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Structure of the sets of weak solutions of an ordinary differential equation in a Banach space

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Abstract. Using the measure of weak non-compactness introduced by de Blasi [5], we prove that if $f\colon I\times B\to E$ (where E is a Banach space, $I=[0,a]$, $B=\{x\in E\colon \|x-x_0\|\leqslant b\}$) is bounded by some constant M and weakly-weakly continuous, and if $\beta(f(J\times V))\leqslant k\beta(V)$ for every subset V of B, then the set of all weak solutions of the Cauchy problem $x'=f(t,x)$, $x(0)=x_0$, defined on $[0,h]$ ($h=\min(a,b/M)$) is non-empty compact and connected in $C_w(J,E)$, the space of weakly continuous functions $u\colon J\to E$ endowed with the topology of weak uniform convergence. This generalizes former results of Szep [9], Szufla [12] and Cramer, Lakshmikantham and Mitchell [3].

Many theorems on the existence of weak solutions of ordinary differential equations in reflexive Banach spaces [9], as well as in non-reflexive ones [3], are known. If the function on the right-hand side of the equation is merely continuous or even uniformly continuous on bounded sets, the equations need not possess any solution [7].

Throughout this paper $(E,\|\ \|)$ will denote a real Banach space, E^* its dual, and E_w the space E endowed with the weak topology.

Assume that $I=[0,a]$, $B=\{x\in E\colon \|x-x_0\|\leqslant b\}$, and that $f\colon I\times B\to E$ is a weakly-weakly continuous function such that $\|f(t,x)\|\leqslant M$ on $I\times B$; moreover, we assume that E_w is sequentially weakly complete (for the definitions see [3]). Let $J=[0,h]$, where $h=\min(a,b/M)$.

We shall denote by β the measure of weak non-compactness of sets in E; the properties of β are analogous to those of the measure of non-compactness α; for the properties of both see [1], [3]–[5], [8], [10].

We shall deal with the Cauchy problem

$$x'=f(t,x),\qquad x(0)=x_0, \tag{1}$$

where x' denotes the weak derivative. It is known [9] that (1) is equivalent to the integral equation

$$x(t)=x_0+\int_0^t f(s,x(s))\,ds\qquad \text{for } t\in I,$$

the integral being taking in Pettis sense.

If $f\colon J\times B\to E$ is weakly-weakly continous and

$$\beta\big(f(J\times V)\big)\leqslant k\beta(V) \tag{2}$$

for every subset V of B, and $0\leqslant hk<1$, then (as in [11]) we introduce the set

$$H = x_0+\overline{\bigcup_{0\leqslant\lambda\leqslant h}\lambda\operatorname{conv} f(J\times H)}.$$

By (2) and by the properties of β, $\beta(H)=0$. Since the set H is closed and convex, by the theorem of Mazur it is weakly closed and consequently weakly compact.

Given any $\eta>0$, let us denote by S_η the set of all functions $u\colon J\to E$ such that

(a) $u(0)=x_0$, $\|u(t)-u(s)\|\leqslant M|t-s|$ for $t, s\in J$;

(b) $\sup\limits_{t\in J}\big\|u(t)-x_0-\int\limits_0^t f(s, u(s))\,ds\big\|<\eta$;

(c) $u(t)\in x_0+\bigcup\limits_{0\leqslant\lambda\leqslant t}\lambda\operatorname{conv} f(I\times H)$.

THEOREM. *Let $f\colon J\times B\to E$ be weakly-weakly continuous, and let*

$$\beta\big(f(J\times V)\big)\leqslant k\beta(V)$$

for every subset V of B; suppose that $0\leqslant hk<1$. Then the set of all weak solutions of the Cauchy problem (1) *defined on J is non-empty, compact and connected in $C_w(J, E)$.*

Proof. For $0<\varepsilon\leqslant h$ let x_ε denote the Euler polygonal line, i.e., the mapping defined by:

$$\begin{aligned} x_\varepsilon(t) &= x_0 && \text{for } 0\leqslant t\leqslant\varepsilon,\\ x_\varepsilon(t) &= x_\varepsilon(t_i)+(t-t_i)f\big(t_i, x_\varepsilon(t_i)\big) && \text{for } t\in[t_i, t_{i+1}], \end{aligned}$$

where $t_i=i\varepsilon$, $i=1, 2, \ldots$, $n_\varepsilon=[h/\varepsilon]$, $t_{n_\varepsilon+1}=h$. Write

$$V=\{x_\varepsilon(\cdot)\colon 0<\varepsilon\leqslant h\}.$$

By definition, $x_\varepsilon(t)\in B$ for every $t\in J$ and

$$\|x_\varepsilon(t)-x_\varepsilon(s)\|\leqslant M|t-s|.$$

First we prove that the set S_η is non-empty; namely, for every $\eta>0$ there exists an $\varepsilon(\eta)>0$ such that $x_\varepsilon\in S_\eta$ for any $\varepsilon<\varepsilon(\eta)$.

Each $x_\varepsilon(\cdot)$ obviously satisfies (a) and for $t\in[t_i, t_{i+1}]$ and $x^*\in E^*$, $\|x^*\|\leqslant 1$, we have

$$\begin{aligned} &\Big|x^*\Big[x_\varepsilon(t)-x_0-\int\limits_0^t f\big(s, x_\varepsilon(s)\big)\,ds\Big]\Big|\\ &\quad=\Big|x^*\Big[x_0+(t_2-t_1)f\big(t_1, x_\varepsilon(t_1)\big)+\ldots+ \end{aligned}$$

$$+(t-t_i)f(t_i,\, x_\varepsilon(t_i))-x_0-\int_0^t f(s,\, x_\varepsilon(s))\,ds]|$$

$$\leqslant \int_0^{t_1} \|f(s,\, x_0)\|\,ds+\int_{t_1}^{t_2} |x^*[f(t_1,\, x_\varepsilon(t_1))-f(s,\, x_\varepsilon(s))]|\,ds+$$

$$+\ldots+\int_{t_i}^{t} |x^*[f(t_i,\, x_\varepsilon(t_i))-f(s,\, x_\varepsilon(s))]|\,ds$$

$$\leqslant M\varepsilon+\delta_1 h.$$

Since $\|x_\varepsilon(s)-x_\varepsilon(t_k)\| \leqslant M|s-t_k|$ and since f is uniformly weakly-weakly continuous (since H is weakly compact), we infer that for any $\delta_1 > 0$ there exists $\delta > 0$ such that $|t-s| < \delta$, $\|x_\varepsilon(t)-x_\varepsilon(s)\| < \delta$ implies

$$\big|x^*\big(f(t,\, x_\varepsilon(t))-f(s,\, x_\varepsilon(s))\big)\big| < \delta_1 \quad \text{for all } x^*,\ \|x^*\| = 1.$$

Therefore for any $\eta > 0$ we can choose $\delta_1 > 0$ and $\varepsilon > 0$ so that $M\varepsilon+\delta_1 h < \eta$. From the above it follows that

$$\|x_\varepsilon(t)-x_0-\int_0^t f(s,\, x_\varepsilon(s))\,ds\| \leqslant M\varepsilon+\delta_1 h < \eta.$$

Obviously

$$x_\varepsilon(t)\in x_0+\bigcup_{0\leqslant\lambda\leqslant t} \lambda\operatorname{conv} f(J\times H) \subset H.$$

Now we prove that, for any $\varepsilon\in(0,\, \min(h,\, \varepsilon(\eta))]$, $x_\delta(t) \to x_\varepsilon(t)$ as $\delta \to \varepsilon$, weakly uniformly on J. Let $t\in[t_i,\, t_{i+1}]$, where $t_i = i\varepsilon$. Since $\delta \to \varepsilon$ implies $i\delta \to t_i$ and $x_\delta(i\delta) \to x_\varepsilon(t_i)$ weakly, by the weak uniform continuity of f we have

$$\begin{aligned}
|x^*(x_\delta(t)-x_\varepsilon(t))| &\leqslant |x^*(x_\delta(i\delta)-x_\varepsilon(t_i))|+\\
&\quad +|x^*[(t-i\delta)f(i\delta,\, x_\delta(i\delta))-(t-t_i)f(t_i,\, x_\varepsilon(t_i))]|\\
&\leqslant |x^*(x_\delta(i\delta)-x_\varepsilon(t_i))|+|x^*[(t-i\delta)f(i\delta,\, x_\delta(i\delta))-\\
&\quad -(t-t_i)f(i\delta,\, x_\delta(i\delta))+(t-t_i)f(i\delta,\, x_\delta(i\delta))-\\
&\quad -(t-t_i)f(t_i,\, x_\varepsilon(t_i))]|\\
&\leqslant |x^*(x_\delta(i\delta)-x_\varepsilon(t_i))|+M|i\delta-t_i|+\\
&\quad +|t-t_i|+|x^*(f(i\delta,\, x_\delta(i\delta))-f(t_i,\, x_\varepsilon(t_i)))| \to 0
\end{aligned}$$

as $\delta \to \varepsilon$ for each $t\in J$ and $x^*\in E^*$, $\|x^*\| = 1$.

Consequently $\varepsilon \mapsto x_\varepsilon(\cdot)$ is continuous from $(0,\, \beta)$ to $C_w(J,\, E)$, whence the set $V=\{x_\varepsilon(\cdot)\colon\ 0<\varepsilon<\beta\}$ is connected in $C_w(J,\, E)$.

Next we prove that the set S_η is connected in $C_w(J,\, E)$.

Let $0 \leqslant p \leqslant h$, $z\in S\eta$, $\varepsilon_0 < \varepsilon(\eta)$. We define a set T_z by the formula:

$$T_z = \{z_p(\cdot)\colon\ p\in J\},$$

where

$$z_p(t) = \begin{cases} z(t) & \text{for } 0 \leqslant t \leqslant p, \\ z(p) & \text{for } p \leqslant t \leqslant ([p/\varepsilon_0]+1) = t_l, \\ z_p(t_i)+(t-t_i)f(t_i, z_p(t_i)) & \text{for } t_i \leqslant t \leqslant t_{i+1}, \end{cases}$$

where $i = [p/\varepsilon_0]+1, \ldots, [h/\varepsilon_0] = n(\varepsilon_0)$, $t_i = i\varepsilon_0$, $t_{n(\varepsilon_0)+1} = h$.

Observe that $z_p(\cdot) \in S\eta$. Obviously, z_p satisfies (a). To prove (b) let us choose $\delta_2 > 0$ and $\varepsilon > 0$ such that

$$\sup_{t\in J} \left\|z(t) - x_0 - \int_0^t f(s, x(s))\,ds\right\| + M\varepsilon + h\delta_2 < \eta. \tag{4}$$

The function z_p satisfies condition (b) for $t \leqslant p$, since in this case $z_p = z$ and $z \in S\eta$. For $t > p$ let $t \in [t_i, t_{i+1}]$, $x^* \in E^*$, $\|x^*\| = 1$; then

$$\begin{aligned} &\left|x^*\Big(z_p(t) - x_0 - \int_0^t f(s, z_p(s))\,ds\Big)\right| \\ &\qquad \leqslant \left|x^*\Big(z(p) - x_0 - \int_0^p f(s, z(s))\,ds\Big)\right| + \left|x^* \int_p^{t_l} f(s, z(p))\,ds\right| + \\ &\qquad\quad + \left|x^*\Big(\int_{t_l}^{t_{l+1}} [f(t_l, z_p(t_l)) - f(s, z_p(s))]\,ds\Big)\right| + \ldots + \\ &\qquad\quad + \left|x^*\Big(\int_{t_i}^{t} [f(t_i, z_p(t_i)) - f(s, z_p(s))]\,ds\Big)\right|. \end{aligned}$$

Since $\|z_p(s) - z_p(t)\| \leqslant M|t-s|$ and f is weakly-weakly uniformly continuous, we see that for δ_2 satisfing (4) we can find a $\delta' > 0$ such that $\varepsilon_0 < \delta'$, $\varepsilon < \varepsilon_0$ implies

$$\left|x^*\big(f(t_i, z_p(t_i)) - f(s, z_p(s))\big)\right| < \delta_2 \tag{5}$$

for every i, $s \in [t_i, t_{i+1}]$, and $x^* \in E^*$, $\|x^*\| = 1$.

From (4), (5) it follows that

$$\left|x^*\Big(z_p(t) - x_0 - \int_0^t f(s, z_p(s))\,ds\Big)\right| \leqslant \sup_{t\in J}\left\|z(t) - x_0 - \int_0^t f(s, x(s))\,ds\right\| + \varepsilon M + h\delta_2 < \eta.$$

We now prove that z_p satisfies condition (c). Since $z \in S\eta$, we have

$$z_p(t) \in x_0 + \bigcup_{0\leqslant\lambda\leqslant t} \lambda \operatorname{conv} f(J \times H) \quad \text{for } 0 \leqslant t \leqslant t_l.$$

If $t_i \leqslant t \leqslant t_{i+1}$ $(i = l, l+1, \ldots, n(\varepsilon))$, then

$$\begin{aligned} z_p(t) &= z(p) + (t_{l+1} - t_l)f(t_l, z_p(t_l)) + \ldots + (t - t_i)f(t_i, z_p(t_i)) \\ &\in x_0 + \bigcup_{0\leqslant\lambda\leqslant p} \lambda \operatorname{conv} f(J \times H) + (t_{l+1} - t_l)f(t_l, z_p(t_l)) + \ldots + \\ &\qquad\qquad + (t - t_i)f(t_i, z_p(t_i)). \end{aligned}$$

Thus

$$\begin{aligned} z_p(t) &= x_0+\lambda[a_1x_1+\ldots+a_nx_n]+(t_{l+1}-t_l)f(t_l, z_p(t_l))+ \\ &\qquad\qquad +\ldots+(t-t_i)f(t_i, z_p(t_i)) \\ &= x_0+a\left[\left(\frac{\lambda a_1}{a}x_1+\ldots+\frac{\lambda a_n}{a}x_n\right)\right]+ \\ &\qquad +\frac{t_{l+1}-t_l}{a}f(t_l, z_p(t_l))+\ldots+\frac{t-t_i}{a}f(t_i, z_p(t_i)) \\ &\in x_0+(\lambda+t-t_l)\operatorname{conv} f(J\times H) = x_0+\lambda_1\operatorname{conv} f(J\times H), \end{aligned}$$

where $a_1+\ldots+a_n = 1$, $a = \lambda+(t_{l+1}-t_l)+\ldots+(t-t_i)$ and $\lambda_1 \leqslant t$. So $z_p\in S\eta$.

We now prove that $z_q(t)\to z_p(t)$ as $q\to p$ weakly uniformly on J.

Let $0\leqslant p\leqslant q\leqslant h$; then for $t\leqslant p$, $\|z_q(t)-z_p(t)\| = 0$. If $t>p$, we can choose q such that $[p/\varepsilon] = [q/\varepsilon]$. Let $t\in[t_i, t_{i+1}]$; then

$$\begin{aligned} |x^*(z_q(t)-z_p(t))| &= \big|x^*\big(z(q)+(t_{l+1}-t_l)f(t_l, z(q))+ \\ &\quad +\ldots+(t-t_i)f(t_i, z_q(t_i))-z(p)-(t_{l+1}-t_l)f(t_p, z(p))- \\ &\quad -\ldots-(t-t_i)f(t_i, z_p(t_i))\big)\big| \\ &\leqslant |x^*(z(q)-z(p))+(t_{l+1}-t_l)|\,|x^*(f(t_l, z(q))-f(t_l, z(p)))|+ \\ &\quad +\ldots+(t-t_i)\,|x^*(f(t_i, z_q(t_i))-f(t_i, z_p(t_i)))|. \end{aligned}$$

By induction and by the uniform weakly-weakly continuity of f we obtain

$$x^*(z_q(t)-z_p(t))\to 0 \quad \text{for every } t\in J \text{ if } q\to p.$$

This proves that the map $p\mapsto z_p$ is continuous from $[0, h]$ to $C_w(J, E)$; consequently the set T_z is connected in $C_w(J, E)$. Since $z_0 = x_\varepsilon\in V\cap T_z$, the set $V\cap T_z$ is connected and therefore the set $W = \bigcup_{z\in S\eta} T_z\cup V$ is connected in $C_w(J, E)$. Moreover, $S\eta\subset W$, since $z = z_h\in T_z$ for each $z\in S\eta$. On the other hand, $W\subset S\eta$ since $T_z\subset S\eta$ and $V\subset S\eta$. Finally, $S\eta = W$ is connected in $C_w(J, E)$.

The closure $\bar{S}\eta$ of $S\eta$ in $C_w(J, E)$ is composed of all functions $u\colon J\to E$ which satisfy:

$$\begin{gathered} u(0) = x_0, \quad \|u(s)-u(t)\|\leqslant M|t-s|, \\ \Big\|u(t)-x_0-\int_0^t f(s, u(s))\,ds\Big\|\leqslant\eta \quad \text{for } t, s\in J. \end{gathered} \tag{6}$$

We see that $\bar{S}\eta$ is a closed equicontinuous subset of $C_w(J, E)$ and $u(t)\in H$ for each $u\in\bar{S}\eta$ and $t\in J$. Since H is weakly compact, we infer by Ascoli's theorem that $\bar{S}\eta$ is compact in $C_w(J, E)$.

Put $V_n = \bar{S}_{1/n}$ for $n = 1, 2, \ldots$ Then V_n is a decreasing sequence of non-empty compact connected subsets of $C_w(J, E)$; therefore the intersection $S = \bigcap_{n=1}^{\infty} V_n$ is non-empty, compact and connected in $C_w(J, E)$. From (6) it follows that S is the set of all solutions of the Cauchy problem (1). This completes our proof.

Remark. If E is a reflexive Banach space, then B is weakly compact and we need not suppose (2). Thus our result is a generalization of the result of Szufla [12] and Szep [9].

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