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Linear problems for systems of difference equations

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1. In the theory of differential equations it is well known ([3], [4], [7]-[10]) that the existence of solutions of general linear problems for differential equations with single-valued right-hand sides is closely related to the uniqueness of solutions of homogeneous problems for differential equations with multi-valued right-hand sides (contingent equations).

It turns out that analoguous theorems may be proved for difference equations. The aim of this paper is to present discrete analogues of the Lasota theorems given in [4] (Theorem 2.1 and Corollary 2.1) and concerning the existence and uniqueness of solutions of general linear problems for non-linear vectorial differential equations.

It is to notice that recently Szafraniec [11] obtained similar results for difference equations, but with difference operators of special form. Owing to the general form of difference operator we deal with in this paper, the theorems we prove in the sequel generalize his results.

Similarly, much like as the above mentioned theorems of Lasota have been successfully applied to many particular boundary value problems for differential equations ([1], [6], [12]), the theorems we give in this note can be used to various particular boundary value problems for difference equations (cf. [2], see also Section 5).

In Section 2 we fix notations and introduce some notions. Next, in Section 3 we quote a generalization of the first theorem of Fredholm due to Lasota [4] and we state a lemma which will be needed in the sequel. The main results are contained in Section 4. At last in Section 5 we show how, by a simple application of these results, one can easily get theorems of Lasota obtained in [5] on the other way.

2. Let B be a Banach space and $c(B)$ — the set of all non-empty convex subsets of B.

For $a \in B$, $A \subset B$, as usual, $\|a\|$, $\delta(a, A)$, and $|A|$ will denote the norm of a vector a, its distance to set A and $\sup\{\|a\|: a \in A\}$, respectively.

A map $H: B \to c(B)$ will be called *homogeneous* if, for every $a \in B$ and any real λ, $H(\lambda a) = \lambda H(a)$, *compact* if, for any bounded subset D of B, the closure of the set $\bigcup_{a \in D} H(a)$ is compact in B, *upper semi-continuous* if its graph $\{(a, b): a \in B, b \in H(a)\}$ is closed in $B \times B$ and *completely continuous* if it is compact and upper semi-continuous.

It is easy to see that a homogeneous map H is compact if and only if the closure of the set $\bigcup_{\|a\|=1} H(a)$ is compact. A map $h: B \to B$ will be called *completely continuous* if the map $B \ni a \to \{h(a)\} \in c(B)$ is completely continuous, $\{h(a)\}$ denoting the set consisting only of $h(a)$.

In the sequel R^m will denote the m-dimensional Euclidean space with the Euclidean norm $|p|$, and $cf(R^m)$ — the set of all non-empty, closed and convex subsets of R^m.

The set $\{0, \ldots, n\}$ will be denoted by N. For $k \in N$ we define the difference operators $\Delta^{(k)}: R^{n+1} \to R^{n+1}$, $\nabla^{(k)}: R^{n+1} \to R^{n+1}$ as follows:

$$\begin{aligned} \Delta^{(k)} v &= (\Delta^{(k)} v_0, \ldots, \Delta^{(k)} v_n) \\ \nabla^{(k)} v &= (\nabla^{(k)} v_0, \ldots, \nabla^{(k)} v_n) \end{aligned} \qquad (v = (v_0, \ldots, v_n) \in R^{n+1})$$

and, for $i \in N$,

$$\Delta^{(0)} v_i = \nabla^{(0)} v_i = v_i,$$

$$\Delta v_i = \Delta^{(1)} v_i = \begin{cases} v_{i+1} - v_i, & i = 0, \ldots, n-1, \\ 0, & i = n, \end{cases}$$

$$\nabla v_i = \nabla^{(1)} v_i = \begin{cases} 0, & i = 0, \\ v_i - v_{i-1}, & i = 1, \ldots, n, \end{cases}$$

and, for an integer k fulfilling the inequality $2 \leqslant k \leqslant n$

$$\Delta^{(k)} v_i = \begin{cases} \Delta(\Delta^{(k-1)} v_i), & i = 0, \ldots, n-k, \\ 0, & i = n-k+1, \ldots, n, \end{cases}$$

$$\nabla^{(k)} v_i = \begin{cases} 0, & i = 0, \ldots, k-1, \\ \nabla(\nabla^{(k-1)} v_i), & i = k, \ldots, n. \end{cases}$$

Composing several times in arbitrary succession the difference operators defined above we get so-called mixed difference operators. For instance, if ν is an odd integer and $k_1, \ldots, k_\nu$ are integers such that

$$k_2 + k_4 + \ldots + k_{\nu-1} \leqslant \nu - (k_1 + k_3 + \ldots + k_\nu),$$

then, for $i \in N$, we have

$$\Delta^{(k_\nu)} \nabla^{(k_{\nu-1})} \ldots \nabla^{(k_2)} \Delta^{(k_1)} v_i$$

$$= \begin{cases} 0, \\ \quad i = 0, \ldots, k_2 + k_4 + \ldots + k_{\nu-1} - 1,\ n - (k_1 + k_3 + \ldots + k_\nu) + 1, \ldots, n, \\ \Delta(\Delta^{(k_\nu - 1)} \nabla^{(k_{\nu-1})} \ldots \nabla^{(k_2)} \Delta^{(k_1)} v_i), \\ \quad i = k_2 + k_4 + \ldots + k_{\nu-1}, \ldots, n - (k_1 + k_3 + \ldots + k_\nu). \end{cases}$$

Finally, for any fixed multi-index $s = (s_1, \ldots, s_m)$ $(s_i \in \{-1, 1\})$ we define the difference operator Δ_s: $(R^m)^{n+1} \to (R^m)^{n+1}$ by the formula

$$\Delta_s(u_0, \ldots, u_n) = (\Delta_s u_0, \ldots, \Delta_s u_n) \qquad ((u_0, \ldots, u_n) \in (R^m)^{n+1}),$$

where for $u_i = (u_i^1, \ldots, u_i^m) \in R^m$, $i \in N$, we set

$$\Delta_s(u_i^1, \ldots, u_i^m) = (\Delta_{s_1} u_i^1, \ldots, \Delta_{s_m} u_i^m) \qquad (\text{with } \Delta_{-1} = \nabla \text{ and } \Delta_1 = \Delta).$$

Thus Δu_i^k and ∇u_i^k $(i \in N,\ k \in \{1, \ldots, m\})$ which appear in the right-hand sides of the last formula are simply i-th coordinates of the vectors $\Delta(u_0^k, \ldots, u_n^k)$ and $\nabla(u_0^k, \ldots, u_n^k)$ of R^{n+1}.

In the sequel, a map f: $N \times R^m \to R^m$ will be called *continuous* if it is continuous as a map of the topological space $N \times R^m$ (in N-discrete topologie) into the topological space R^m.

A map F: $N \times R^m \to cf(R^m)$ will be called *upper semi-continuous* if, for every fixed $i \in N$, the mapping $F(i, \cdot)$: $R^m \to cf(R^m)$ is upper semi-continuous.

As usual, mappings f^k: $N \times R^m \to R$, F^k: $N \times R^m \to cf(R)$ $(k = 1, \ldots, m)$ such that $f = (f^1, \ldots, f^m)$ and $F^k(i, q) = p_k(F(i, q))$ $(i \in N,\ q \in R^m,\ p_k$ denotes k-th projection in $R^m)$ will be called the *components* of f and F, respectively.

For the maps f and F defined above we admit the following definitions:

A vector

$$u = (u_0, \ldots, u_n) \in (R^m)^{n+1} \qquad (u_i = (u_i^1, \ldots, u_i^m) \in R^m \text{ for } i \in N)$$

satisfying the condition

$$\Delta_s u_i = f(i, u_i) \qquad (i \in N)$$

(called the *difference equation*) will be called *solution* of this equation.

Similarly, the condition

$$\Delta_s u_i \in F(i, u_i) \qquad (i \in N),$$

where $u = (u_0, \ldots, u_n) \in (R^m)^{n+1}$ $(u_i = (u_i^1, \ldots, u_i^m) \in R^m$ for $i \in N)$ will be called *difference equation with a multi-valued right-hand side* or shortly *difference contingent equation,* and a vector u satisfying this condition — solution of this equation.

3. The following theorem is due to Lasota [4].

THEOREM 3.1. *Let B be a Banach space, H: $B \to c(B)$ a homogeneous and completely continuous map and let h: $B \to B$ be a completely continuous map satisfying the condition*

$$\lim_{\|a\| \to \infty} \frac{1}{\|a\|} \delta(h(a), H(a)) = 0. \tag{3.1}$$

Under these assumptions, if $a = 0$ is the unique vector of B satisfying the condition

$$a \in H(a), \tag{3.2}$$

then there exists at least one solution of the equation

$$a = h(a). \tag{3.3}$$

Now, we state a simple lemma which will be needed in the proof of theorems contained in the next section.

LEMMA. *For any integer k, a homogeneous and upper semi-continuous map $H\colon R^k \to c(R^k)$ is completely continuous.*

Proof. It suffices to show that the set $\bigcup_{|a|=1} H(a)$ is bounded. Suppose, on the contrary, that there is a sequence $\{p_\nu\} \subset \bigcup_{|a|=1} H(a)$ satisfying the condition

$$|p_\nu| \to \infty \quad (\nu \to \infty).$$

Then there exists a sequence $\{u_\nu\}$ such that

$$p_\nu \in H(u_\nu), \quad |u_\nu| = 1.$$

Setting $q_\nu = p_\nu/|p_\nu|$ $(\nu = 1, 2, \ldots)$, by the homogeneouity of H we get

$$q_\nu \in H\left(\frac{u_\nu}{|p_\nu|}\right) \quad (\nu = 1, 2, \ldots).$$

Passing to suitable subsequences, if necessary, we may assume that

$$q_\nu \to q_0 \quad (\nu \to \infty).$$

Hence we obtain $|q_0| = 1$ what is impossible because the condition

$$\frac{1}{|p_\nu|} u_\nu \to 0 \quad (\nu \to \infty)$$

and the upper semi-continuity of H imply that $q_0 \in H(0)$ and, consequently, by the homogeneouity of H, we should have $|q_0| = 0$. This contradiction completes the proof.

4. For maps $F\colon N \times R^m \to cf(R^m)$, $f\colon N \times R^m \to R^m$ and $L\colon (R^m)^{n+1} \to R^m$ we make following assumptions:

(i) The components of F satisfy the condition $F^k(0, p) = \{0\}$ if $s_k = -1$ and $F^k(n, p) = \{0\}$ if $s_k = 1$ $(k = 1, \ldots, m)$, where $p \in R^m$ and s_k denotes the k-th component of multi-index s which appears in the definition of Δ_s.

F is upper semi-continuous and homogeneous with respect to p $(F(i, \lambda p) = \lambda F(i, p)$ for $\lambda \in R)$.

(ii) The components of f satisfy the condition $f^k(0, p) = 0$ if $s_k = -1$ and $f^k(n, p) = 0$ if $s_k = 1$ $(k = 1, \ldots, m)$, where p, s_k are defined as above.

f is continuous and satisfies the condition

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{n} \sup_{|p|\leqslant t} \delta\big(f(i, p), F(i, p)\big) = 0. \tag{4.1}$$

(iii) L is continuous and homogeneous, i.e. $L\lambda u = \lambda Lu$ $(\lambda \in R, u \in (R^m)^{n+1})$,

(iv) f satisfies condition (ii) with (4.1) replaced by

$$f(i, p) - f(i, q) \in F(i, p-q) \quad (i \in N, p, q \in R^m), \tag{4.2}$$

(v) L is linear (additive and homogeneous) and continuous.

Now we shall consider two linear problems.

The first of them consists in the search for a solution $u = (u_0, \ldots, u_n) \in (R^m)^{n+1}$ of the difference equation

$$\Delta_s u_i = f(i, u_i) \quad (i \in N), \tag{4.3}$$

satisfying the condition

$$Lu = r \quad (r \in R^m). \tag{4.4}$$

The second one consists in the search for a solution of the difference contingent equation

$$\Delta_s u_i \in F(i, u_i) \quad (i \in N), \tag{4.5}$$

satisfying the homogeneous condition

$$Lu = 0. \tag{4.6}$$

The interdependence between these two problems is given by theorems we state below.

THEOREM 4.1. *If the maps F, f, L satisfy conditions* (i), (ii), (iii) *and problem* (4.5), (4.6) *has the unique solution $u = 0$, then for any $r \in R^m$ problem* (4.3), (4.4) *has at least one solution.*

THEOREM 4.2. *If the maps F, f, L satisfy conditions* (i), (iv), (v) *and problem* (4.5), (4.6) *has the unique solution $u = 0$, then for any $r \in R^m$ problem* (4.3), (4.4) *has exactly one solution.*

Proof of Theorem 4.1. We set $B = (R^m)^{n+1}$. There is a simple isomorphic map of B onto R^k, where $k = m(n+1)$, so B is a Banach space.

We put $\sum_{j=\nu}^{\mu} \lambda_j = 0$ if, $\mu < \nu$ $(\lambda_j \in R)$.

We define a mapping $h\colon B \to B$ setting $h(u) = \bar{u}$, where the vector $\bar{u} = (\bar{u}_0, \ldots, \bar{u}_n)$ of B is defined by the formula

$$\bar{u}_i^k = \begin{cases} \sum_{j=0}^{i-1} f^k(j, u_j) + u_0^k + (Lu)^k - r^k & \text{if } s_k = 1, \\ \sum_{j=1}^{i} f^k(j, u_j) + u_0^k + (Lu)^k - r^k & \text{if } s_k = -1 \end{cases}$$

$$(i \in N,\ k = 1, \ldots, m)$$

($\bar{u}_i^k, u_i^k, f^k(j, u_j), (Lu)^k, r^k$ denote the k-th components of vectors $\bar{u}_i, u_i, f(j, u_j), Lu, r$ of R^m, respectively).

From the definition of h and from the condition put on components $f^k(0, p), f^k(n, p)$ in (ii) it easily follows that the fixed point of h is a solution of problem (4.3), (4.4). Thus to complete the proof it suffices to apply Theorem 3.1. To this end we define a map $H\colon B \to c(B)$ as follows:

Let $H(u)$ be the set of all vectors $\bar{\bar{u}}$ of B for which there is a vector $w = (w_0, \ldots, w_n) \in B$ such that $w_j \in F(j, u_j)$ $(j \in N)$ and

$$\bar{\bar{u}}_i^k = \begin{cases} \sum_{j=0}^{i-1} w_j^k + u_0^k + (Lu)^k & \text{if } s_k = 1 \\ \sum_{j=1}^{i} w_j^k + u_0^k + (Lu)^k & \text{if } s_k = -1 \end{cases} \qquad (i \in N, k = 1, \ldots, m)$$

(notations — as in the definition of h).

From the assumption that the sets $F(j, u_j)$ $(j \in N)$ are non-empty and convex it follows that $H(u)$ is a non-empty and convex set.

Thus, for to apply Theorem 3.1 it remains to verify that

1° H is homogeneous and completely continuous,

2° h is completely continuous,

3° h and H satisfy condition (3.1),

4° $u = 0$ is the unique vector of B satisfying the condition $u \in H(u)$.

For 1° it suffices to show that H is completly continuous, since the homogeneouity of H is a simple consequence of the definition of H and of condition (i).

First of all we show that H is upper semi-continuous. To this end suppose that sequences $\{\overset{\nu}{\bar{\bar{u}}}\}, \{\overset{\nu}{u}\} \subset B$ satisfy the condition

$$(4.7) \qquad \overset{\nu}{\bar{\bar{u}}} \to \overset{0}{\bar{\bar{u}}}, \quad \overset{\nu}{u} \to \overset{0}{u} \quad (\nu \to \infty), \quad \overset{\nu}{\bar{\bar{u}}} \in H(\overset{\nu}{u}) \ (\nu = 1, 2, \ldots),$$

where $\overset{0}{\bar{\bar{u}}}$ and $\overset{0}{u}$ are fixed vectors of B.

Then there exists a sequence $\{\overset{\nu}{w}\} \subset B$ such that

$$(4.8) \qquad \overset{\nu}{\bar{\bar{u}}}{}_i^k = \begin{cases} \sum_{j=0}^{i-1} \overset{\nu}{w}{}_j^k + \overset{\nu}{u}{}_0^k + (L\overset{\nu}{u})^k & \text{if } s_k = 1, \\ \sum_{j=1}^{i} \overset{\nu}{w}{}_j^k + \overset{\nu}{u}{}_0^k + (L\overset{\nu}{u})^k & \text{if } s_k = -1, \end{cases} \qquad \overset{\nu}{w}_j \in F(j, \overset{\nu}{u}_j)$$

$$(i \in N, j \in N, k = 1, \ldots, m).$$

Hence, and from the condition put on components $F^k(0, p)$, $F^k(n, p)$ in (i) we have

$$(4.9) \qquad \Delta_s \overset{\nu}{\bar{\bar{u}}}_i = \overset{\nu}{w}_i, \quad \overset{\nu}{w}_i \in F(i, \overset{\nu}{u}_i) \quad (i \in N, \nu = 2, 1, \ldots).$$

Conditions (4.7) and (4.9), in view of the continuity of the difference operator Δ_s, imply

$$(4.10) \qquad \Delta_s \overset{\nu}{\bar{\bar{u}}}_i \to \Delta_s \overset{0}{\bar{\bar{u}}}_i, \quad \overset{\nu}{u}_i \to \overset{0}{u}_i \quad (\nu \to \infty), \quad \Delta_s \overset{\nu}{\bar{\bar{u}}}_i \in F(i, \overset{\nu}{u}_i)$$

$$(i \in N, \nu = 1, 2, \ldots).$$

Hence, setting $\overset{0}{w}_i = \Delta_s \overset{0}{\bar{\bar{u}}}_i$, we get by (i)

$$\overset{0}{w}_i \in F(i, \overset{0}{u}_i).$$

Thus formula (4.8) has at the limit (as $\nu \to \infty$) the following form

$$\overset{0}{\bar{\bar{u}}}{}_i^k = \begin{cases} \sum_{j=0}^{i-1} \overset{0}{w}{}_j^k + \overset{0}{u}{}_0^k + (L\overset{0}{u})^k & \text{if } s_k = 1, \\ \sum_{j=1}^{i} \overset{0}{w}{}_j^k + \overset{0}{u}{}_0^k + (L\overset{0}{u})^k & \text{if } s_k = -1, \end{cases} \qquad \overset{0}{w}_j \in F(j, \overset{0}{u}_j)$$

$$(i \in N, k = 1, \ldots, m)$$

what means that $\overset{0}{\bar{\bar{u}}} \in H(\overset{0}{u})$ and, in consequence, shows that H is upper semi-continuous. Now, a simple application of Lemma proves that H is completely continuous.

For 2° it suffices to observe that condition (ii) implies the continuity of h and the continuity is equivalent to the complete continuity for mappings in finite dimensional spaces.

Condition (4.1), assumed in (ii), simply implies 3°.

For to prove 4°, suppose that $u \in H(u)$. Then from the definition of H and from (i) we get

$$\Delta_s u_i = w_i \quad (i \in N), \quad Lu = 0,$$

for $w = (w_0, \ldots, w_n) \in B$ such that

$$w_i \in F(i, u_i) \quad (i \in N).$$

Hence the vector w is a solution of problem (4.5), (4.6) and by assumption of our theorem we have $u = 0$, what proves 4° and completes the proof of Theorem 1.

Proof of Theorem 4.2. It is easy to verify that conditions (i), (iv) and (v) imply conditions (i), (ii) and (iii). Indeed, setting $q = 0$ in (4.2) we get

$$f(i, p) - f(i, 0) \in F(i, p) \qquad (i \in N),$$

what immediately yields (4.1).

Thus the existence of a solution of problem (4.3) and (4.4) is simply a consequence of Theorem (4.1).

For to prove its uniqueness, suppose that vectors u, v of $(R^m)^{n+1}$ are solutions of problem (4.3) and (4.4). Then, for the vector $w = u - v$, we have

$$\Delta_s w_i \in F(i, w_i) \qquad (i \in N), Lw = 0,$$

what, by the assumption of our theorem, gives $w = 0$ and completes the proof of the theorem.

5. In order to illustrate the above theorems we give a simple application.

Lasota [5] shows that if the function g: $N \times R^2 \to R$ satisfies the following inequality:

$$|g(i, p_1, p_2)| \leqslant A|p_1| + B|p_2| + C, \tag{5.1}$$

where the non-negative constans A, B, C fulfil the inequality

$$A\frac{1}{4\sin^2\frac{\pi}{2n}} + B\cdot\frac{1}{2}\left[\frac{n+1}{2}\right] < 1 \tag{5.2}$$

($[x]$ denotes the whole part of x), then there exists at least on solution of problem

$$\nabla\Delta v_i = g(i, v_i, \Delta v_i) \qquad (i = 1, \ldots, n-1), \tag{5.3}$$

$$v_0 = \alpha, \qquad v_n = \beta, \tag{5.4}$$

and provided condition (5.1) is replaced by a Lipschitz condition of the form

$$|g(i, p_1, p_2) - g(i, \bar{p}_1, \bar{p}_2)| \leqslant A|p_1 - \bar{p}_1| + B|p_2 - \bar{p}_2|, \tag{5.5}$$

the solution is unique.

Observe that setting $v_i = u_i^1$, $\Delta v_i = u_i^2$, $s = (1, -1)$ (now we have $\Delta_s u_i = (\Delta u_i^1, \nabla u_i^2)$ $(i \in N)$) and defining mappings f: $N \times R^2 \to R^2$,

$L: (R^2)^{n+1} \to R^2$ by the formulae

$$f(i, w^1, w^2) = \big(w^2, g(i, w^1, w^2)\big), \quad i = 1, \dots, n-1 \; (w = (w^1, w^2) \in R^2),$$

$$f(0, w^1, w^2) = (w^2, 0), \quad f(n, w^1, w^2) = (0, 0),$$

$$Lu = (u_0^1, u_n^1), \quad \text{where } u = (u_0, \dots, u_n) \in (R^2)^{n+1}$$

we can write problem (5.3) and (5.4) in the following form:

$$\Delta_s u_i = f(i, u_i) \quad (i \in N) \quad Lu = (a, \beta).$$

Now, in order to obtain any theorem concerning the existence or uniqueness of solutions of this problem it suffices to apply Theorem 4.1 or Theorem 4.2, fixing in advance a condition which assures that $u = 0$ is the unique solution of the problem

$$\Delta_s u_i \in F(i, u_i) \quad (i \in N) \quad Lu = 0,$$

where the map $F: N \times R^2 \to cf(R^2)$ is defined by the formulae

$$F(i, w) = \{(p^1, p^2) \in R^2 \colon p^1 = w^2, |p^2| \leqslant A|w^1| + B|w^2|\}$$
$$(i = 1, \dots, n-1),$$
$$F(0, w) = \{(w^2, 0)\}, \quad F(u, w) = \{(0, 0)\}.$$

In the case of the Lasota result mentioned at the begining of this section, such a condition is given by inequality (5.2) (see [5]).

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