(3.26)\qquad \tilde\mu^{(i)}\{\Omega^{(i)}\} = \sum_{n=0}^{\infty} \tilde\mu^{(i)}\{\Omega^{(i)}[n]\} = 1 \quad (i = 1, 2).$$

Therefore, from (3.24) and (3.25) we obtain (3.8). This completes the proof.

Let $\Omega_x^{(i)}[n, s]$ $(x\in\Omega_A^*\setminus\Omega_0;\ s = 0, 1)$ be the subset of $\Omega_x^{(i)}[n]$ consisting of trajectories of the i-th process having the form

$$(3.27)\qquad x\to x_1\to \dots \to x_n \quad (x_n\in\Omega_s;\ x, x_1, \dots, x_{n-1}\in\Omega_A^*\setminus\Omega_0).$$

It is clear that

$$(3.28)\qquad \Omega_x^{(i)}[n] = \Omega_x^{(i)}[n, 0]\cup\Omega_x^{(i)}[n, 1] \quad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2),$$

$$(3.29)\qquad \tilde\mu_x^{(i)}\{\Omega_x^{(i)}[n]\} = \sum_{s=0}^{1} \tilde\mu_x^{(i)}\{\Omega_x^{(i)}[n, s]\} \quad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2).$$

Hence, from (2.25) we get

$$t_2^{(i)}(x_0, x_1, \ldots, x_n) = \begin{cases} C_1^{(i)} + nC_2 & \text{if } (x \to x_1 \to \ldots \to x_n) \in \Omega_x^{(i)}[n, 0], \\ 0 & \text{if } (x \to x_1 \to \ldots \to x_n) \in \Omega_x^{(i)}[n, 1] \end{cases} \quad (x \in \Omega_A^* \setminus \Omega_0;\ n \geqslant 1). \tag{3.30}$$

LEMMA 3.2. *Under the assumptions of Lemma* 3.1 *and* $\varphi \in L_1(\Omega)$, *the expected values* $Mt_2^{(i)}$ $(i = 1, 2)$ *exist and are finite, and*

$$Mt_2^{(i)} = C_1^{(i)}(\sqrt{\pi}, \sqrt{\pi})_{\Omega_0} + C_1^{(i)}(\pi, v^{(i)})_{\Omega \setminus \Omega_0} + C_2(\pi, w^{(i)})_{\Omega \setminus \Omega_0} \quad (i = 1, 2), \tag{3.31}$$

where $v^{(i)}(x)$ *and* $w^{(i)}(x)$ *are the solutions in* $L_\infty(\Omega \setminus \Omega_0)$ *of the equations*

$$v^{(i)}(x) - [T^{(p)} v^{(i)}](x) = [T_0^{(p)} \cdot 1](x) + \frac{g_i(x)}{\mu(\Omega_0)} \left[\tilde{T}_0^{(p)} \left(\frac{1}{g_i} \right) \right](x) \quad (x \in \Omega \setminus \Omega_0), \tag{3.32}$$

$$w^{(i)}(x) - [T^{(p)} w^{(i)}](x) = v^{(i)}(x) \quad (x \in \Omega \setminus \Omega_0). \tag{3.33}$$

Proof. Put

$$v^{(i)}(x) = \sum_{n=1}^{\infty} \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} \quad (x \in \Omega_A^* \setminus \Omega_0;\ i = 1, 2). \tag{3.34}$$

From (3.29) and (3.17) we deduce

$$v^{(i)}(x) \leqslant \sum_{n=1}^{\infty} \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n]\} = \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}\} = 1 \quad (x \in \Omega_A^* \setminus \Omega_0),$$

i.e. $v_x^{(i)}(x) \in L_\infty(\Omega \setminus \Omega_0)$. From (3.7) we obtain

$$\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[1, 0]\} = \int_{\Omega_0} P_i(1, x, dx_1) \quad (x \in \Omega_A^* \setminus \Omega_0), \tag{3.35}$$

$$\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} = \int_{\Omega_A^* \setminus \Omega_0} \tilde{\mu}_{x_1}^{(i)}\{\Omega_{x_1}^{(i)}[n-1, 0]\} P_i(1, x, dx_1) \quad (x \in \Omega_A^* \setminus \Omega_0;\ n \geqslant 2). \tag{3.36}$$

Hence, from (3.34) we have

$$\int_{\Omega_A^* \setminus \Omega_0} v^{(i)}(x_1) P_i(1, x, dx_1) = \sum_{n=2}^{\infty} \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} \quad (x \in \Omega_A^* \setminus \Omega_0). \tag{3.37}$$

Using (3.34), (3.35) and (3.37) we obtain

$$v^{(i)}(x) = \int_{\Omega_A^* \setminus \Omega_0} v^{(i)}(x_1) P_i(1, x, dx_1) + \int_{\Omega_0} P_i(1, x, dx_1) \quad (x \in \Omega_A^* \setminus \Omega_0). \tag{3.38}$$

Therefore, from (3.21) and (2.7) we get

$$v^{(i)}(x) = \int_{\Omega\setminus\Omega_0} K(x, x_1)p(x, x_1)v^{(i)}(x_1)\mu(dx_1) + \frac{g_i(x)}{\mu(\Omega_0)} \int_{\Omega_0} \frac{p(x, x_1)}{g_i(x_1)}\mu(dx_1) + \\ + \int_{\Omega_0} K(x, x_1)p(x, x_1)\mu(dx_1) \qquad (x\in\Omega\setminus\Omega_0 \ (\mathrm{mod}\,\mu)),$$

i.e. $v^{(i)}(x)$, defined by (3.34), is the unique solution in $L_\infty(\Omega\setminus\Omega_0)$ of equation (3.32) (7).

Put

$$(3.39) \qquad w^{(i)}(x) = \sum_{n=1}^{\infty} n\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} \qquad (x\in\Omega_A^*\setminus\Omega_0;\ i = 1, 2).$$

By (3.29), (3.10) and (3.15) we deduce

$$w^{(i)}(x) \leqslant \sum_{n=1}^{\infty} n\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n]\} = z^{(i)}(x) \leqslant q^{-2} \qquad (x\in\Omega_A^*\setminus\Omega_0),$$

i.e. $w^{(i)}(x)\in L_\infty(\Omega\setminus\Omega_0)$. From (3.39) and (3.36) it follows that

$$\int_{\Omega_A^*\setminus\Omega_0} w^{(i)}(x_1)P_i(1, x, dx_1) = \sum_{n=1}^{\infty} n\tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} - \sum_{n=1}^{\infty} \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} \\ (x\in\Omega_A^*\setminus\Omega_0).$$

Hence, from (3.21), (2.7), (3.34) and (3.39) we have

$$\int_{\Omega\setminus\Omega_0} K(x, x_1)p(x, x_1)w^{(i)}(x_1)\mu(dx_1) = w^{(i)}(x) - v^{(i)}(x) \qquad (x\in\Omega\setminus\Omega_0 \ (\mathrm{mod}\,\mu)),$$

i.e. $w^{(i)}(x)$, defined by (3.39), is the unique solution in $L_\infty(\Omega\setminus\Omega_0)$ of equation (3.33).

Since $t_2^{(i)} = t_2^{(i)}(x_0, x_1, \ldots, x_l) \geqslant 0$ (see (2.25)), $Mt_2^{(i)}$ exists and is defined by the following formula (see (2.27), (2.16) and (3.7)):

$$(3.40) \qquad Mt_2^{(i)} \equiv \int_{\tilde{\Omega}^{(i)}} t_2^{(i)} d\tilde{\mu}^{(i)} = \int_{\Omega^{(i)}} t_2^{(i)} d\tilde{\mu}^{(i)} \\ = \int_{\Omega^{(i)}[0]} t_2^{(i)}(x_0)P(dx_0) + \sum_{n=1}^{\infty} \int_{\Omega^{(i)}[n]} t_2^{(i)}(x_0, x_1, \ldots, x_n)d\tilde{\mu}^{(i)} \\ = \int_{\Omega_0\cup\Omega_1} t_2^{(i)}(x_0)P(dx_0) + \\ + \sum_{n=1}^{\infty} \int_{\Omega_A^*\setminus\Omega_0} \Big\{ \int_{\Omega_x^{(i)}[n]} t_2^{(i)}(x, x_1, \ldots, x_n)d\tilde{\mu}_x^{(i)}\Big\} P(dx).$$

(7) It follows from (3.22) that equations (3.32) and (3.33) have the unique solutions in $L_\infty(\Omega\setminus\Omega_0)$.

From (3.28) and (3.30) we deduce

$$(3.41)\qquad \int\limits_{\Omega_x^{(i)}[n]} t_2^{(i)}(x, x_1, \ldots, x_n)\, d\tilde{\mu}_x^{(i)} = (C_1^{(i)} + nC_2) \int\limits_{\Omega_x^{(i)}[n]} d\tilde{\mu}_x^{(i)}$$

$$= (C_1^{(i)} + nC_2)\, \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} \qquad (x \in \Omega_A^* \setminus \Omega_0;\ n \geqslant 1).$$

From (2.25) and (2.11) we have

$$(3.42)\qquad \int\limits_{\Omega_0 \cup \Omega_1} t_2^{(i)}(x_0) P(dx_0) = C_1^{(i)} P(\Omega_0) = C_1^{(i)} \int\limits_{\Omega_0} \pi(x) \mu(dx).$$

By (3.40)-(3.42), it follows clearly that

$$Mt_2^{(i)} = C_1^{(i)} \int\limits_{\Omega_0} \pi(x)\mu(dx) + C_1^{(i)} \int\limits_{\Omega_A^* \setminus \Omega_0} \sum_{n=1}^{\infty} \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} P(dx) +$$

$$+ C_2 \int\limits_{\Omega_A^* \setminus \Omega_0} \sum_{n=1}^{\infty} n \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} P(dx).$$

Hence, from (3.34), (3.39) and (2.11) we obtain (3.31). This completes the proof.

LEMMA 3.3. *Under the assumptions of Lemma* 3.1, *the expected values* $M\tilde{t}_4^{(i)}$ $(i = 1, 2)$ *exist and are finite, and*

$$(3.43)\qquad M\tilde{t}_4^{(i)} = (\sqrt{\pi}, \sqrt{\pi})_{\Omega_0} + (\pi, v^{(i)})_{\Omega \setminus \Omega_0} \qquad (i = 1, 2),$$

where $v^{(i)}(x)$ *is the solution in* $L_\infty(\Omega \setminus \Omega_0)$ *of equation* (3.32).

Proof. Since $\tilde{t}_4^{(i)} = \tilde{t}_4^{(i)}(x_0, x_1, \ldots, x_l) \geqslant 0$ (see (2.28)), there exists $M\tilde{t}_4^{(i)}$ and we have (see (2.27))

$$(3.44)\qquad M\tilde{t}_4^{(i)} \equiv \int\limits_{\tilde{\Omega}^{(i)}} t_4^{(i)} d\tilde{\mu}^{(i)} = \int\limits_{\Omega^{(i)}} t_4^{(i)} d\tilde{\mu}^{(i)}$$

$$= \int\limits_{\Omega^{(i)}[0]} \tilde{t}_4^{(i)}(x_0) d\tilde{\mu}^{(i)} + \sum_{n=1}^{\infty} \int\limits_{\Omega^{(i)}[n]} \tilde{t}_4^{(i)}(x_0, \ldots, x_n) d\tilde{\mu}^{(i)}.$$

From (2.28) we deduce

$$(3.45)\qquad \tilde{t}_4^{(i)}(x_0) = \begin{cases} 1 & \text{if } x_0 \in \Omega_0, \\ 0 & \text{if } x_0 \in \Omega_1. \end{cases}$$

$$(3.46)\qquad \tilde{t}_4^{(i)}(x, x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } (x \to x_1 \to \ldots \to x_n) \in \Omega_x^{(i)}[n, 0], \\ 0 & \text{if } (x \to x_1 \to \ldots \to x_n) \in \Omega_x^{(i)}[n, 1]. \end{cases}$$

Hence, by (3.44), (2.16) and (3.7), it is not difficult to see that

$$Mt_4^{(i)} = \int\limits_{\Omega_0} P(dx_0) + \int\limits_{\Omega_A^* \setminus \Omega_0} \sum_{n=1}^{\infty} \tilde{\mu}_x^{(i)}\{\Omega_x^{(i)}[n, 0]\} P(dx).$$

Therefore, from (2.11) and (3.34), we obtain (3.43). This completes the proof.

By formula (3.1) and Lemmas 3.1-3.3, the following theorem is evident:

THEOREM 3.1. *Let the assumptions of Lemma* 3.2 *and condition* (2.22) *be satisfied. Then the mean time* T_ε *for solving the problem* (1.2) *by the* ε*-scheme* S_ε *is finite and defined by the formula*

$$T_\varepsilon = \sum_{i=1}^{2} N_i Q^{(i)} + 2C_d + C_s, \tag{3.47}$$

where N_i *is given by formula* (2.23),

$$Q^{(i)} \equiv \tau_0 + \bar{C}_1^{(i)}(\sqrt{\pi}, \sqrt{\pi})_{\Omega_0} + (\pi, \tau_p^{(i)} z + \bar{C}_p^{(i)} v^{(i)} + C_2 w^{(i)})_{\Omega \setminus \Omega_0} \quad (i = 1, 2), \tag{3.48}$$

$$\bar{C}_1^{(i)} \equiv C_1^{(i)} + C_s = C_\varphi + C_{g_i} + C_\pi + C_m + C_d + C_s, \tag{3.49}$$

and $z(x)$, $v^{(i)}(x)$ *and* $w^{(i)}(x)$ *are the solutions in* $L_\infty(\Omega \setminus \Omega_0)$ *of equations* (3.9), (3.32) *and* (3.33), *respectively.*

By this theorem, it is easy to estimate the upper bound of the mean time T_ε for solving the problem (1.2) by the ε-scheme (see [10], p. 65-67).

In order to obtain more concrete results, we estimate $D\eta^{(i)}$ in formula (2.23).

Let T_p, $T_p^{(0)}$ and $\tilde{T}_p^{(0)}$ be the integral operators defined by the following formulae:

$$\begin{gathered} [T_p f](x) = \int_{\Omega \setminus \Omega_0} \frac{K(x, y)}{p(x, y)} f(y) \mu(dy), \\ [T_p^{(0)} f](x) = \int_{\Omega_0} \frac{K(x, y)}{p(x, y)} f(y) \mu(dy), \quad [\tilde{T}_p^{(0)} f](x) = \int_{\Omega_0} \frac{f(y) \mu(dy)}{p(x, y)}. \end{gathered} \tag{3.50}$$

LEMMA 3.4. *Under the assumptions of Lemma* 3.2, *suppose that* $\varphi^2(x)/\pi(x) \in L_1(\Omega)$ *and that the following conditions are satisfied:*

(P$_4$) $T_p \in [L_\infty(\Omega \setminus \Omega_0) \to L_\infty(\Omega \setminus \Omega_0)]$ *and the series* $\sum_{n=0}^{\infty} T_p^n f$ *converges in* $L_\infty(\Omega \setminus \Omega_0)$ (*for all* $f \in L_\infty(\Omega \setminus \Omega_0)$);

(P$_5$) $T_p^{(0)}, \tilde{T}_p^{(0)} \in [L_\infty(\Omega_0) \to L_\infty(\Omega \setminus \Omega_0)]$.

Then we have (2.22) *and*

$$D\eta^{(i)} = \left(\frac{\varphi^2}{\pi}, g_i\right)_{\Omega_0} + \left(\frac{\varphi^2}{\pi}, u_p^{(i)}\right)_{\Omega \setminus \Omega_0} - (u^{(i)}, \varphi)^2, \tag{3.51}$$

where $u^{(i)}(x)$ *is the solution in* $L_\infty(\Omega)$ *of equation* (2.18) *and* $u_p^{(i)}(x)$ *is the solution in* $L_\infty(\Omega \setminus \Omega_0)$ *of the equation*

$$u_p^{(i)}(x) - [T_p u_p^{(i)}](x) = [T_p^{(0)} (g_i)^2](x) + \frac{g_i(x)}{\mu(\Omega_0)} [\tilde{T}_p^{(0)} g_i](x) \quad (x \in \Omega \setminus \Omega_0). \tag{3.52}$$

Proof. Put

$$(3.53)\quad H_i(x) = [T_p^{(0)}(g_i)^2](x) + \frac{g_i(x)}{\mu(\Omega_0)}[\tilde{T}_p^{(0)} g_i](x) \qquad (x \in \Omega\setminus\Omega_0;\ i = 1, 2).$$

Since $g_i(x) \in L_\infty(\Omega)$, it follows, by condition (P_5), that $H_i(x) \in L_\infty(\Omega\setminus\Omega_0)$. Moreover, from (3.21) and (2.7) we obtain

$$(3.54)\quad H_i(x) \equiv \int\limits_{\Omega_0} \frac{g_i^2(y)}{p^2(x, y)}\left[K(x, y)p(x, y) + \frac{g_i(x)p(x, y)}{\mu(\Omega_0)g_i(y)}\right]\mu(dy) = \int\limits_{\Omega_0} \frac{g_i^2(y)}{p(x, y)} P_i(1, x, dy).$$

From condition (P_4), we deduce that the solution in $L_\infty(\Omega\setminus\Omega_0)$ of equation (3.52) is of the form

$$(3.55)\quad u_p^{(i)}(x) = \sum_{n=0}^{\infty} [T_p^n H_i](x) \qquad (x \in \Omega\setminus\Omega_0).$$

It is known (see condition (P_2) and [10] or [11], Lemma (2.2)) that

$$(3.56)\quad H_i(x) \geqslant 0, \quad \frac{K(x, y)}{p(x, y)} \geqslant 0 \quad (x \in \Omega\setminus\Omega_0 \,(\mathrm{mod}\,\mu),\ y \in \Omega_0 \,(\mathrm{mod}\,\mu)).$$

Hence, from (3.54), (3.55), (3.21), (2.7) and (2.11) it is easy to deduce that

$$\int\limits_{\Omega\setminus\Omega_0} \frac{\varphi^2(x)}{\pi(x)} u_p^{(i)}(x)\mu(dx) = \sum_{n=1}^{\infty} \underbrace{\int\limits_{\Omega\setminus\Omega_0} \cdots \int\limits_{\Omega\setminus\Omega_0}}_{(n)} \left[\int\limits_{\Omega_0} \frac{\varphi^2(x) g_i^2(x_n) P(1, x_{n-1}, dx_n)}{\pi^2(x)p^2(x, x_1)\ldots p^2(x_{n-1}, x_n)}\right] \times$$
$$\times P_i(1, x_{n-2}, dx_{n-1}) \ldots P_i(1, x, dx_1) P(dx).$$

Therefore, from (2.15) and (2.16) we obtain

$$(3.57)\quad \int\limits_{\Omega\setminus\Omega_0} \frac{\varphi^2(x)}{\pi(x)} u_p^{(i)}(x)\mu(dx) = \sum_{n=1}^{\infty} \int\limits_{\Omega^{(i)}[n]} (F^{(i)})^2 d\tilde{\mu}^{(i)}.$$

Moreover, it is evident (see (2.11) and (2.15)) that

$$(3.58)\quad \int\limits_{\Omega_0} \frac{\varphi^2(x)}{\pi(x)} g_i^2(x)\mu(dx) = \int\limits_{\Omega_0} \frac{\varphi^2(x) g_i^2(x)}{\pi^2(x)} P(dx) = \int\limits_{\Omega^{(i)}[0]} (F^{(i)})^2 d\tilde{\mu}^{(i)}.$$

From (2.27), (3.57) and (3.58) we have

$$(3.59)\qquad M(\eta^{(i)})^2 \equiv \int\limits_{\tilde{\Omega}^{(i)}} (F^{(i)})^2 d\tilde{\mu}^{(i)} = \int\limits_{\Omega^{(i)}} (F^{(i)})^2 d\tilde{\mu}^{(i)}$$

$$= \sum_{n=0}^{\infty} \int\limits_{\Omega^{(i)}{}_{[n]}} (F^{(i)})^2 d\tilde{\mu}^{(i)} = \left(\frac{\varphi^2}{\pi}, g_i^2\right)_{\Omega_0} + \left(\frac{\varphi^2}{\pi}, u_p^{(i)}\right)_{\Omega\setminus\Omega_0}.$$

Since $\varphi^2/\pi \in L_1(\Omega)$, $g_i \in L_\infty(\Omega)$ and $u_p^{(i)} \in L_\infty(\Omega\setminus\Omega_0)$, so from (3.59) we deduce that $M(\eta^{(i)})^2 < +\infty$. Moreover, by (2.17), we get $(M\eta^{(i)})^2 = (u^{(i)}, \varphi)^2 < +\infty$. Hence, the variance $D\eta^{(i)}$ exists and is finite, and (see (3.59) and (2.17))

$$D\eta^{(i)} = M(\eta^{(i)})^2 - (M\eta^{(i)})^2 = \left(\frac{\varphi^2}{\pi}, g_i\right)_{\Omega_0} + \left(\frac{\varphi^2}{\pi}, u_p^{(i)}\right) - (u^{(i)}, \varphi)^2.$$

This completes the proof.

It follows from (3.51) and (3.52) that we can diminish the value of the variance $D\eta^{(i)}$ by choosing a function $p(x, y)$ (or $\pi(x)$) such that its absolute value be great on $\Omega\times\Omega$ (or on Ω). We can also estimate the upper bound of $D\eta^{(i)}$ (see [10], p. 72-77). Moreover, from Lemma 3.4 and Theorem 3.1 it is easy to deduce

THEOREM 3.2. *Assume that* $\varphi \in L_1(\Omega)$, *and that conditions* (A), (B) *and* (C) *are satisfied. Suppose that the function* $p(x, y)$ *is bounded, measurable on* $\Omega\times\Omega$ *and satisfies conditions* (P_1)-(P_5). *Let the function* $\pi(x)$ *satisfy conditions* (Π_1), (Π_2) *and*

$$(\Pi_3)\qquad \varphi^2\pi^{-1} \in L_1(\Omega).$$

Then the mean time T_s *for solving the problem* (1.2) *by* S_s *is finite and defined by the formula* (8)

$$(3.60)\qquad T_s = \sum_{i=1}^{2} Q^{(i)}([L^{(i)}/\varepsilon_i^2 + 1]) + 2C_d + C_s,$$

where the constants ε_i $(i = 1, 2)$ *are given by condition* (E),

$$(3.61)\qquad L^{(i)} \equiv D\eta^{(i)} = \left(\frac{\varphi^2}{\pi}, g_i^2\right)_{\Omega_0} + \left(\frac{\varphi^2}{\pi}, u_p^{(i)}\right) - (u^{(i)}, \varphi)^2 \qquad (i = 1, 2),$$

$u_p^{(i)}$ *is the solution in* $L_\infty(\Omega\setminus\Omega_0)$ *of equation* (3.52), *and* $u^{(i)}$ *is the solution in* $L_\infty(\Omega)$ *of equation* (2.18).

COROLLARY. *Under the assumptions of Theorem* 3.2, *we have*

$$(3.62)\qquad T_s < 2C_d + C_s +$$
$$+\left[\tau_0 + \left(\tau_p + 2\bar{C}_1^{(2)} + \frac{C_2}{1-a^*}\right)(1-a^*)^{-1}\right]\left[2 + \bar{D}\sum_{i=1}^{2}\varepsilon_i^{-2} - \sum_{i=1}^{2}\varepsilon_i^{-2}(u^{(i)}, \varphi)^2\right],$$

(8) The symbol $[L^{(i)}/\varepsilon_i^2]$ denotes the integral part of the value $L^{(i)}/\varepsilon_i^2$.

where

(3.63)

$$\bar{D} \equiv \left(2\|g\| + \frac{2GM}{1-\alpha^*} + \Delta\right)^2 \left[1 + \frac{(\|T_p\|^{n_p} - 1)\big(\mu(\Omega_0)\|T_p^{(0)}\| + \|\tilde{T}_p^{(0)}\|\big)}{\mu(\Omega_0)(1-\beta_p)(\|T_p\|-1)}\right] \left\| \frac{\varphi^2}{\pi} \right\|,$$

(3.64)

$$\tau_p \equiv \max(\tau_p^{(1)}, \tau_p^{(2)}), \qquad \left\|\frac{\varphi^2}{\pi}\right\| \equiv \int_{\Omega} \frac{\varphi^2(x)}{\pi(x)}\, \mu(dx),$$

$$\|g\| = \operatorname*{vrai\,sup}_{\mu \quad x\in\Omega} \{|g(x)|\},$$

(3.65)

$$\|T_p^{(0)}\| \equiv \operatorname*{vrai\,sup}_{\mu \quad x\in\Omega\setminus\Omega_0} \left\{ \int_{\Omega_0} \frac{K(x,y)}{p(x,y)}\, \mu(dy) \right\},$$

$$\|\tilde{T}_p^{(0)}\| \equiv \operatorname*{vrai\,sup}_{\mu \quad x\in\Omega\setminus\Omega_0} \left\{ \int_{\Omega_0} \frac{\mu(dy)}{p(x,y)} \right\}, \quad \|T_p\| = \operatorname*{vrai\,sup}_{\mu \quad x\in\Omega\setminus\Omega_0} \left\{ \int_{\Omega\setminus\Omega_0} \frac{K(x,y)}{p(x,y)}\, \mu(dy) \right\},$$

and the natural number is chosen such that the following condition is satisfied:

(3.66)

$$\beta_p \equiv \|T_p^{n_p}\|_{L_\infty(\Omega\setminus\Omega_0)} < 1.$$

Proof. It follows from condition (P_4) that there exists a natural number n_p defined by condition (3.66). Moreover, from (3.52) and (3.53) we have also

$$u_p^{(i)} = \sum_{n=0}^{\infty} T_p^n H_i = \sum_{n=0}^{\infty} (T_p^{n_p})^n \sum_{k=0}^{n_p-1} T_p^k H_i.$$

Therefore,

(3.67)

$$\|u_n^{(i)}\|_{L_\infty(\Omega\setminus\Omega_0)} \leqslant (1-\beta_p)^{-1} \frac{\|T_p\|^{n_p} - 1}{\|T_p\| - 1} \|H_i\|_{L_\infty(\Omega\setminus\Omega_0)}.$$

From (3.53) we deduce

(3.68)

$$\|H_i\|_{L_\infty(\Omega\setminus\Omega_0)} \leqslant \left(\|T_p^{(0)}\| + \frac{\|\tilde{T}_p^{(0)}\|}{\mu(\Omega_0)}\right) \|g_i\|^2,$$

where (see (2.9))

(3.69)

$$\|g_i\| \equiv \operatorname*{vrai\,sup}_{\mu \quad x\in\Omega} \{|g_i(x)|\} \leqslant 2\|g\| + \frac{2GM}{1-\alpha^*} + \Delta.$$

From (3.51) we have

(3.70)

$$D\eta^{(i)} \leqslant \left\|\frac{\varphi^2}{\pi}\right\| \left(\|g_i\|^2 + \|u_p^{(i)}\|_{L_\infty(\Omega\setminus\Omega_0)}\right) - (u^{(i)}, \varphi)^2.$$

By (3.61), (3.63) and (3.67)-(3-70) it is easy to deduce

(3.71)

$$L^{(i)} \equiv D\eta^{(i)} \leqslant \bar{D} - (u^{(i)}, \varphi)^2 \qquad (i = 1, 2).$$

From (2.9) we get

$$\frac{g_i(x)}{g(x)} \leqslant \frac{2G(1-a^*)}{2GM+\Delta(1-a^*)} \qquad (x\in \Omega_\Delta^* \setminus \Omega_0;\ y\in\Omega_0;\ i=1,2).$$

Hence, from (2.10) and (3.4) we obtain

$$\left\| \frac{g_i}{\mu(\Omega_0)}\tilde{T}_0^{(p)}\left(\frac{1}{g_i}\right)\right\|_{L_\infty(\Omega\setminus\Omega_0)} \leqslant \frac{2GM(1-a^*)}{2GM+\Delta(1-a^*)} \qquad (i=1,2). \tag{3.72}$$

By (3.3) and conditions (P_1^*), (P_1), we deduce

$$\|T_0^{(p)}(1)\|_{L_\infty(\Omega\setminus\Omega_0)} \leqslant a \leqslant a^* < 1. \tag{3.73}$$

Therefore, from (3.22) and (3.32) it is easy to get

$$\|v^{(i)}\|_{L_\infty(\Omega\setminus\Omega_0)} \leqslant (1-a^*)^{-1}\left(a^* + \frac{2GM(1-a^*)}{2GM+\Delta(1-a^*)}\right) < (1-a^*)^{-1}. \tag{3.74}$$

From (3.22), (3.33) and (3.9) we have

$$\|w^{(i)}\|_{L_\infty(\Omega\setminus\Omega_0)} \leqslant (1-a^*)^{-1}\|v^{(i)}\|_{L_\infty(\Omega\setminus\Omega_0)} < (1-a^*)^{-2}, \tag{3.75}$$

$$\|z\|_{L_\infty(\Omega\setminus\Omega_0)} \leqslant (1-a^*)^{-1}. \tag{3.76}$$

Using the definition of C_{g_i} and (2.9) it is easy to deduce that $C_{g_1} < C_{g_2}$; therefore (see (3.49)),

$$\bar{C}_1^{(1)} < \bar{C}_1^{(2)}. \tag{3.77}$$

From (3.48), (3.74)-(3.77) and condition (Π_2) we have

$$Q^{(i)} < \tau_0 + \left(\tau_p + 2\bar{C}_1^{(2)} + \frac{C_2}{1-a^*}\right)(1-a^*)^{-1} \qquad (i=1,2). \tag{3.78}$$

Hence, by (3.47) and (3.71) it is easy to obtain formula (3.62). This completes the proof.

Note that by Lemmas 3.1 - 3.4 we can obtain an estimation of the upper bound of T_ε more precise than by formula (3.62) (see [10], p. 80).

4. Optimal ε-scheme and the comparison of ε-schemes. Put

$$s_\varepsilon = \{p(x,y),\ \pi(x),\ \Delta,\ \delta,\ \varepsilon_1,\ \varepsilon_2\}, \tag{4.1}$$

where the functions $p(x,y)$, $\pi(x)$ and the constants Δ, δ, ε_1, ε_2 are used in the suitable ε-scheme S_ε (i.e. they satisfy the assumptions of Theorem 3.1). It is clear that each s_ε corresponds in one-to-one way to the ε-scheme:

$$S_\varepsilon = S_\varepsilon(s_\varepsilon). \tag{4.2}$$

From (3.47) we see that the mean time T_ε for solving the problem (1.2) by $S_\varepsilon(s_\varepsilon)$ is the functional depending on s_ε:

$$T_\varepsilon = T_\varepsilon(s_\varepsilon) \equiv \sum_{i=1}^{2}\left([D\eta^{(i)}(s_\varepsilon)/\varepsilon_i^2(s_\varepsilon)]+1\right)Q^{(i)}(s_\varepsilon) + 2C_d + C_s. \tag{4.3}$$

From the practical sense of $T_\varepsilon(s_\varepsilon)$, we can replace the comparison of ε-schemes $S_\varepsilon(s_\varepsilon)$ by the one of values of the functional $T_\varepsilon(s_\varepsilon)$. Now we consider the problem more concretely for a class of special ε-schemes. Put

$$\varphi_\varepsilon = \varphi_1 \times \varphi_2 \times \varphi_3 \times \varphi_4 \times \varphi_5, \tag{4.4}$$

where

$$\varphi_1 = \{p(x, y) \in M(\Omega^2, \Sigma^2, \mu^2)\colon (\mathrm{P}_1)\text{-}(\mathrm{P}_5)\}, \tag{4.5}$$

and $M(\Omega^2, \Sigma^2, \mu^2)$ is a space of the following functions, $(\Sigma \times \Sigma)$-measurable and bounded on $\Omega \times \Omega$:

$$\varphi_2 = \{\pi(x)\colon (\Pi_1)\text{-}(\Pi_3)\}, \tag{4.6}$$

$$\varphi_3 = \{\Delta\colon \Delta > 0\}, \tag{4.7}$$

$$\varphi_4 = \{\delta\colon \delta > 0\}, \tag{4.8}$$

$$\varphi_5 = \{(\varepsilon_1, \varepsilon_2)\colon (\mathrm{E})\}. \tag{4.9}$$

It is easy to see that each

$$s_\varepsilon = \{p(x, y), \pi(x), \Delta, \delta, \varepsilon_1, \varepsilon_2\} \in \varphi_\varepsilon$$

satisfies the assumptions of Theorem 3.2. Therefore, we have

$$T_\varepsilon(s_\varepsilon) \equiv \sum_{i=1}^{2} Q^{(i)}(s_\varepsilon)\big([L^{(i)}(s_\varepsilon)/\varepsilon_i^2(s_\varepsilon)]+1\big)+2C_d+C_s \qquad (s_\varepsilon \in \varphi_\varepsilon), \tag{4.10}$$

where the dependences of $Q^{(i)}$ and $L^{(i)}$ upon s_ε are defined by formulae (3.48) and (3.61), respectively.

Definition 4.1. Let $s_\varepsilon, \tilde{s}_\varepsilon \in \varphi_\varepsilon$. Then the ε-scheme $S_\varepsilon = S_\varepsilon(s_\varepsilon)$ will be called *better* than the ε-scheme $\tilde{S}_\varepsilon = S_\varepsilon(\tilde{s}_\varepsilon)$ if $\tau_\varepsilon(s_\varepsilon) < \tau_\varepsilon(\tilde{s}_\varepsilon)$, where

$$\tau_\varepsilon(s_\varepsilon) = \sum_{i=1}^{\infty} Q^{(i)}(s_\varepsilon)\big([L^{(i)}(s_\varepsilon)/\varepsilon_i^2(s_\varepsilon)]+1\big). \tag{4.11}$$

It is easy to observe that (see (4.10) and (4.11)) if $S_\varepsilon = S_\varepsilon(s_\varepsilon)$ is the ε-scheme better than $\tilde{S}_\varepsilon = S_\varepsilon(\tilde{s}_\varepsilon)$, then the solution of problem (1.2) by S_ε is faster than that by $\tilde{S}_\varepsilon$. Similarly we have

Definition 4.2. $S_\varepsilon^* = S_\varepsilon(s_\varepsilon^*)$ is called the *optimal ε-scheme* in the class of ε-schemes $S = \{S_\varepsilon(s_\varepsilon)\colon s_\varepsilon \in \varphi_\varepsilon\}$ if

$$\tau_\varepsilon(s_\varepsilon^*) = \inf_{s_\varepsilon \in \varphi_\varepsilon}\{\tau_\varepsilon(s_\varepsilon)\}. \tag{4.12}$$

The minimum problem of (4.12) is unsolved in this paper. However, in the special case where $p(x, y) \equiv \bar{p}$ and $\pi(x) \equiv \bar{\pi}$ ($\bar{p}$ and $\bar{\pi}$ are constants), the solution of (4.12) becomes the problem of minimizing a function of six variables. Therefore, we can use the Monte Carlo method for solving this problem (see [12]).

Remark 4.1. Since the number N_i of experiments of each i-th process of S_ε in the practical computation is very great, we can take, by (2.23) and (3.61), the approximation

$$N_i = [D\eta^{(i)}/\varepsilon_i^2]+1 \approx \varepsilon_i^{-2} D\eta^{(i)} = L^{(i)}\varepsilon_i^{-2}.$$

Hence, by (4.11), we have

$$(4.13) \qquad \Phi_\varepsilon(s_\varepsilon) \equiv \sum_{i=1}^{2} Q^{(i)}(s_\varepsilon)\varepsilon_i^{-2}(s_\varepsilon)L^{(i)}(s_\varepsilon) \approx \tau_\varepsilon(s_\varepsilon).$$

Therefore, for the purpose of practical comparison of ε-schemes, in Definitions 4.1 and 4.2 we can replace the functional $\tau_\varepsilon(s_\varepsilon)$ by the simpler functional $\Phi_\varepsilon(s_\varepsilon)$. Now we use the functional $\Phi_\varepsilon(s_\varepsilon)$ to express the time (in practice) for solving the problem (1.2) by $S_\varepsilon = S_\varepsilon(s_\varepsilon)$.

5. Method of decreasing the time needed for the solution of the problem. In order to decrease the time needed for solving the problem (1.2), we now consider a method of decreasing the value of the functional Φ_ε. Let

$$(5.1) \qquad \gamma(\pi) = \underset{\tilde{\mu} \quad x\in\Omega^*}{\text{vrai inf}} \{\pi(x)\} > 0.$$

Then there exists a set $\tilde{A}^* \in \tilde{\Sigma}$ such that $\tilde{A}^* \subset \Omega^*$, $\tilde{\mu}(A^*) = 0$, and

$$(5.2) \qquad \gamma(\pi) \geqslant \gamma_0 \equiv \inf_{x\in\Omega^*\setminus\tilde{A}^*} \{\pi(x)\} > 0.$$

Therefore,

$$(5.3) \qquad \pi(x) \geqslant \gamma_0 > 0 \qquad (x \in \Omega^* \, (\text{mod}\, \tilde{\mu})).$$

LEMMA 5.1. *Under the assumptions of Theorem* 3.2 *and condition* (5.1), *let us put*

$$(5.4) \qquad \pi_\lambda(x) = \begin{cases} \lambda^{-1}\pi(x) & \text{if } x\in\Omega, \\ \pi(x) + \dfrac{\lambda-1}{\delta\lambda}\displaystyle\int_\Omega \pi(y)\mu(dy) & \text{if } x\in\Omega^*, \end{cases}$$

where $s_\varepsilon \equiv \{p(x, y), \pi_\lambda(x), \Delta, \delta, \varepsilon_1, \varepsilon_2\} \in \varphi_\varepsilon$, *and* λ *is a constant satisfying the condition*

$$(5.5) \qquad \lambda_0 = \frac{\int_\Omega \pi(y)\mu(dy)}{\delta\gamma_0 + \int_\Omega \pi(y)\mu(dy)} \leqslant \lambda < 1.$$

Then $s_\varepsilon^{(\lambda)} \equiv \{p(x, y), \pi_\lambda(x), \Delta, \delta, \varepsilon_1, \varepsilon_2\} \in \varphi_\varepsilon$.

Furthermore, if C_π, τ_0 and $\tau_p^{(i)}$ are constants (and we replace $S_\varepsilon = S_\varepsilon(s_\varepsilon)$ by $S_\varepsilon^{(\lambda)} = S_\varepsilon(s_\varepsilon^{(\lambda)})$), we have

$$\Phi_\varepsilon(s_\varepsilon) - \Phi_\varepsilon(s_\varepsilon^{(\lambda)}) = (1-\lambda)\sum_{i=1}^{2} \varepsilon_i^{-2} Q^{(i)}(s_\varepsilon^{(\lambda)})(u^{(i)}, \varphi)^2 + \\ + \frac{\tau_0}{\lambda}(1-\lambda)\sum_{i=1}^{2}\varepsilon_i^{-2} L^{(i)}(s_\varepsilon^{(\lambda)}) + \frac{\tau_0}{\lambda}(1-\lambda)^2 \sum_{i=1}^{2}\varepsilon_i^{-2}(u^{(i)}, \varphi)^2. \tag{5.6}$$

Proof. Since $s_\varepsilon \in \varphi_\varepsilon$, so $p(x, y) \in \varphi_1$, $\pi(x) \in \varphi_2$, $\Delta \in \varphi_3$, $\delta \in \varphi_4$, and $(\varepsilon_1, \varepsilon_2) \in \varphi_5$. It follows from (5.2) and conditions (Π_1) and $\delta > 0$ that

$$\int_\Omega \pi(y)\mu(dy) > 0 \qquad (\lambda_0 > 0). \tag{5.7}$$

Therefore, by (5.5), we obtain

$$0 < \lambda < 1. \tag{5.8}$$

By (5.4), (5.8) and condition (Π_1), we receive

$$0 < \pi_\lambda(x) = \lambda^{-1}\pi(x) < +\infty \qquad (x \in \Omega \pmod{\mu}). \tag{5.9}$$

From (5.5), (5.4) and (5.3) it follows that

$$\pi_\lambda(x) = \pi(x) - \frac{1-\lambda}{\delta\lambda}\int_\Omega \pi(y)\mu(dy) \geqslant \pi(x) - \gamma_0 \geqslant 0 \\ (x \in \Omega^* \pmod{\tilde{\mu}}). \tag{5.10}$$

Hence, by (5.9), we have

$$\pi_\lambda(x) \geqslant 0 \qquad (x \in \Omega \cup \Omega^* = \tilde{\Omega} \pmod{\tilde{\mu}}). \tag{5.11}$$

Taking into account (5.4), (2.2) and condition (Π_2), we easily obtain

$$\int_{\tilde{\Omega}} \pi_\lambda(x)\tilde{\mu}(dx) \\ = \lambda^{-1}\int_\Omega \pi(x)\mu(dx) + \int_{\Omega^*}\pi(x)\tilde{\mu}(dx) + \frac{\lambda-1}{\delta\lambda}\tilde{\mu}(\Omega^*)\int_\Omega \pi(y)\mu(dy) \\ = \int_{\Omega^*}\pi(x)\tilde{\mu}(dx) + \int_\Omega \pi(x)\tilde{\mu}(dx) = \int_{\tilde{\Omega}}\pi(x)\tilde{\mu}(dx) = 1. \tag{5.12}$$

From (5.4) and condition (Π_3) it is clear that

$$\varphi^2(x)\pi_\lambda(x)^{-1} \equiv \lambda\varphi^2(x)\pi(x)^{-1} \in L_1(\Omega). \tag{5.13}$$

For reason of (5.9) and (5.11)-(5.13) we remark that the function $\pi_\lambda(x)$, defined by formula (5.4), satisfies also conditions (Π_1)-(Π_3), i.e. $\pi_\lambda(x) \in \varphi_2$. Therefore,

$$s_\varepsilon^{(\lambda)} \equiv \{p(x, y), \pi_\lambda(x), \Delta, \delta, \varepsilon_1, \varepsilon_2\} \in \varphi_\varepsilon.$$

Since C_π, τ_0 and $\tau_p^{(i)}$ are constants (by hypothesis), so $\bar{C}_1^{(i)}$ and C_2 are also constants (see (2.26) and (3.49)). Hence, from (4.13), (3.48) and (3.61) it follows that

$$\Phi_\varepsilon(s_\varepsilon^{(\lambda)}) = \sum_{i=1}^{2} Q^{(i)}(s_\varepsilon^{(\lambda)})\, \varepsilon_i^{-2} L^{(i)}(s_\varepsilon^{(\lambda)}), \tag{5.14}$$

where

$$Q^{(i)}(s_\varepsilon^{(\lambda)}) = \tau_0 + \bar{C}_1^{(i)}\left(\sqrt{\pi_\lambda}, \sqrt{\pi_\lambda}\right)_{\Omega_0} + \left(\pi_\lambda, \tau_p^{(i)} z + \bar{C}_1^{(i)} v^{(i)} + C_2 w^{(i)}\right)_{\Omega\setminus\Omega_0}, \tag{5.15}$$

$$L^{(i)}(s_\varepsilon^{(\lambda)}) \doteq \left(\frac{\varphi^2}{\pi_\lambda}, g_i^2\right)_{\Omega_0} + \left(\frac{\varphi^2}{\pi_\lambda}, u_p^{(i)}\right)_{\Omega\setminus\Omega_0} - (u^{(i)}, \varphi)^2. \tag{5.16}$$

By virtue of (3.48), (5.15) and (5.4) we deduce that

$$Q^{(i)}(s_\varepsilon^{(\lambda)}) = \lambda^{-1}[Q^{(i)}(s_\varepsilon) + (\lambda - 1)\tau_0] \tag{5.17}$$

or

$$Q^{(i)}(s_\varepsilon) = \lambda Q^{(i)}(s_\varepsilon^{(\lambda)}) + (1 - \lambda)\tau_0. \tag{5.18}$$

Similarly, from (3.61), (5.16) and (5.4) it follows

$$L^{(i)}(s_\varepsilon^{(\lambda)}) = \lambda[L^{(i)}(s_\varepsilon) + (u^{(i)}, \varphi)^2] - (u^{(i)}, \varphi)^2, \tag{5.19}$$

$$L^{(i)}(s_\varepsilon) = \lambda^{-1}[L^{(i)}(s_\varepsilon^{(\lambda)}) + (1 - \lambda)(u^{(i)}, \varphi)^2]. \tag{5.20}$$

By (4.13), (5.18) and (5.20), we have

$$\Phi_\varepsilon(s_\varepsilon) = \sum_{i=1}^{2} Q^{(i)}(s_\varepsilon^{(\lambda)})\varepsilon_i^{-2} L^{(i)}(s_\varepsilon^{(\lambda)}) + (1-\lambda)\sum_{i=1}^{2} \varepsilon_i^{-2} Q^{(i)}(s_\varepsilon^{(\lambda)})(u^{(i)}, \varphi)^2 +$$
$$+ \lambda^{-1}(1-\lambda)\tau_0 \sum_{i=1}^{2} \varepsilon_i^{-2} L^{(i)}(s_\varepsilon^{(\lambda)}) + \lambda^{-1}(1-\lambda)^2 \tau_0 \sum_{i=1}^{2} \varepsilon_i^{-2}(u^{(i)}, \varphi)^2.$$

Finally, from (5.14) it follows (5.6). This completes the proof.

Remark 5.1. By (5.6) and (5.8) it easily follows

$$\Phi_\varepsilon(s_\varepsilon^{(\lambda)}) < \Phi_\varepsilon(s_\varepsilon), \tag{5.21}$$

i.e., under the assumptions of Lemma 5.1, from the ε-scheme $S_\varepsilon = S_\varepsilon(s_\varepsilon)$ we can construct the suitable better ε-scheme $S_\varepsilon^{(\lambda)} = S_\varepsilon(s_\varepsilon^{(\lambda)})$ (in the sense of (5.21)).

Remark 5.2. If condition (5.5) is replaced by the stricter condition

$$\lambda_0 < \lambda < 1, \tag{5.22}$$

then, from (5.4) and (5.2) it follows

$$\gamma(\pi_\lambda) \equiv \operatorname*{vrai\,inf}_{\tilde{\mu}\ \ x\in\Omega^*} \{\pi_\lambda(x)\} > \gamma(\pi) - \gamma_0 > 0. \tag{5.23}$$

It means that the function $\pi(x)$ satisfies conditions having form (5.1). Therefore, we can further use Lemma 5.1 for the ε-scheme $S_\varepsilon^{(\lambda)} = S_\varepsilon(s_\varepsilon^{(\lambda)})$ to receive the better ε-scheme $S_\varepsilon^{(\mu)} = S_\varepsilon(s_\varepsilon^{(\mu)})$. We can also repeat the above-given process for $S_\varepsilon^{(\mu)}$ and so on.

Note that we can choose many values for the constant λ satisfying condition (5.5). Hence, from the given ε-scheme $S_\varepsilon = S_\varepsilon(s_\varepsilon)$ we can construct a better ε-scheme $S_\varepsilon^{(\lambda)} = S_\varepsilon(s_\varepsilon^{(\lambda)})$ by many different methods. The problem is to choose such a value of λ that the value of the functional $\Phi_\varepsilon(s_\varepsilon^{(\lambda)})$ be minimal.

LEMMA 5.2. *Under the assumptions of Lemma* 5.1 *we have*

$$\Phi_\varepsilon(s_\varepsilon^{(\lambda_0)}) = \min_{\lambda_0\leqslant\lambda<1} \{\Phi_\varepsilon(s_\varepsilon^{(\lambda)})\}, \tag{5.24}$$

where the constant λ_0 *is defined by* (5.5).

Proof. Put

$$\Delta\Phi(\lambda) = \Phi_\varepsilon(s_\varepsilon) - \Phi_\varepsilon(s_\varepsilon^{(\lambda)}) \qquad (\lambda_0 \leqslant \lambda < 1). \tag{5.25}$$

From (5.6), (5.17) and (5.19) we get

$$\Delta\Phi(\lambda) = -\lambda^{-1}[(a_2+a_3)\lambda^2 + (a_1 - 2a_3 - a_2)\lambda + (a_3 - a_1)], \tag{5.26}$$

where

$$a_1 \equiv \sum_{i=1}^{2} \varepsilon_i^{-2} Q^{(i)}(s_\varepsilon)(u^{(i)}, \varphi)^2, \qquad a_2 \equiv \tau_0 \sum_{i=1}^{2} \varepsilon_i^{-2} L^{(i)}(s_\varepsilon),$$
$$a_3 \equiv \tau_0 \sum_{i=1}^{2} \varepsilon_i^{-2} (u^{(i)}, \varphi)^2. \tag{5.27}$$

By (5.26), it follows

$$\frac{d\Delta\Phi(\lambda)}{d\lambda} = -\lambda^{-1}[(a_2+a_3)\lambda^2 + (a_1 - a_3)]. \tag{5.28}$$

Since $Q^{(i)}(s_\varepsilon) > \tau_0$ (see (3.48)), we have

$$a_1 = \sum_{i=1}^{2} \varepsilon_i^{-2} Q^{(i)}(s_\varepsilon)(u^{(i)}, \varphi)^2 > \tau_0 \sum_{i=1}^{2} \varepsilon_i^{-2}(u^{(i)}, \varphi)^2 = a_3. \tag{5.29}$$

Moreover, from (3.61) and (5.27) it is easy to deduce

$$a_2+a_3 = \tau_0\sum_{i=1}^{2}\varepsilon_i^{-2}[L^{(i)}(s_\varepsilon)+(u^{(i)},\varphi)^2] > 0. \tag{5.30}$$

Using (5.28)-(5.30), we have

$$\frac{d\Delta\Phi(\lambda)}{d\lambda} < 0 \qquad (\lambda_0 \leqslant \lambda < 1).$$

Therefore,

$$\Delta\Phi(\lambda_0) = \max_{\lambda_0\leqslant\lambda<1}\{\Delta\Phi(\lambda)\}. \tag{5.31}$$

By virtue of (5.31) and (5.25), we get (5.24). This completes the proof.

Remark 5.3. It follows from Lemma 5.2 that in order to decrease most quickly the value of $\Phi_\varepsilon(s_\varepsilon)$ (using Lemma 5.1), the constant λ in condition (5.5) must be chosen such that $\lambda = \lambda_0$. Then formula (5.4) has the form

$$\pi_{\lambda_0}(x) = \begin{cases} \lambda_0^{-1}\pi(x) & \text{if } x\in\Omega, \\ \pi(x)-\gamma_0 & \text{if } x\in\Omega^*. \end{cases} \tag{5.32}$$

Remark 5.4. If we replace condition (5.1) by the stricter condition

$$\gamma(\pi) \equiv \operatorname{vrai}_{\tilde{\mu}}\inf_{x\in\Omega^*}\{\pi(x)\} > \gamma_0 \equiv \inf_{x\in\Omega^*\setminus\tilde{A}_0^*}\{\pi(x)\} > 0, \tag{5.33}$$

where $\tilde{A}_0^*\in\tilde{\Sigma}$, $\tilde{A}_0^*\subset\Omega^*$ and $\tilde{\mu}(\tilde{A}_0^*) = 0$. Then from (5.32) we have

$$\gamma(\pi_{\lambda_0}) \equiv \operatorname{vrai}_{\tilde{\mu}}\inf_{x\in\Omega^*}\{\pi_{\lambda_0}(x)\} = \gamma(\pi)-\gamma_0 > 0, \tag{5.34}$$

i.e. the function $\pi_{\lambda_0}(x)$, defined by (5.32), satisfies also the condition having form (5.1). Therefore, we can futher use Lemma 5.1 for $S_\varepsilon^{(\lambda_0)} = S_\varepsilon(s_\varepsilon^{(\lambda_0)})$ to receive a better ε-scheme $S_\varepsilon^{(\lambda_1)} = S_\varepsilon(s_\varepsilon^{(\lambda_1)})$.

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CZAS POTRZEBNY DO OSZACOWANIA PEWNEGO FUNKCJONAŁU METODĄ MONTE CARLO

STRESZCZENIE

W pracach [10] i [11] skonstruowano klasę probabilistycznych modeli dla obliczenia wartości funkcjonału

$$(u, \varphi) = \int_{\Omega} u(x)\varphi(x)\mu(dx),$$

gdzie $\varphi(x)$ jest funkcją μ-całkowalną na Ω, $u(x)$ zaś rozwiązaniem równań całkowych postaci

$$u(x) - \int_{\Omega} K(x, y)u(y)\mu(dy) = g(x)$$

w przestrzeni funkcji Σ-mierzalnych i ograniczonych na Ω (mod μ).

W pracy tej oszacowano czas potrzebny do obliczeń według każdego z modeli probabilistycznych poprzedniej klasy. Rozpatrzono także metodę zmniejszenia czasu obliczeń.