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ASSIGNMENT PROBLEM WITH FUZZY ESTIMATES OF EFFECTIVENESS

1. Introduction. The classical assignment problem, which in the literature is usually identified with the problem of assigning n workers to n jobs, comprises a parameter a_{ij} defined as the effectiveness of the i-th worker at the j-th job. In other words, it is a facility estimate of any given worker to any job. Only in a few cases the estimates may be derived explicitly and precisely. It is possible only when they have concrete interpretations (such as assignment cost or task accomplishment time).

The above-mentioned estimates are quite often obtained on the basis of subjective and verbal opinions of people (experts). Obviously, these opinions are rather qualitative than quantitative.

This paper formulates the assignment problem and indicates a way of its solution when the effectiveness estimates are unprecisely determined and represented in the form of fuzzy sets in an estimate space [3].

Let us formulate classical problems of the optimal assignment. The solution methods of these problems will then be used in algorithms solving the assignment problem with non-sharp effectiveness estimates.

Assignment problem (Problem 1). For given estimates a_{ij}, where $i, j = 1, 2, \ldots, n$, find such an assignment (an n-element permutation $P = (p_1, p_2, \ldots, p_n)$) for which the total effectiveness is maximal:

$$\sum_{i=1}^{n} a_{ip_i} \to \max.$$

Bottleneck assignment problem (Problem 2). For given estimates a_{ij} $(i, j = 1, 2, \ldots, n)$ find such an assignment which maximizes the least effectiveness:

$$\min_{1 \leqslant i \leqslant n} a_{ip_i} \to \max.$$

There are known several methods for solving both problems. A simple algorithm solving the bottleneck assignment problem is given in [2].

2. Non-sharp effectiveness estimates. In a number of cases there is a lack of explicit estimate criteria as well as of precise methods for their determination. The estimates are many times performed by experts stating their opinions in a very implicit and vague way. The methodology proposed by Zadeh (see [3] and [4]), in which the values of the so-called language variables are treated as fuzzy sets in the meaning space, can be a tool for formal presentation of these estimates.

For convenience of the reader let us provide some basic properties of fuzzy sets.

Intuitively, a fuzzy set is a class of objects in which there is no sharp boundary between objects belonging and not belonging to the class.

Definitions. Let $X = \{x\}$ denote the space set. Then $A \subset X$ is said to be a *fuzzy set* in X if there exists a membership function μ_A: $X \to [0,1]$, where $\mu_A(x)$ is called the grade of membership of x to A.

The *support* of a fuzzy set A is a set $S(A)$ such that $x \in S(A)$ if and only if $\mu_A(x) > 0$.

A fuzzy set A is *contained* in a fuzzy set B, written $A \subset B$, if and only if $\mu_A \leqslant \mu_B$.

A' is said to be a *complement* of a fuzzy set A if and only if $\mu_{A'} = 1 - \mu_A$.

The *intersection* of fuzzy sets A and B, denoted by $A \cap B$, is the set for which the membership function is defined by

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)], \qquad x \in X.$$

The intersection operation is given the meaning of the connective AND. This is (as quoted in [1]) a "hard" meaning, since the operation does not include the difference between $\mu_A(x)$ and $\mu_B(x)$ as long as $\mu_A(x) > \mu_B(x)$ and conversely. Another meaning, i.e. "more soft", is imposed on the connective AND by the algebraic product operation where the difference is — to some degree — included.

AB is called the *algebraic product* of fuzzy sets A and B if

$$\mu_{AB}(x) = \mu_A(x) \cdot \mu_B(x), \qquad x \in X.$$

$A \cup B$ is said to be the *union* of fuzzy sets A and B if

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)], \qquad x \in X.$$

A^a $(a \geqslant 0)$ is called the *a-th power* of a fuzzy set A if

$$\mu_{A^a}(x) = [\mu_A(x)]^a, \qquad x \in X.$$

In general, the value of a linguistic variable is a composite term $X = \langle x_1, x_2, \ldots, x_p \rangle$ which is a connection of atomic terms $x_1, x_2, \ldots, x_p$ (see [4]).

These atomic terms may be divided into four categories:

1. primary terms which are labels of specified fuzzy sets, the meanings of these terms (e.g., young, small, etc.);
2. the negation NOT and the connectives AND and OR;
3. hedges such as: very, less than, etc.
4. marks such as parentheses.

The meaning of a term X is derived from meanings of the atomic terms by substitution of the negation NOT by the complementation operation as well as by substitution of the connectives AND and OR by the intersection and union operations. The power operation is usually applied for finding meanings of the hedges, e.g., very, much, slightly, more or less, etc.

For example, the following operations correspond mostly to actions of some hedges [4]:

very: $x = x^2$;
a little more than: $x = x^{1.25}$;
a little less than: $x = x^{0.75}$.

The language variable in the case under consideration is the effectiveness of the i-th worker at the j-th job. Values A_{ij} of the variable may be expressions such as: low, high, mean, a bit higher than mean, very high, etc. The meanings of these expressions can be represented in the form of fuzzy sets in a defined space U. In our case, the space U can be an ordered number set: $U = \{u_1, u_2, \ldots, u_m\}$ (scale of the estimates)

3. Formal problem model. Let A_{ij} be effectivenesses of particular workers at particular jobs defined by values of a linguistic variable "effectiveness". Assume that fuzzy sets in U are defined as meanings of these values. Those fuzzy sets will be denoted also by A_{ij}. The membership functions related to the sets A_{ij} are denoted by μ_{ij}:

$$\mu_{ij}\colon U \to [0,1].$$

The assignment problem under consideration is formulated as follows.

Each worker is to be assigned to a job in the way to have all corresponding effectivenesses possibly near the desired effectiveness G. The effectiveness G is also a value of the linguistic variable "effectiveness" defined by a fuzzy set in U with the membership function μ_G. For example, G may be defined as "high" or "not much low", etc. In the sequel, G will be called an *assignment evaluation criterion.*

For the so formulated problem it seems to be natural to evaluate the assignment, determined as the permutation $P = (p_1, p_2, \ldots, p_n)$, by the meaning of the value W_P of the linguistic variable "effectiveness", where W_P is equal to A_{1p_1} and A_{2p_2} and, $\ldots$, and A_{np_n} and G.

Identifying names of the variables with names of fuzzy sets, we define the meaning of the variable W_P as

$$W_P = A_{1p_1} \dot{\cap} A_{2p_2} \dot{\cap} \ldots \dot{\cap} A_{np_n} \dot{\cap} G, \tag{1}$$

where the symbol $\dot{\cap}$ denotes either the intersection $\cap$ or the algebraic product $\cdot$, according to the assumed interpretation of the connective AND.

In view of the definitions of the intersection and algebraic product operations the membership function is defined by

$$\mu_{W_P}(u) = \big[\bigwedge_i \mu_{ip_i}(u)\big] \wedge \mu_G(u), \qquad u \in U, \tag{2}$$

where $\wedge$ means minimization when (1) involves the intersection operation or stands for $\cdot$ (multiplication) when (1) contains the algebraic product.

The solution for which a support of the set W_P is empty ($S(W_P) = \emptyset$) is referred to as unfeasible.

The solution, say P^*, for which the maximal value of the membership function in the set W_P is the largest one, i.e.

$$\mathop{\forall}_{P \neq P^*} [\max_{u \in U} \mu_{W_{P^*}}(u) \geqslant \max_{u \in U} \mu_{W_P}(u)], \tag{3}$$

is called *optimal* in the sense of the criterion G. The search for the permutation P^* leads to the maximization of the function $\mu_{W_P}(u)$ over all permutations $P \in \Pi$ (Π — the n-element permutation set) and over all values of the estimates $u \in U$.

Expression (3) may be rewritten as

$$\begin{aligned} \max_{P \in \Pi} \max_{u \in U} \mu_{W_P}(u) &= \max_{P \in \Pi} \max_{u \in U} \big[\big(\bigwedge_{1 \leqslant i \leqslant n} \mu_{ip_i}(u)\big) \wedge \mu_G(u)\big] \\ &= \max_{u \in U} \max_{P \in \Pi} \big[\big(\bigwedge_{1 \leqslant i \leqslant n} \mu_{ip_i}(u)\big) \wedge \mu_G(u)\big] \\ &= \max_{u \in U} \max_{P \in \Pi} \big\{\big[\bigwedge_{1 \leqslant i \leqslant n} \mu_{ip_i}(u)\big] \wedge \mu_G(u)\big\}. \end{aligned} \tag{4}$$

Consider the case where equation (1) contains the intersection operation. Then in formulae (2) and (4) the symbol $\wedge$ denotes the minimization operation. The part of equation (4), embraced within square brackets, for fixed $u \in U$ takes then the form

$$\max_{P \in \Pi} \min_{1 \leqslant i \leqslant n} \mu_{ip_i}(u) = z(u). \tag{5}$$

This form is identical to the classical criterion (Problem 2) if we assume that $a_{ij} = \mu_{ij}(u)$ $(i, j = 1, 2, \ldots, n)$.

This fact together with the full form of equation (4) forces the following method for searching the optimal permutation P^* with respect to the criterion G:

(i) Solve m tasks of Problem-2-type assuming $a_{ij} = \mu_{ij}(u_k)$ for the k-th task ($k = 1, 2, \ldots, m$). The value of function (5) and the corresponding permutation are denoted by $z(u_k)$ and P_k^*, respectively.

(ii) Among the permutations P_k^* ($k = 1, 2, \ldots, m$) select the permutation P_r^* satisfying the condition

$$\min[z(u_r), \mu_G(u_r)] = \max_{1 \leqslant k \leqslant m} \min[z(u_k), \mu_G(u_k)].$$

The permutation P_r^* satisfies all conditions of the optimal assignment P^*.

In the case of the algebraic product the part of equation (4) within square brackets takes the form

$$\max_{P \in \Pi} \prod_{i=1}^{n} \mu_{ip_i}(u) = w(u). \tag{6}$$

Since the function $\log x$ is increasing, it is possible to put (6) in the equivalent form

$$\max_{P \in \Pi} \sum_{i=1}^{n} \log \mu_{ip_i}(u) = \log w(u). \tag{7}$$

The form of equation (7) is identical with the classical criterion for Problem 1 if $a_{ij} = \log \mu_{ij}(u)$ ($i, j = 1, 2, \ldots, m$). So in this case the search for the optimal permutation P^* in the sense of the criterion G may be accomplished in the following way:

(i) Solve m tasks of Problem-1-type assuming $a_{ij} = \log \mu_{ij}(u_k)$ for the k-th task ($k = 1, 2, \ldots, m$). The value of function (7) and the corresponding permutation are denoted by $L(u_k)$ and P_k^*, respectively.

(ii) Among the permutations P_k^* ($k = 1, 2, \ldots, m$) select the permutation P_r^* satisfying the condition

$$w(u) \cdot \mu_G(u) = \max_{1 \leqslant k \leqslant m} [w(u_k) \cdot \mu_G(u_k)], \quad \text{where } w(u) = \exp[L(u)].$$

The permutation P_r^* satisfies all conditions of the optimal assignment P^*.

Tables 1-6 present results of solving a numerical example for $n = 4$ by both methods.

Table 1 shows the membership function μ_{ij} for estimates A_{ij} in the space $U = \{0, 0.1, 0.2, \ldots, 0.9, 1.0\}$.

Table 2 comprises membership functions for three exemplary criteria.

Tables 3 and 5 include computational results for stage (i) of the first and second methods, respectively.

Tables 4 and 6 present computational results for stage (ii) corresponding to the above-mentioned criteria.

TABLE 1

u_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
A_{11}	0.1	0.3	0.5	0.6	0.7	0.8	0.9	1.0	0.7	0.5	0.3
A_{12}	0.4	0.6	0.8	1.0	0.9	0.8	0.6	0.5	0.3	0.3	0.2
A_{13}	0	0.1	0.1	0.2	0.3	0.4	0.6	0.7	0.8	1.0	0.8
A_{14}	0.2	0.2	0.2	0.3	0.5	0.6	0.7	0.8	1.0	0.9	0.7
A_{21}	0.5	0.6	0.7	0.8	1.0	0.8	0.6	0.5	0.4	0.2	0.2
A_{22}	1.0	0.9	0.8	0.7	0.7	0.7	0.6	0.5	0.4	0.3	0.3
A_{23}	0	0	0.1	0.1	0.2	0.4	0.6	0.8	0.9	1.0	0.9
A_{24}	0	0	0.1	0.2	0.4	0.6	0.8	1.0	0.9	0.8	0.6
A_{31}	0.8	0.9	1.0	0.9	0.8	0.7	0.6	0.4	0.2	0.1	0
A_{32}	0.1	0.1	0.1	0.3	0.5	0.6	0.7	0.9	1.0	0.9	0.8
A_{33}	0.2	0.3	0.5	0.7	0.9	1.0	0.9	0.8	0.7	0.6	0.5
A_{34}	0.1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.9	1.0	0.8
A_{41}	0.7	0.8	0.9	1.0	0.9	0.7	0.5	0.3	0.3	0.2	0.2
A_{42}	0	0	0.1	0.2	0.2	0.3	0.4	0.5	0.6	0.8	1.0
A_{43}	0.3	0.3	0.3	0.5	0.6	0.7	0.8	1.0	0.9	0.8	0.7
A_{44}	0.5	0.6	0.7	0.8	0.9	1.0	0.9	0.8	0.7	0.6	0.5

TABLE 2

u_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
G_1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
G_2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
G_3	0	0.2	0.4	0.6	0.8	1.0	0.8	0.6	0.4	0.2	0

TABLE 3

u_k	Solutions			$z(u_k)$
0	2, 1, 3, 4	4, 2, 1, 3	4, 2, 3, 1	0.2
0.1	1, 2, 3, 4	2, 1, 3, 4		0.3
0.2	1, 2, 3, 4	2, 1, 3, 4		0.5
0.3	1, 2, 3, 4	optimal for G_2		0.7
0.4	2, 1, 3, 4			0.9
0.5	2, 1, 3, 4	optimal for G_3		0.8
0.6	1, 4, 2, 3			0.7
0.7	1, 4, 2, 3			0.9
0.8	1, 4, 2, 3	optimal for G_1		0.7
0.9	1, 3, 2, 4	1, 4, 3, 2	1, 4, 2, 3	0.5
1.0	1, 3, 2, 4	1, 4, 3, 2	1, 4, 2, 3	0.3

TABLE 4

u_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\min\{\mu_{G_1}(u_k), z(u_k)\}$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.7	0.5	0.3
$\min\{\mu_{G_2}(u_k), z(u_k)\}$	0.2	0.3	0.5	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
$\min\{\mu_{G_3}(u_k), z(u_k)\}$	0	0.2	0.4	0.6	0.8	0.8	0.7	0.6	0.4	0.2	0

TABLE 5

u_k	Solutions		$-\log L(u_k)$
0	4, 2, 1, 3		1.319
0.1	3, 2, 1, 4		1.314
0.2	2, 1, 3, 4		0.708
0.3	2, 1, 3, 4		0.349
0.4	2, 1, 3, 4	optimal for G_2	0.138
0.5	2, 1, 3, 4	optimal for G_3	0.194
0.6	1, 4, 2, 3		0.395
0.7	1, 4, 2, 3	optimal for G_1	0.046
0.8	1, 4, 2, 3		0.246
0.9	1, 3, 4, 2		0.398
1.0	1, 3, 4, 2		0.666

TABLE 6

u_k	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$-L(u_k)-\log\mu_{G_1}(u_k)$	–	2.314	1.407	0.872	0.516	0.495	0.617	0.201	0.344	0.444	0.666
$-L(u_k)-\log\mu_{G_2}(u_k)$	1.319	1.360	0.805	0.504	0.360	0.495	0.793	0.569	0.946	1.398	–
$-L(u_k)-\log\mu_{G_3}(u_k)$	–	2.013	1.106	0.571	0.235	0.194	0.492	0.268	0.646	1.397	–

In the case of the second method, for convenience of calculations, a decimal logarithm is used and, also, instead of the maximization criterion — the equivalent minimization criterion for functions taken with a negative sign. Tables 3 and 4 indicate the corresponding optimal solutions.

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ZAGADNIENIE PRZYPORZĄDKOWANIA PRZY NIEOSTRYCH OCENACH PRZYDATNOŚCI

STRESZCZENIE

W pracy rozważono problem przyporządkowania n robotników n stanowiskom, w którym oceny przydatności A_{ij} $(i, j = 1, 2, \ldots, n)$ są nieprecyzyjne i w modelu zagadnienia wyrażone w postaci zbiorów rozmytych (*fuzzy sets* [3]) w skończonej przestrzeni ocen $U = \{u_1, u_2, \ldots, u_m\}$ poprzez funkcję uczestnictwa $\mu_{ij}: U \to [0, 1]$.

Przy danym kryterium G preferencji ocen (G — zbiór rozmyty w przestrzeni U) określa się rozwiązanie optymalne jako permutację n-elementową $P = (p_1, p_2, \ldots, p_n)$, dla której maksymalna wartość funkcji uczestnictwa μ_{W_P} zbioru rozmytego (1) jest największa. We wzorze (1) symbol $\dot{\cap}$ oznacza jedną z dwu operacji: iloczyn mnogościowy $\cap$ lub iloczyn algebraiczny $\cdot$ (por. [3]). Dla obu przypadków opisano algorytmy poszukiwania optymalnego rozwiązania.