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Mixed covariant derivative and conjugate connections

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Abstract. This note is composed with two parts. In the first one we introduce the notion of a mixed covariant derivative $\nabla_v^{(\Gamma_1,\Gamma_2)}A$ of a geometric object A, in the direction of v, with respect to a pair (Γ_1, Γ_2) of connections in $P(M, G)$. This definition generalizes the classical definitions in the following sense: (1) if $\Gamma = \Gamma_1 = \Gamma_2$, our definition coincides with the definition of $\nabla_v^{\Gamma}A$ due to R. Crittenden; (2) if Γ_1, Γ_2 are linear connections on M and A is a tensor field on M, then our definition coincides with the definition of a mixed covariant derivative introduced by A. P. Norden.

In this part we prove also some properties of the operator $\nabla_v^{(\Gamma_1,\Gamma_2)}$.

In the second part we prove a necessary and sufficient condition under which two connections Γ_1, Γ_2 are φ-conjugate, where φ is an endomorphism of the structural group G. This condition has the form $\nabla_v^{(\Gamma_1,\Gamma_2)}A = 0$, where A is some geometric object which type depends of φ.

Last of all we consider the special case of connections of the order r.

Introduction. This work is composed of two parts. The first concerns the problem of a generalization of the notion of the mixed covariant derivative and the second concerns applications of this operator to the study of couples of conjugate connections.

The notion of the mixed covariant derivative was introduced by A. P. Norden [9] for tensors and linear connections. Let Γ_1 and Γ_2 be two linear connections in M and let g be a metric tensor on M. The condition

$$\nabla^{\Gamma_1\Gamma_2}g = 0$$

has very interesting geometric interpretation, namely, this condition means that for each curve $\gamma(t)$ in M and for each vector fields v and w defined along γ and parallel with respect to Γ_1 and Γ_2, respectively, the equality $g(v\otimes w) = 0$ at one point of γ implies this equality at each point of γ, i.e., if v and w are orthogonal at one point of γ, they must be orthogonal at each point of γ.

Later this operator was generalized to the case where g was a tensor density [4]. In this paper we define the mixed covariant derivative $\nabla^{\Gamma_1\Gamma_2}A$ in the case of arbitrary connections Γ_1, Γ_2 (defined in some principal fibre bundle) and an arbitrary geometrical object A which satisfies some hypothesis concerning the transformation formula of A. This generalized covariant derivative $\nabla^{\Gamma_1\Gamma_2}A$ has the following properties:

(a) If Γ_1, Γ_2 are two linear connections and A is a tensor or a tensor density, then this definition coincides with the definition of A. P. Norden [9] or with the definition given in [4] (see Section 3).

(b) $\Gamma_1 = \Gamma_2$ our definition coincides with the definition of R. Crittenden [2] (see Proposition 2.13).

(c) $\nabla^{\Gamma_1\Gamma_2}$ satisfies a special form of Leibniz's formula (see Propositions 4.7, 4.11, 4.14).

The second part of this work concerns an application of this operator to the investigation of couples of conjugate connections. The fundamental theorem gives a sufficient and necessary condition under which two connections Γ_1 and Γ_2 are φ-conjugate. This condition concerns the existence of some geometric object A (whose type depends only on φ) such that $\nabla^{\Gamma_1\Gamma_2}A = 0$.

Last of all we shall consider the case of connections of order r, i.e. connections in the principal fibre bundle L^rM. For $r = 2$ we obtain the most interesting, which will be considered in Section 10.

In this work differentiability always means differentiability of class C^∞. We shall use the following notations.

If M is a differentiable manifold and x is a point of M, then T_xM denotes a vector space tangent to M at the point x, and $TM = \bigcup_x T_xM$ denotes a bundle tangent to M. If $f\colon M \to N$ is a differentiable mapping, then for $x \in M$ we denote by

$$d_xf\colon T_xM \to T_{f(x)}M$$

a linear homomorphism induced by f (at x), and by

$$df\colon TM \to TN, \quad (df)|T_xM = d_xf,$$

a homomorphism of tangent bundles induced by f. It is clear that

$$d_x(g\circ f) = d_{f(x)}g\circ d_xf, \quad d(g\circ f) = dg\circ df.$$

If N is a submanifold of M, then we identify T_xN with a subspace of T_xM (by the canonical monomorphism) for x in N. If M and M' are two manifolds, then we also identify $T_{(x,x')}(M\times M')$ with $T_xM \oplus T_{x'}M'$ (by the canonical isomorphism). We shall never differentiate $T_{(x,x')}(M\times M')$ from $T_xM \oplus T_{x'}M'$.

In the proofs of our propositions we shall very often use the following formula of Leibniz (see [6], Proposition 1.4, p. 11).

PROPOSITION 0. *Let $f\colon M\times N \to K$ be mapping and $(p, q) \in M\times N$. Then for each vector*

$$w = w_M \oplus w_N \in T_{(p,q)}(M \times N) = T_p M \oplus T_q N$$

we have

$$(d_{(p,q)} f)(w) = (d_p{}' f)(w_M) + (d_q'' f)(w_N),$$

where

$$'f\colon M \to K, \qquad 'f(x) = f(x, q), \qquad x \in M,$$

$$''f\colon N \to K, \qquad ''f(y) = f(p, y), \qquad y \in N.$$

(The notations $'f$ and $''f$ will be used in the whole work.)

We shall also use the following notations:

(1) If $f\colon M \to K$ and $g\colon N \to L$ we write

$$f \times g\colon M \times N \to K \times L, \qquad (f \times g)(x, y) = \big(f(x), g(y)\big).$$

(2) If $f\colon M \to K$ and $h\colon M \to L$ we write

$$(f, h)\colon M \to K \times L, \qquad (f, h)(x) = \big(f(x), h(x)\big).$$

1. Geometric objects and their covariant derivative. Suppose we are given a principal fibre bundle $P(M, G)$, a manifold F and an action on the left of the structural group G on F. We denote by L_ξ the left translation of the group G on F. Now we can construct a fibre bundle

$$E = E(P, M, G, F, L_\xi)$$

associated with $P(M, G)$, with the standard fibre F and the action L_ξ of G on F. We recall only that (see [6], p. 54–55):

$$E = P \times f/G,$$

where the action (on the right) of G on $P \times F$ is defined as follows:

$$(p, f)\cdot \xi = \big(p \cdot \xi, L_{\xi^{-1}}(f)\big).$$

For $(p, f) \in P \times F$ we denote by $\langle p, f\rangle$ the equivalence class of (p, f) in E. Thus,

(1.1) $$\langle p, f\rangle = \langle p', f'\rangle \Leftrightarrow \exists\, \xi \in G\colon\ p' = p \cdot \xi, \quad f' = L_{\xi^{-1}}(f).$$

Now we have the following lemma (see [2], Lemma 1).

LEMMA 1.2. *There is a one-to-one correspondence between sections of $E = E(P, M, G, F, L_\xi)$ and mappings $A\colon P \to F$ such that*

$$A \circ R_\xi = L_{\xi^{-1}} \circ A \quad \textit{for all } \xi \in G,$$

where R_ξ is the right translation of G on P. If a section $\sigma\colon M \to E$ and a mapping $A\colon P \to F$ are associated, then

$$\sigma\big(\pi(p)\big) = \langle p, A(p)\rangle.$$

The proof of this lemma is trivial (see [2] and [10]). The above lemma permits the following definition:

DEFINITION 1.3. A mapping $A: P \to F$ is called a *geometric object of the type* (F, L_ξ), or shortly an (F, L_ξ)*-object on* $P(M, G)$ if

$$A \circ R_\xi = L_{\xi^{-1}} \circ A \qquad \text{for all } \xi \in G.$$

The family $\{L_\xi\}$ is called the *transformation formula of* A.

Next we shall define the covariant derivative of a geometric object.

We are given a connection Γ in $P(M, G)$ and a vector field $v: M \to TM$ on M. We denote by H_v^Γ the horizontal lift of v, i.e., $H_v^\Gamma: P \to TP$ is a vector field on P uniquely defined by the conditions:

(i) $H_v^\Gamma(p) \in \Gamma_p$ — the space of horizontal vectors at the point p,

(ii) $d\pi \circ H_v^\Gamma = v \circ \pi$, where $\pi: P \to M$ is the projection.

Now we set

DEFINITION 1.4. Let $A: P \to F$ be an (F, L_ξ)-object on $P(M, G)$;

$$\nabla_v^\Gamma A = dA \circ H_v^\Gamma: P \to TF$$

is called a *covariant derivative of* A in the direction of v and with respect to the connection Γ (see [2], [10]).

It is easy to verify that (see [2], [10]):

$$(\nabla_v^\Gamma A) \circ R_\xi = dL_{\xi^{-1}} \circ \nabla_v^\Gamma A. \tag{1.5}$$

This means that

PROPOSITION 1.6. *If* $A: P \to F$ *is a* (F, L_ξ)*-object on* $P(M, G)$*, then* $\nabla_v^\Gamma A$ *is a* (TF, dL_ξ)*-object on* $P(M, G)$.

2. A mixed covariant derivative. Suppose we are given a manifold F, a Lie group G and two actions on the left of G on F. We denote by λ_ξ and Λ_ξ the left translations of G on F for these actions, and we suppose that for all $\xi, \eta \in G$,

$$\lambda_\xi \circ \Lambda_\eta = \Lambda_\eta \circ \lambda_\xi. \tag{2.1}$$

This is the foundamental hypothesis adopted throughout this paper. (2.1) permits us to define two actions (on the left) of $G \times G$ on F as follows:

$$L_{(\xi,\eta)}: F \to F, \quad L_{(\xi,\eta)} = \lambda_\xi \circ \Lambda_\eta, \tag{2.2.1}$$

$$\tilde{L}_{(\xi,\eta)}: F \to F, \quad \tilde{L}_{(\xi,\eta)} = \Lambda_\xi \circ \lambda_\eta, \tag{2.2.2}$$

and a new action (also on the left) of G on F

$$L_\xi = L_{(\xi,\xi)} = \tilde{L}_{(\xi,\xi)} = \lambda_\xi \circ \Lambda_\xi: F \to F. \tag{2.3}$$

In order to define a mixed covariant derivative we introduce the following construction.

We consider the Whitney sum $P+P$ (see [6], p. 82). Let us recall that

$$P+P = \{(p, q)\in P\times P\colon \pi(p) = \pi(q)\},$$

where $\pi\colon P\to M$ is the projection. $P+P$ is a principal fibre bundle over M with the structural group $G\times G$ which acts on $P+P$ as follows:

$$R_{(\xi,\eta)}(p, q) = (p, q)\cdot(\xi, \eta) = (p\cdot\xi, q\cdot\eta).$$

According to the definition of $P+P$ we can define the mapping

$$g\colon P+P\to G, \quad q\cdot g(p, q) = p. \tag{2.4}$$

Let us remark that condition (2.4) defines uniquely the element $g(p, q)$ in G and g is of class C^∞.

We are given an (F, L_ξ)-object A on $P(M, G)$, where L_ξ is defined by (2.3). We put

$$\bar{A}\colon P+P\to F, \quad \bar{A}(p, q) = \Lambda_{g(p,q)}(A(p)), \tag{2.5.1}$$

$$\tilde{A}\colon P+P\to F, \quad \tilde{A}(p, q) = \lambda_{g(p,q)}(A(p)). \tag{2.5.2}$$

Now we have

LEMMA 2.6. *$\bar{A}$ is a $(F, L_{(\xi,\eta)})$-object and $\tilde{A}$ is a $(F, \tilde{L}_{(\xi,\eta)})$-object on $P+P$.*

Proof. To begin with, let us remark that from

$$(q\cdot\eta)\cdot g(p\cdot\xi, q\cdot\eta) = p\cdot\xi = q\cdot g(p, q)\xi,$$

we obtain

$$g(p\cdot\xi, q\cdot\eta) = \eta^{-1}g(p, q)\xi. \tag{2.6.1}$$

Thus we have

$$\begin{aligned}[\bar{A}\circ R_{(\xi,\eta)}](p, q) &= \bar{A}(p\cdot\xi, q\cdot\eta) \quad \text{from (2.5.1), (2.6.1), we have}\\ &= \Lambda_{\eta^{-1}g(p,q)\xi}(A(p\cdot\xi)) \quad \text{from } A\circ R_\xi = L_{\xi^{-1}}\circ A,\ \text{we have}\\ &= (\Lambda_{\eta^{-1}}\circ\Lambda_{g(p,q)}\circ\Lambda_\xi\circ L_{\xi^{-1}})(A(p)) \ \text{from (2.3), (2.2.1), (2.1)}\\ &= L_{(\xi^{-1},\eta^{-1})}(\Lambda_{g(p,q)}(A(p))) = (L_{(\xi^{-1},\eta^{-1})}\circ\bar{A})(p, q).\end{aligned}$$

Analogously we verify that $\tilde{A}$ is a $(F, \tilde{L}_{(\xi,\eta)})$-object on $P+P$.

Suppose we are given two connections Γ_1 and Γ_2 in $P(M, G)$. They induce a connection, denoted by $\Gamma_1+\Gamma_2$, in $P+P$ uniquely determined by the condition that the projections of $P+P$ onto the first and the seconds component transforms $\Gamma_1+\Gamma_2$ onto Γ_1 and Γ_2, respectively. If

ω_1, ω_2 and ω are the connection forms of Γ_1, Γ_2 and $\Gamma_1+\Gamma_2$ respectively, then for any point $(p, q)\in P+P$ and for a vector v

$$v = v_1 \oplus v_2 \in T_{(p,q)}(P+P) \subset T_{(p,q)}(P\times P) = T_pP\oplus T_qP$$

we have

$$\omega_{(p,q)}(v) = (\omega_1)_p(v_1)+(\omega_2)_q(v_2) \tag{2.7}$$

(see [6], Proposition 6.3, p. 82). This formula implies immediately that

LEMMA 2.8. $H_v^{\Gamma_1+\Gamma_2}(p, q) = H_v^{\Gamma_1}(p)\oplus H_v^{\Gamma_2}(q)$.

Now we can define a mixed covariant derivative.

DEFINITION 2.9. Let Γ_1 and Γ_2 be two connections in $P(M, G)$ and let v: $M\to TM$ be a vector field on M. If $A: P\to F$ is an (F, L_ξ)-object on P, where L_ξ is given by (2.3), then a mixed covariant derivative of A with respect to the couple (Γ_1, Γ_2) in the direction of v is defined as follows:

$$\nabla_v^{\Gamma_1\Gamma_2}A = \Delta_v^{\Gamma_1+\Gamma_2}\bar{A}\circ \Delta = d\bar{A}\circ H_v^{\Gamma_1+\Gamma_2}\circ \Delta\colon P\to TF,$$

where $\bar{A}$ is the $(F, L_{(\xi,\eta)})$-object given by (2.5.1) and Δ: $P\to P+P$ is the diagonal mapping.

Next we prove

PROPOSITION 2.10. *If A is an (F, L_ξ)-object, then $\nabla_v^{\Gamma_1\Gamma_2}A$ is an (TF, dL_ξ)-object.*

Proof.

$$\begin{aligned}(\nabla_v^{\Gamma_1\Gamma_2}A)\circ R_\xi &= \nabla_v^{\Gamma_1+\Gamma_2}\bar{A}\circ \Delta\circ R_\xi\\ &= \nabla_v^{\Gamma_1+\Gamma_2}\bar{A}\circ R_{(\xi,\xi)}\circ \Delta \quad \text{by using (1.5), (2.6)}\\ &= dL_{(\xi^{-1},\xi^{-1})}\circ \nabla_v^{\Gamma_1\Gamma_2}\bar{A}\circ \Delta\\ &= dL_{\xi^{-1}}\circ \nabla_v^{\Gamma_1\Gamma_2}A.\end{aligned}$$

To end this section we prove some properties of the operator of a mixed covariant derivative. The proofs of these properties are based on the "local formula" for $\nabla_v^{\Gamma_1\Gamma_2}A$, which we formulate below.

Let us fix a point $p_0\in P$. There exists a section σ: $U\to P$ defined in some neighbourhood U of $\pi(p_0)$ such that $\sigma(\pi(p_0)) = p_0$. We also fix this section σ. If we write

$$A^+\colon P+P\to F, \quad A^+(p, q) = A(p), \tag{2.11.1}$$

then we have

$$A = \Lambda\circ(g, A^+),$$

where Λ: $G\times F\to F$, $\Lambda(\xi,f) = \Lambda_\xi(f)$. Applying Leibniz's formula (see Proposition 0) to this, we obtain

$$d_{(p_0,p_0)}\bar{A} = d_0{}'\Lambda \circ d_{(p_0,p_0)}g + d_{A(p_0)}{}''\Lambda \circ d_{(p_0,p_0)}A^+,$$

where

$$'\Lambda\colon G \to F, \quad '\Lambda(\xi) = \Lambda(\xi, A(p_0)) = \Lambda_\xi(A(p_0))$$

$$''\Lambda\colon F \to F, \quad ''\Lambda(f) = \Lambda(g(p_0, p_0), f) = \Lambda(e, f) = f = \mathrm{id}(f).$$

On the other hand, it is easy to see that

$$d_{(p_0,p_0)}A^+ = d_{p_0}A \circ \pi_1, \tag{2.11.2}$$

where π_1 is the restriction to the subspace $T_{(p_0,p_0)}(P+P) \subset T_{(p_0,p_0)}(P \times P)$ of the canonical projection

$$T_{(p_0,p_0)}(P \times P) = T_{p_0}P \oplus T_{p_0}P \to T_{p_0}P$$

onto the first component.

Thus we obtain

$$d_{(p_0,p_0)}\bar{A} = d_0{}'\Lambda \circ d_{(p_0,p_0)}g + d_{p_0}A \circ \pi_1. \tag{2.11.3}$$

Now we define a function $\tilde{g}\colon P|U \times P|U \to G$ (among other things we need for this purpose the section σ) setting

$$\tilde{g}(q_1, q_2) = \xi_2^{-1}\xi_1, \tag{2.11.4}$$

where $q_i = \sigma(\pi(q_i))\cdot\xi_i$, $i = 1, 2$. $\tilde{g}$ is of class C^∞, and it is easy to show that $g = \tilde{g}|(P+P)$. Hence, for $(p, q) \in (P+P)|U$,

$$d_{(p,q)}g = d_{(p,q)}\tilde{g}\,|\,T_{(p,q)}(P+P).$$

Applying Leibniz's formula to $\tilde{g}$ and to the vector

$$v = v_1 \oplus v_2 \in T_{(p_0,p_0)}(P+P) \subset T_{p_0}P \oplus T_{p_0}P,$$

we have

$$(d_{(p_0,p_0)}g)(v) = (d_{p_0}{}'\tilde{g})(v_1) + (d_{p_0}{}''\tilde{g})(v_2),$$

where $'\tilde{g}$, $''\tilde{g}\colon P|U \to G$, $'\tilde{g}(p) = \tilde{g}(p, p_0)$, $''\tilde{g}(p) = \tilde{g}(p_0, p)$. By the definition of g, $''\tilde{g}(p) = ('\tilde{g}(p))^{-1}$, and hence $d_{p_0}{}''\tilde{g} = -d_{p_0}{}'\tilde{g}$. Thus

$$(d_{(p_0,p_0)}g)(v) = (d_{p_0}{}'\tilde{g})(v_1 - v_2). \tag{2.11.5}$$

Now from (2.11.2), (2.11.3), (2.11.5), Definition 2.9 and Lemma 2.8 we obtain

$$\begin{aligned}(\nabla_v^{\Gamma_1\Gamma_2}A)(p_0) &= (d_{(p_0,p_0)}\bar{A})\big(H_v^{\Gamma_1}(p_0) \oplus H_v^{\Gamma_2}(p_0)\big) \\ &= d_{p_0}({}'\Lambda \circ {}'g)\big(H_v^{\Gamma_1}(p_0) - H_v^{\Gamma_2}(p_0)\big) + d_{p_0}A\big(H_v^{\Gamma_1}(p_0)\big).\end{aligned} \tag{2.12}$$

Using formula (2.12), we can now prove the following properties of the operator $\nabla_v^{\Gamma_1\Gamma_2}$.

PROPOSITION 2.13. *We have the formula* $\nabla_v^{\Gamma\Gamma} A = \nabla_v^{\Gamma} A$.

It follows immediately from (2.12).

PROPOSITION 2.14. *We have* $\nabla_v^{\Gamma_1\Gamma_2} A = \nabla_v^{\Gamma_2+\Gamma_1} \tilde{A} \circ \Delta$, *where* $\tilde{A}$ *is given by* (2.5.2).

Proof. Let

$$f = \nabla_v^{\Gamma_1\Gamma_2} A - \nabla_v^{\Gamma_2+\Gamma_1} \tilde{A} \circ \Delta$$
$$= \nabla_v^{\Gamma_1+\Gamma_2} \bar{A} \circ \Delta - \nabla_v^{\Gamma_2\Gamma_1} \tilde{A} \circ \Delta.$$

Let us fix a point $p_0 \in P$ and a section $\sigma\colon U \to P$ defined in some neighbourhood U of $\pi(p_0)$ such that $\sigma(\pi(p_0)) = p_0$. For $\nabla_v^{\Gamma_1+\Gamma_2} \tilde{A} \circ \Delta$ we can find a formula analogous to formula (2.12) for $\nabla_v^{\Gamma_1\Gamma_2} A = \nabla_v^{\Gamma_1+\Gamma_2} \bar{A} \circ \Delta$. We need only to replace Λ by λ, $H_v^{\Gamma_1}(p_0)$ by $H_v^{\Gamma_2}(p_0)$ and inversely. Thus we have

$$(2.14.1) \qquad (\nabla_v^{\Gamma_2+\Gamma_1} \tilde{A} \circ \Delta)(p_0)$$
$$= d_{p_0}('\lambda \circ '\tilde{g})\big(H_v^{\Gamma_2}(p_0) - H_v^{\Gamma_1}(p_0)\big) + d_{p_0} A\big(H_v^{\Gamma_2}(p_0)\big),$$

where $'\lambda\colon G \to F$, $'\lambda(\xi) = \lambda_\xi\big(A(p_0)\big)$. From (2.14.1) and (2.12) we obtain

$$f(p_0) = [d_{p_0}('\Lambda \circ '\tilde{g}) + d_{p_0}('\lambda \circ 'g) - d_{p_0} A]\big(H_v^{\Gamma_1}(p_0) - H_v^{\Gamma_2}(p_0)\big).$$

Since from Leibniz's formula it follows that

$$d_{p_0} S^* = d_{p_0}{'\Lambda} + d_{p_0}{'\lambda},$$

where $S^*\colon G \to F$, $S^*(\xi) = L_\xi{'A(p_0)} = (\lambda_\xi \circ \Lambda_\xi)\big(A(p_0)\big)$, if we write $S = S^* \circ 'g$, then

$$f(p_0) = (d_{p_0} S - d_{p_0} A)\big(H_v^{\Gamma_1}(p_0) - H_v^{\Gamma_2}(p_0)\big).$$

Next, let us remark that

$$A = L \circ ('g, B),$$

where $B(p) = L_{('g(p))^{-1}}\big(A(p)\big)$ and $L\colon G \times F \to F$, $L(\xi, f) = L_\xi(f)$. Since $B(p_0) = A(p_0)$ and $'g(p_0) = e$, applying Leibniz's formula, we have

$$d_{p_0} A = d_{p_0} S + d_{p_0} B,$$

and hence

$$f(p_0) = -(d_{p_0} B)\big(H_v^{\Gamma_1}(p_0) - H_v^{\Gamma_2}(p_0)\big).$$

It is easy to see that $B \circ R_\xi = B$, that is, B is constant on the fibre of $P(M, G)$. Thus $f(p_0) = 0$ because the vector $H_v^{\Gamma_1}(p_0) - H_v^{\Gamma_2}(p_0)$ is vertical (tangent to a fibre). This completes our proof.

PROPOSITION 2.15. *Let* $A\colon P \to F$ *be an* (F, L_ξ)*-object, where* $L_\xi = \lambda_\xi \circ \Lambda_\xi$. *If* $\Lambda_\xi = \mathrm{id}$ *for all* ξ, *then*

$$\nabla_v^{\Gamma_1\Gamma_2} A = \nabla_v^{\Gamma_1} A.$$

Analogously, if $\lambda_\xi = \mathrm{id}$ *for all* ξ, *then*

$$\nabla_v^{\Gamma_1\Gamma_2} A = \nabla_v^{\Gamma_2} A.$$

Proof. We prove this is the case of $\Lambda_\xi = \mathrm{id}$ (the second case is analogous). Let us remark that

$$\bar{A}(p, q) = \Lambda_{g(p,q)}\big(A(p)\big) = A(p) = A^+(p, q),$$

and hence, applying (2.11.2), we obtain

$$\begin{aligned}(\nabla_v^{\Gamma_1\Gamma_2} A)(p) &= d_{(p,p)}\bar{A}\big(H_v^{\Gamma_1}(p) \oplus H_v^{\Gamma_2}(p)\big)\\ &= (d_p A \circ \pi_1)\big(H_v^{\Gamma_1}(p) \oplus H_v^{\Gamma_2}(p)\big)\\ &= d_p A\big(H_v^{\Gamma_1}(p)\big)\\ &= (\nabla_v^{\Gamma_2} A)(p).\end{aligned}$$

PROPOSITION 2.16. *We have the formula*

$$\nabla_v^{\Gamma_1\Gamma_2} A + \nabla_v^{\Gamma_2\Gamma_1} A = \nabla_v^{\Gamma_1} A + \nabla_v^{\Gamma_2} A.$$

This proposition follows immediately from (2.12) and Proposition 2.13.

3. The case of tensors. In this section we shall show that in the case of linear connections and tensors our definition of a mixed covariant derivative coincides with the definition of A. P. Norden [9].

Let

$$F = R^{n^{r+s}} = \{(f^{i_1\ldots i_r}_{j_1\ldots j_s}):\ i_a, j_\beta = 1, \ldots, n,\ a = 1, \ldots, r,\ \beta = 1, \ldots, s\}$$

and let us fix two integers p and q such that $0 \leqslant p \leqslant r$, $0 \leqslant q \leqslant s$. For $A = [a^i_j] \in GL(n, R)$ we write

$$A^{-1} = [\dot{a}^i_j] \in GL(n, R), \tag{3.1}$$

and we define two actions on the left of the linear group $GL(n, R)$ on F as follows:

$$\lambda_A,\ \Lambda_A\colon F \to F.$$

$$\lambda_A(f) = (a^{i_1}_{k_1} \ldots a^{i_p}_{k_p} f^{k_1\ldots k_p i_{p+1}\ldots i_r}_{l_1\ldots l_q j_{q+1}\ldots j_s} \dot{a}^{l_1}_{j_1} \ldots \dot{a}^{l_q}_{j_q}), \tag{3.2.1}$$

$$\Lambda_A(f) = (a^{i_{p+1}}_{k_{j+1}} \ldots a^{i_r}_{k_r} f^{i_1\ldots i_q k_{p+1}\ldots k_r}_{j_1\ldots j_q l_{q+1}\ldots l_s} \dot{a}^{l_{q+1}}_{j_{q+1}} \ldots \dot{a}^{l_s}_{j_s}) \tag{3.2.2}$$

for $f = (f^{i_1\ldots i_r}_{j_1\ldots j_s}) \in F$. It is clear that

$$\lambda_A \circ \Lambda_B = \Lambda_B \circ \Lambda_A.$$

Let S be a tensor field on M of the type (r, s), that is, in our general terminology, S is an $(L, \lambda_A \circ \Lambda_A)$-object on the bundle of linear frames LM.

If we fix a local coordinate system $(U, \varphi) \in at\, l(M)$, then it induced a trivialization of $LM|U$ which permits us to identify $LM|U$ with $U \times \times GL(n, R)$, i.e.,

$$LM|U = U \times GL(n, R).$$

Now the tensor field S is uniquely determined on $LM|U$ by the function $t(x) = S(x, I)$ because

$$S(x, A) = (S \circ R_A)(x, I) = (\lambda_{A^{-1}} \circ \Lambda_{A^{-1}} \circ S)(x, I) \tag{3.3}$$
$$= (\lambda_{A^{-1}} \circ \Lambda_{A^{-1}} \circ t)(x).$$

The functions $t(x) = \left(t^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x)\right)$ are called the *coordinates of* S with regard to (U, φ). The identification $LM|U = U \times GL(n, R)$ gives also the following identification:

$$(LM + LM)|U = U \times GL(n, R) \times GL(n, R).$$

In this case the function g defined by (2.4) is given by the formula

$$g(x, A, B) = B^{-1}A, \tag{3.4}$$

and for the object $\bar{S}$ defined by (2.5.1) we now have the formula

$$\bar{S}(x, A, B) = \Lambda_{g(x,A,B)}\left(S(x, A)\right) = (\Lambda_{B^{-1}} \circ \lambda_{A^{-1}} \circ t)(x), \tag{3.5.1}$$

that is,

$$\bar{S}(x, A, B) \tag{3.5.2}$$
$$= \left(\overset{*}{a}{}^{i_1}_{k_1} \ldots \overset{*}{a}{}^{i_p}_{k_p} \overset{*}{b}{}^{i_{p+1}}_{k_{p+1}} \ldots \overset{*}{b}{}^{i_r}_{k_r} t^{k_1 \ldots k_r}_{l_1 \ldots l_s}(x) a^{l_1}_{j_1} \ldots a^{l_q}_{j_q} b^{l_{q+1}}_{j_{q+1}} \ldots b^{l_s}_{j_s}\right).$$

Let Γ and $\check{\Gamma}$ be two linear connections on M, and let Γ^i_{jk} and $\check{\Gamma}^i_{jk}$ be Christoffel's symbols of Γ and $\check{\Gamma}$, respectively. We consider a vector field v on M. The identification $LM|U = U \times GL(n, R)$ implies also the identification $TM|U = U \times R^n$, and now for $x \in U$

$$v(x) = \left(x, v^k(x) a_k\right),$$

where $\{e_k\}$ is the canonical base of R^n. Since the functions Γ^i_{jk} are defined as coefficients of the decomposition

$$(\sigma_0^* \omega)(a_k) = \Gamma^i_{jk}(x) E^j_i,$$

where $\{E^j_i\}$ is the canonical base of $T_I\left(GL(n, R)\right) = R^{n^2}$, $\sigma_0 \colon U \to LM$, $\sigma_0(x) = (x, I)$ and ω is the connection form of Γ, the condition $\omega_{(x,I)} \cdot \left(H_v(x, I)\right) = 0$ implies

$$H^\Gamma_v(x, I) = \left(x, I, v^i(x) e_i, -\Gamma^\alpha_{k\beta}(x) E^\beta_\alpha\right)$$

(let us remark that $T(LM|U) = LM|U \times R^n \times R^{n^2}$). Analogously,

$$H^{\check{\Gamma}}_v(x, I) = \left(x, I, v^i(x) e_i, -\check{\Gamma}^\alpha_{k\beta}(x) E^\beta_\alpha\right).$$

In order to find $\nabla_v^{\Gamma\check{\Gamma}}\, S$, we need only to calculate (see Proposition 2.10)

$$
\begin{aligned}
(\nabla_v^{\Gamma\check{\Gamma}} S)(x, I) &= (d_{(x,I,I)}\overline{S})(H_v^{\Gamma}(x, I) \oplus H_v^{\check{\Gamma}}(x, I) \\
&= (d_{(x,I,I)}\overline{S})\big(x, I, I, v^i(x)e_i,\ -\Gamma^a_{k\beta}(x)v^k(x)E^\beta_a,\ -\check{\Gamma}^a_{k\beta}(x)v^k \times \\
&\qquad \times (x)E^\beta_a\big) \\
&= \big(t(x), [(\partial_k \overline{S}^{i_1\dots i_r}_{j_1\dots j_s})(x, I, I) - \Gamma^a_{k\beta}(x)(\partial^\beta_a S^{i_1\dots i_r}_{j_1\dots i_s})(x, I, I) - \\
&\qquad - \check{\Gamma}^a_{k\beta}(x)(\check{\partial}^\beta_a \overline{S}^{i_1\dots j_r}_{j_1\dots j_s})(x, I, I)]v^k(x)E^{j_1\dots j_s}_{i_1\dots i_r}\big),
\end{aligned}
$$

where $\{E^{j_1\dots j_s}_{i_1\dots i_r}\}$ is the canonical base of $F = R^{n^{r+s}}$, and

$$
\partial_k = \partial/\partial x^k, \quad \partial^a_\beta = \partial/\partial a^a_\beta\ , \quad \check{\partial}^a_\beta = \partial/\partial b^a_\beta .
$$

Since

$$
(\partial^\beta_a \dot{a}^\lambda_\eta)(I) = -\delta^\beta_\eta \delta^\lambda_a, \qquad (\check{\partial}^\beta_a \dot{b}^\lambda_\eta)(I) = -\delta^\beta_\eta \delta^\lambda_a,
$$

we have

$$
\begin{aligned}
(\partial_k \overline{S}^{i_1\dots i_r}_{j_1\dots j_s})(x, I, I) &= (\partial_k t^{i_1\dots i_r}_{j_1\dots j_s})(x), \\
(\partial^\beta_a \overline{S}^{i_1\dots i_r}_{j_1\dots j_s})(x, I, I) &= -\sum_{\lambda=1}^{p} \delta^{i_\lambda}_a t^{i_1\dots,\overset{(\lambda)}{a},\dots i_r}_{j_1\dots j_s}(x) + \sum_{\lambda=1}^{2} \delta^\beta_{j_\lambda} t^{i_1\dots i_r}_{j_1\dots,\underset{(\lambda)}{\beta},\dots j_s}(x), \\
(\check{\partial}^\beta_a \overline{S}^{i_1\dots i_r}_{j_1\dots j_s})(x, I, I) &= -\sum_{\lambda=p+1}^{r} \delta^{i_\lambda}_a t^{i_1\dots,\overset{(\lambda)}{a},\dots i_r}_{j_1\dots j_s}(x) + \sum_{\lambda=q+1}^{s} \delta^\beta_{j_\lambda} t^{i_1\dots i_r}_{j_1\dots,\underset{(\lambda)}{\beta},\dots j_s},
\end{aligned}
$$

where $t^{i_1\dots,\overset{(\lambda)}{a},\dots i_r}_{j_1\dots j_s}(x)$ means that at the λ-th place there is a and at the other places there are indices $j_1, \dots, i_r$ (instead of i_λ). From these formulae we obtain

$$
\begin{aligned}
(\nabla_v^{\Gamma\check{\Gamma}} S)(x, I) = \Big(t(x), \Big[&(\partial_k t^{i_1\dots i_r}_{j_1\dots j_s})(x) + \sum_{\lambda=1}^{p} \Gamma^{i_\lambda}_{ka}(x) t^{i_1\dots,\overset{(\lambda)}{a},\dots i_r}_{j_1\dots j_s}(x) + \\
&+ \sum_{\lambda=p+1}^{r} \check{\Gamma}^{i_\lambda}_{ka}(x) t^{i_1\dots,\overset{(\lambda)}{a},\dots i_r}_{j_1\dots j_s}(x) - \sum_{\lambda=1}^{q} \Gamma^\beta_{kj_\lambda}(x) t^{i_1\dots i_r}_{j_1\dots,\underset{(\lambda)}{\beta},\dots j_s}(x) - \\
&- \sum_{\lambda=q+1}^{s} \check{\Gamma}^\beta_{kj_\lambda}(x) t^{i_1\dots i_r}_{j_1\dots,\underset{(\lambda)}{\beta},\dots j_s}(x)\Big]\, v^k(x) E^{j_1\dots j_s}_{i_1\dots i_r}\Big).
\end{aligned}
$$

The expression in the brackets [...] is the definition of a mixed covariant derivative introduced by A. P. Norden [9].

Analogously, we can show that in the case of tensor densities our definition coincides with the definition given in [4].

4. Leibniz's formulas for a mixed covariant derivative. Suppose there are three manifolds F_1, F_2, F_2 and two actions on the left of a Lie group G on each F_i

$$\lambda^{(i)}_\xi,\ \Lambda^{(i)}_\xi\colon F_i \to F_i, \quad i = 1, 2, 3,$$

such that $\lambda^{(i)}_\xi \circ \Lambda^{(i)}_\eta = \Lambda^{(i)}_\eta \circ \lambda^{(i)}_\xi$ for all $\xi, \eta \in G$. We introduce the notation (as in Section 2, see (2.1), (2.2.1)):

(4.1.1) $$L^{(i)}_{(\xi,\eta)} = \lambda^{(i)}_\xi \circ \Lambda^{(i)}_\eta,$$

(4.1.2) $$\tilde{L}^{(i)}_{(\xi,\eta)} = \Lambda^{(i)}_\xi \circ \lambda^{(i)}_\eta,$$

(4.1.3) $$L^{(i)}_\xi = L^{(i)}_{(\xi,\xi)} = \tilde{L}^{(i)}_{(\xi,\xi)} = \lambda^{(i)}_\xi \circ \Lambda^{(i)}_\xi.$$

Now suppose there is a mapping

(4.2) $$\psi\colon F_1 \times F_2 \to F_3$$

such that

(4.2.1) $$\psi \circ (\lambda^{(1)}_\xi \times \mathrm{id}_{F_2}) = \lambda^{(3)}_\xi \circ \psi,$$

(4.2.2) $$\psi \circ (\mathrm{id}_{F_1} \times \Lambda^{(2)}_\xi) = \Lambda^{(3)}_\xi \circ \psi,$$

(4.2.3) $$\psi \circ (\Lambda^{(1)}_\xi \times \lambda^{(2)}_\xi) = \psi.$$

In order to simplify the notation we set

$$\psi(f_1, f_2) = f_1 \cdot f_2 \in F_3 \quad \text{for } f_1 \in F_1,\ f_2 \in F_2.$$

Let us remark that if $F_1 = F_2 = F_3$, $\lambda^{(1)}_\xi = \lambda^{(2)}_\xi = \lambda^{(3)}_\xi$, $\Lambda^{(1)}_\xi = \Lambda^{(2)}_\xi = \Lambda^{(3)}_\xi$, then ψ is a multiplication law in F_1.

If $A_i\colon P \to F_i$ is a $(F_i, L^{(i)}_\xi)$-object on $P(M, G)$, then we consider objects $\bar{A}_i$, $\tilde{A}_i$ on $P+P$ (in the same way as in Section 2, see (2.5.1), (2.5.2) and Lemma 2.6)

(4.3.1) $$\bar{A}_i(p, q) = \Lambda^{(i)}_{g(p,q)}\big(A_i p)\big),$$

(4.3.1) $$\tilde{A}_i(p, q) = \lambda^{(i)}_{g(p,q)}\big(A_i(p)\big),$$

where g is given by (2.4). We now prove

LEMMA 4.4. *If $A_i\colon P \to F_i$ is a $(F_i, L^{(i)})$-object on $P(M, G)$, $i = 1, 2$, then*

$$A = A_1 \cdot A_2 = \psi \circ (A_1, A_2)\colon P \to F_3$$

is an $(F_3, L^{(3)}_\xi)$-object on $P(M, G)$.

Proof.

$$\begin{aligned}
A \circ R_\xi &= \psi \circ (A_1, A_2) \circ R_\xi \\
&= \psi \circ (A_1 \circ R_\xi, A_2 \circ R_\xi) \quad \text{since } A_i \text{ is a } (F_i, L^{(i)}_\xi)\text{-object} \\
&= \psi \circ (L^{(1)}_{\xi^{-1}} \circ A_1, L^{(2)}_{\xi^{-1}} \circ A_2) \quad \text{from (4.1.1), (4.1.3)} \\
&= \psi \circ (\lambda^{(1)}_{\xi^{-1}} \times \mathrm{id}_{F_2}) \circ (\mathrm{id}_{F_1} \times \Lambda^{(2)}_{\xi^{-1}}) \circ (\Lambda^{(1)}_{\xi^{-1}} \circ \lambda^{(2)}_{\xi^{-1}}) \circ (A_1, A_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{from (4.2.1)–(4.2.2)} \\
&= \lambda^{(3)}_{\xi^{-1}} \circ \Lambda^{(3)}_{\xi^{-1}} \circ \psi \circ (A_1, A_2) = L^{(3)}_{\xi^{-1}} \circ A.
\end{aligned}$$

LEMMA 4.5. *If A_i and A are the same as in Lemma* 4.4, *then*

$$A^+ = \bar{A}_1 \circ \tilde{A}_2 = \psi\circ(\bar{A}_1, \tilde{A}_2),$$

where A^+ is given by (2.11.1).

Proof.

$$\begin{aligned}(\bar{A}_1\cdot\tilde{A}_2)(p, q) &= \psi(\bar{A}_1(p, q), \tilde{A}_2(p\, q)) \quad \text{from (4.3.1), (4.3.2)}\\ &= \psi\big(A^{(1)}_{g(p,q)}(A_1(p)), \lambda^{(2)}_{g(p,q)}(A_2(p))\big) \quad \text{from (4.2.3)}\\ &= \psi\big(A_1(p), A_2(p)\big) = A(p) \stackrel{\mathrm{df}}{=} A^+(p, q).\end{aligned}$$

In order to formulate Leibniz's formula we introduce the following notation. We define the mapping

$$\psi^*\colon (TF_1\times F_2)\cup(F_1\times TF_2)\to TF_3,$$

(4.6.1) $\qquad \psi^*(v, f_2) = (dRF_{f_2})(v) \quad$ for $v\in TF_1, f_2\in F_2$,

(4.6.2) $\qquad \psi^*(f_1, w) = (dL_{f_1})(w) \quad$ for $f_1\in F_1, w\in TF_2$,

where $R_{f_2}\colon F_1\to F_3$, $L_{f_1}\colon F_2\to F_3$, $R_{f_2}(f_1) = L_{f_1}(f_2) = (f_1, f_2)$ and next we set

(4.6.3) $\qquad \psi^*(v, f_2) \doteq v\cdot f_2, \quad \psi^*(f_1, w) = f_1\cdot w.$

Now we have

PROPOSITION 4.7 (Leibniz's formula). *If $A_i\colon P\to F_i$ is an $(F_i, L^{(i)}_\xi)$-object on $P(M, G)$ for $i = 1, 2$, then, using the notation of* (4.2.4), (4.6.3), *for any connections Γ_1, Γ_2 in $P(M, G)$ and for any vector field v on M, we have*

$$\nabla^{\Gamma_1}_v(A_1\cdot A_2) = (\nabla^{\Gamma_1\Gamma_2}_v A_1)\cdot A_2 + A_1\cdot(\nabla^{\Gamma_2\Gamma_1}_v A_2).$$

Proof. From Lemma 4.5 and Proposition 0, for an arbitrary fixed point $p\in P$ we obtain

$$d_{(p,p)}A^+ = dR_{A_2(p)}\circ d_{(p,p)}\bar{A}_1 + dL_{A_1(p)}\circ d_{(p,p)}\tilde{A}_2$$

(let us remark that $\tilde{A}_2(p, p) = A_2(p)$ and $\bar{A}_1(p, p) = A_1(p)$). Thus, Definition 2.9 and Proposition 2.14 imply

$$\begin{aligned}&(\nabla^{\Gamma_1\Gamma_2}_v A_1)\cdot A_2 + A_1\cdot(\nabla^{\Gamma_2\Gamma_1}_v A_2)(p)\\ &\qquad = [dR_{A_2(p)} d_{(p,p)}\tilde{A}_1 + dL_{A_1(p)} d_{(p,p)}\bar{A}_2]\big(H^{\Gamma_1}_v(p) + H^{\Gamma_2}_v(p)\big)\\ &\qquad = d_{(p,p)}A^+\big(H^{\Gamma_1}_v(p) + H^{\Gamma_2}_v(p)\big) \quad \text{from (2.11.2)}\\ &\qquad = d_p A\big(H^{\Gamma_1}_v(p)\big) = (\nabla^{\Gamma_1}_v A)(p),\end{aligned}$$

and now the proof is complete.

We also prove other forms of Leibniz's formula. In order to do this, suppose there are manifolds F, H_1, H_2, and actions on the left of G

$$\lambda_\xi,\ \Lambda_\xi\colon F \to F,$$
$$h_\xi^{(i)}\colon H_i \to H_i, \quad i = 1, 2,$$

on F, H_1 and H_2, respectively. We suppose that $\lambda_\xi \circ \Lambda_\eta = \Lambda_\eta \circ \lambda_\xi$. Next, we assume a mapping

$$\Phi\colon H_1 \times F \to H_2 \tag{4.8}$$

such that

$$\Phi \circ (h^{(1)} \times \lambda_\xi) = \Phi, \tag{4.8.1}$$

$$\Phi \circ (\mathrm{id}_{H_1} \times \Lambda_\xi) = h^{(2)} \circ \Phi. \tag{4.8.2}$$

Let us remark that if $H_1 = H_2$ and $h_\xi^{(1)} = h_\xi^{(2)}$, then Φ is an exterior multiplication law on H_1. For this reasons we shall note

$$\Phi(h, f) = h \cdot f, \quad h \in H_1,\ f \in F, \tag{4.8.3}$$

also in the general case.

It is easy to show

LEMMA 4.9. *If $A\colon P \to F$ is a (F, L_ξ)-object on $P(M, G)$, where $L_\xi = \lambda_\xi \circ \Lambda_\xi$, and $a\colon P \to H_1$ is an $(H_1, h_\xi^{(1)})$-object on $P(M, G)$, then*

$$b = a \cdot A = \Phi \circ (a, A)$$

is an $(H_2, h^{(2)})$-object, because

$$\begin{aligned}
b \circ R_\xi &= \Phi \circ (a, A) \circ R_\xi = \Phi \circ (a \circ R_\xi, A \circ R_\xi) \\
&= \Phi \circ (h_{\xi^{-1}}^{(1)} \circ a, L_{\xi^{-1}} \circ A) \\
&= \Phi \circ (h_{\xi^{-1}}^{(1)} \times \lambda_{\xi^{-1}}) \circ (\mathrm{id}_{H_1} \times \Lambda_{\xi^{-1}}) \circ (a, A) \\
&= h_{\xi^{-1}}^{(2)} \circ \Phi \circ (a, A) = h_{\xi^{-1}}^{(2)} \circ b.
\end{aligned}$$

In the same way as for ψ, we define

$$\Phi^*\colon (H_1 \times TF) \cup (TH_1 \times F) \to TH_2,$$

$$\Phi^*(h, v) = dL_h(v) \quad \text{for } h \in H_1,\ v \in TF, \tag{4.10.1}$$

$$\Phi^*(w, f) = dR_f(w) \quad \text{for } w \in TR_1,\ f \in F, \tag{4.10.2}$$

where $L_h\colon F \to H_2$, $R_f\colon H_1 \to H_2$, $L_h(f) = R_f(h) = \Phi(h, f) = h \cdot f$, and we note

$$\Phi^*(h, v) = h \cdot v, \quad \Phi^*(w, f) = w \cdot f. \tag{4.10.3}$$

Now we can prove

PROPOSITION 4.11. *If $a\colon P \to H_1$ is an $(H_1, h_\xi^{(1)})$-object and $A\colon P \to F$ is an (F, L_ξ)-object on $P(M, G)$, where $L_\xi = \lambda_\xi \circ \Lambda_\xi$, then, using the notation of* (4.8.3), (4.10.3), *we have*

$$\nabla_v^{\Gamma_1}(a \cdot A) = (\nabla_v^{\Gamma_2} a) \cdot A + a \cdot (\nabla_v^{\Gamma_1 \Gamma_2} A).$$

Proof. We define the trivial action of G on H_i, denoted by $H^{(i)}_\xi$, that is,

$$H^{(i)}_\xi = \mathrm{id}_{H_i} \quad \text{for all } \xi \in G.$$

Now each $(H_i, h^{(i)}_\xi)$-object is also an $(H_i, h^{(i)}_\xi \circ H^{(i)}_\xi)$-object and if

$$\begin{aligned}(F_1, \lambda^{(1)}_\xi, \Lambda^{(1)}_\xi) &= (H_1, H^{(1)}_\xi, h^{(1)}_\xi),\\ (F_2, \lambda^{(2)}_\xi, \Lambda^{(2)}_\xi) &= (F, \lambda_\xi, \Lambda_\xi),\\ (F_3, \lambda^{(3)}_\xi, \Lambda^{(3)}_\xi) &= (H_2, H^{(2)}_\xi, h^{(2)}_\xi),\end{aligned}$$

then Φ satisfies conditions (4.2.1)–(4.2.3), and thus, by Proposition 4.7, we have

$$\nabla^{\Gamma_1}_v(a\cdot A) = (\nabla^{\Gamma_1\Gamma_2}_v a)\cdot A + a\cdot(\nabla^{\Gamma_2\Gamma_1}_v A).$$

Now, from Proposition 2.15, we obtain our assertion.

Using the above Leibniz's formulas, we can prove the following interpretation of the condition $\nabla^{\Gamma_1\Gamma_2}_v A = 0$:

Suppose we are given a manifold F and two actions on the left of the group G on F; let λ_ξ, Λ_ξ denote left translations for these actions. As usual, we suppose that $\lambda_\xi \circ \Lambda_\eta = \Lambda_\eta \circ \lambda_\xi$. Next, suppose we are given a manifold H, an action (on the left) of G on H and an exterior multiplication law $H \times F \to H$ which satisfies conditions (4.8.1), (4.8.2). (This is the case $(H_1, h^{(1)}_\xi) = (H_2, h^{(2)}_\xi) = (H, h_\xi)$.) Now we have

PROPOSITION 4.12. *Let A be an $(F, \lambda_\xi \circ \Lambda_\xi)$-object on $P(M, G)$ and $v\colon M \to TM$ be a vector field on M. If $\nabla^{\Gamma_1\Gamma_2}_v A = 0$, then for each (H, h_ξ)-object t on $P(M, G)$ we have*

$$\nabla^{\Gamma_1}_v t = 0 \Rightarrow \nabla^{\Gamma_2}_v(t\cdot A) = 0.$$

Furthermore, if we suppose that F is a semi-group with the unity and $(h\cdot a)\cdot b = h\cdot(a\cdot b)$, $h\cdot 1 = h$ for all $a, b \in F$, $h \in H$, then for a non-singular $(F, \lambda_\xi \circ \Lambda_\xi)$-object A (that is, $A(p)$ is an invertible element in F for all p) the condition $\nabla^{\Gamma_1\Gamma_2}_v A = 0$ implies

$$\nabla^{\Gamma_1}_v t = 0 \Leftrightarrow \nabla^{\Gamma_2}_v(t\cdot A) = 0.$$

(*We suppose also that $F \times F \ni (f_1, f_2) \to f_1\cdot f_2 \in F$ satisfies* (4.2.1)–(4.2.3).)

Proof. If we suppose that $\nabla^{\Gamma_1}_v t = 0$, then from Leibniz's formula (Proposition 4.11) we obtain

$$\nabla^{\Gamma_2}_v(t\cdot A) = (\nabla^{\Gamma_1}_v t)\cdot A + t\cdot(\nabla^{\Gamma_1\Gamma_2}_v A) = 0.$$

If we suppose additionally that F is a semi-group and A is non-singular, then the condition $\nabla^{\Gamma_2}_v(t\cdot A) = 0$ implies $(\nabla^{\Gamma_1}_v t)\cdot A = 0$. Multiplying that by A^{-1}, where $A^{-1}(p) = \big(A(p)\big)^{-1}$, we obtain $\nabla^{\Gamma_1}_v t = 0$.

5. r-affinors. In this section we shall consider some objects on L^rM. First of all we define the group L_n^r and the fibre bundle L^rM.

Let F_n^r be the set of all r-jets (at 0) of mappings $\varphi\colon R^n \to R^n$ such that $\varphi(0) = 0 \cdot F_n^r$ is a manifold diffeomorphic with R^N, where $N = n\left[\binom{n+r}{r} - 1\right]$. We define the semi-group structure (with the unity) on F_n^r, setting

$$[\varphi][\psi] = [\varphi \circ \psi].$$

Let L_n^r be the set of invertible elements in F_n^r, that is,

$$L_n^r = \{[\varphi] \in F_n^r\colon \varphi \text{ is a local diffeomorphism in some neighbourhoof of } 0\}.$$

L_n^r is an open subset of F_n^r. It is a Lie group.

DEFINITION 5.1. L_n^r is called a *differential r-group*.

Next we shall construct the principal fibre bundle L^rM called the *bundle of r-frames*. This construction is the following:

Let M be a manifold and let $\operatorname{atl}(M)$ denote an atlas in M. For two charts (U, φ), $(U', \varphi') \in \operatorname{atl}(M)$ and for a point $x \in U \cap U'$ we write

$$j_x^r(\varphi' \circ \varphi^{-1}) = [T_{-\varphi'(x)} \circ \varphi' \circ \varphi^{-1} \circ T_{\varphi(x)}],$$

where $T_v\colon R^n \to R^n$, $T_v(x) = x + v$, $v \in R^n$, $j_x^r(\varphi' \circ \varphi^{-1})$ is an element of L_n^r, $n = \dim M$. In the set

$$Z = \bigcup \{U \times \{\varphi\} \times L_n^r\colon (U, \varphi) \in \operatorname{atl}(M)\}, \quad n = \dim M,$$

we define an equivalence relation

$$(5.2) \qquad (x, \varphi, a) \sim (x', \varphi', a') \Leftrightarrow x = x', \quad a' = j_x^r(\varphi' \circ \varphi^{-1})a,$$

and let

$$(5.3) \qquad L^rM = Z/\sim, \quad \pi\colon L^rM \to M, \quad \pi([x, \varphi, a]) = x,$$

where $[x, \varphi, a]$ denotes the equivalence class of (x, φ, a) in L^rM. A differential structure on L^rM is defined as follows. If (U, ψ) is a chart of $\operatorname{atl}(M)$, then we consider

$$(5.4) \qquad \tilde{\psi}\colon \pi^{-1}(U) = L^rM \,|\, U \ni [x, \varphi, a] \to \big(x, j_x^r(\psi \circ \varphi^{-1})a\big) \in U \times L_n^r.$$

$\tilde{\psi}$ is a well-defined injective mapping because

$$[x, \varphi, a] = [x', \varphi', a'] \Leftrightarrow x = x', \quad a' = j_x^r(\varphi' \circ \varphi^{-1})a \Leftrightarrow x = x',$$

$$j_x^r(\psi \circ \varphi^{-1})a' = j_x^r(\psi \circ \psi^{-1}) j_x^r(\psi' \circ \varphi^{-1})a = j_x^r(\psi \circ \varphi^{-1})a.$$

$\tilde{\psi}$ is also surjective because $\tilde{\psi}([x, \psi, a]) = (x, a)$. It is easy to see that for two charts (U, ψ), $(U', \psi') \in \operatorname{atl}(M)$

$$\tilde{\psi}' \circ \tilde{\psi}^{-1}\colon (U \cap U') \times L_n^r \to (U \cap U') \times L_n^r$$

is a C^∞-mapping (precisely, it is easy to verify that $(\tilde{\psi}'\circ\tilde{\psi}^{-1})(x, a) = (x, j^r_x(\psi'\circ\psi^{-1})a))$. Thus there is a differential structure, and only one, on L^rM such that $\tilde{\psi}$ given by (5.4) is a diffeomorphism for all $(U, \psi) \in \mathrm{atl}(M)$. The action of L^r_n on L^rM, $n = \dim M$, is defined by

$$(5.5) \qquad [x, \varphi, a]\cdot\beta = [x, \varphi, a\beta].$$

In this way we obtain a principal fibre bundle L^rM.

DEFINITION 5.6. L^rM is called the *bundle of r-frames*. $L^1M = LM$ is the *bundle of linear frames*.

We shall define some objects in L^rM. In order to do this, we assume two action of the group L^r_n on F^r_n

$$(5.7.1) \qquad \lambda_a\colon F^r_n \to F^r_n, \quad \lambda_a(f) = af,$$

$$(5.7.2) \qquad \Lambda_a\colon F^r_n \to F^r_n, \quad \Lambda_a(f) = fa^{-1},$$

It is easy to see that $\lambda_a\circ\Lambda_\beta = \Lambda_\beta\circ\lambda_a$ for all a, $\beta \in L^r_n$, thus

$$(5.7.3) \qquad ad_a = \lambda_a\circ\Lambda_a$$

defines also an action of L^r_n on F^r_n.

DEFINITION 6.8. (F^r_n, ad_a)-objects on L^rM, $n = \dim M$, are called *r-affinors on M*. 1-affinors on M are *tensor fields of the type* $(1, 1)$ *on M*.

Next, let H^r_n be a set of all r-jets at 0 of functions $t\colon R^n \to R$ such that $t(0) = 0$. H^r_n is a manifold diffeomorphic with R^L, where $L = \binom{n+r}{r} - 1$. We define an action on the left of L^r_n on H^r_n, setting

$$(5.9) \qquad h_a\colon H^r_n \to H^r_n, \quad h_a([t]) = [t\circ\varphi^{-1}],$$

where $a = [\varphi] \in L^r_n$.

DEFINITION 5.10. (H^r_n, h_a)-objects on L^rM are called *r-jet fields*, or shortly *r-jets*, on M. 1-jet fields on M are covector fields on M.

We can define

$$\psi\colon F^r_n \times F^r_n \to F^r_n, \quad \Phi(a, \beta) = a\beta,$$

$$\Phi\colon H^r_n \times F^r_n \to F^r_n, \quad \Phi([t], [\varphi]) = [t\circ\varphi],$$

and it is easy to verify that conditions (4.2.1)–(4.2.3), (4.8.1) and (4.8.2) are satisfied.

In the case of r-affinors and r-jets, Proposition 4.7 can be proved in the following form.

PROPOSITION 5.11. *Let A be an r-affinor on M, let Γ_1, Γ_2 be two connections in L^rM and let v: $M \to TM$ be a vector field on M. $\nabla_v^{\Gamma_1\Gamma_2} A = 0$ if and only if for each r-jet t on M and for each integral curve γ of v (i.e., $\gamma(s) = v^{\gamma(s)}$)*

$$\nabla_v^{\Gamma_1} t = 0 \ \text{along}\ \gamma \Rightarrow \nabla_v^{\Gamma_2}(t\cdot A) = 0 \ \text{along}\ \gamma \tag{5.11.1}$$

(we say that $\nabla_v^{\Gamma} t = 0$ along γ if $(\nabla_v^{\Gamma} t)(\gamma(s)) = 0$ for all s).

If A is non-singular (i.e., $A(p) \in L_n^r$ for $p \in L^rM$), then in condition (5.11.1) *the symbol of implication "$\Rightarrow$" can be remplaced by the symbol of equivalence "$\Leftrightarrow$".*

Proof. The same proof as in Proposition 4.7 shows that the condition $\nabla_v^{\Gamma_1\Gamma_2} A$ implies (5.11.1).

Inversely, we suppose that condition (5.11.1) is satisfied. Let us fix a point p_0 in L^rH and let γ be the integral curve of v such that $\gamma(0) = p_0$. For each $a \in H_n^r$ there is an r-jet t on M such that $t(p_0) = a$ and $\nabla_v^{\Gamma_1} t = 0$ along γ. From Leibniz's formula (Proposition 4.14) we obtain

$$\nabla_v^{\Gamma_2}(t\cdot A) = \nabla_v^{\Gamma_1} t\cdot A + t\cdot \nabla_v^{\Gamma_1\Gamma_2} A = 0 \quad \text{along } \gamma,$$

and hence, $a\cdot(\nabla_v^{\Gamma_1\Gamma_2} A)(p_0) = 0$. Since a is any element of H_n^r, we have $(\nabla_v^{\Gamma_1\Gamma_2} A)(p_0) = 0$. This completes our proof because p_0 has been arbitrary.

6. Principal objects. Let us suppose we are given a Lie group G and two endomorphisms φ and ψ of G. We define two actions (on the left) of G on G, setting

$$\lambda_a^{(\varphi)}\colon G \to G, \quad \lambda_a^{(\varphi)}(x) = \varphi(a)x, \tag{6.1.1}$$

$$\Lambda_a^{(\psi)}\colon G \to G, \quad \Lambda_a^{(\psi)}(x) = x\psi(a^{-1}). \tag{6.1.2}$$

It is easy to see that $\lambda_a^{(\varphi)} \circ \Lambda_\beta^{(\psi)} = \Lambda_\beta^{(\psi)} \circ \lambda_a^{(\varphi)}$, and hence we can define another action (on the left) of G on G

$$L_a^{(\varphi,\psi)} = \lambda_a^{(\varphi)} \circ \Lambda_a^{(\psi)}\colon G \to G. \tag{6.1.3}$$

DEFINITION 6.2. $(G, L_a^{(\varphi,\psi)})$-objects on $P(M,G)$ are called *principal objects of the type* (φ,ψ), or shortly *principal* (φ,ψ)*-objects*, on $P(M,G)$. Principal (φ, id)-objects are called *covariant objects of the type* φ and principal (id, φ)-objects are called *contravariant objects of the type* φ. Principal $(\mathrm{id}, \mathrm{id})$-objects are called *$G$-affinors.*

If $G = L_n^r$ and $P = L^rM$, then G-affinors are (non-singular) r-affinors introduced in Section 5.

Let us remark that in the case of the linear group $L_n^1 = GL(n, R)$ and $P = LM$ principal objects on P are tensors (precisely tensor fields) or tensor densities whose types depend only on (φ,ψ). Namely, we have:

Our terminology	Classical terminology
principal (id, id)-object L^1_n-affinor 1-affinor	tensor of the type (1, 1)
covariant object of the type φ_0, where $\varphi_0(X) = (X^{-1})^t$	tensor of the type (0, 2)
contravariant object of the type φ_0	tensor of the type (2, 0)
covariant object of the type φ_1, where $\varphi_1(X) = \varepsilon(\det X)\lvert\det X\rvert^{-a}X$, $\varepsilon \equiv 1$ or $\varepsilon(t) = \operatorname{sgn} t$	tensor W-density (if $\varepsilon = 1$) or tensor G-density (if $\varepsilon(t) = \operatorname{sgn} t$) of the weight a and type (1, 1)
contravariant object of the type φ_1	tensor W-density (if $\varepsilon = 1$) or tensor G-density (if $\varepsilon(t) = \operatorname{sgn} t$) of the weight $-a$ and type (1, 1)
covariant object of the type φ_2, where $\varphi_2(X) = \varepsilon(\det X)\lvert\det X\rvert^{-a}(X^{-1})^t$	tensor W-density (if $\varepsilon = 1$) or tensor G-density (if $\varepsilon(t) = \operatorname{sgn} t$) of the weight a and type (0, 2)
contravariant object of the type φ_2	tensor W-density (if $\varepsilon = 1$) or tensor G-density (if $\varepsilon(t) = \operatorname{sgn} t$) of the weight $-a$ and type (2, 0)

(Let us remark that φ_0, φ_1 and φ_2 are all endomorphisms of $GL(n, R)$ instead of "scalar endomorphisms"; see [7]. For definitions of W- and G-densities, see [5].)

It is easy to see the following

PROPOSITION 6.3. (a) *If A is a principal (φ, ψ)-object and B is a principal (φ, χ)-object on $P(M, G)$, then $A\cdot B$, $(A\cdot B)(p) = A(p)\cdot B(p)$, is a principal (φ, χ)-object on $P(M, G)$.*

(b) *If A is a principal (φ, ψ)-object on $P(M, G)$, then $A^{-1}\colon P\to G$, $A^{-1}(p) = \big(A(p)\big)^{-1}$ is a principal (ψ, φ)-object, and furthermore*

$$\nabla_v^{\Gamma_1\Gamma_2}A = 0 \Leftrightarrow \nabla_v^{\Gamma_1\Gamma_2}A^{-1} = 0.$$

Proof. We prove only the last formula. In order to do this, let us remark that $A\cdot A^{-1} = \delta$ is a G-affinor such that $\delta(p) = 1$ for $p\in P$. Thus, applying Leibniz's formula, from $\nabla_v^{\Gamma_1}\delta = 0$ we obtain

$$(\nabla_v^{\Gamma_1\Gamma_2}A)\cdot A^{-1} + A\cdot(\nabla_v^{\Gamma_1\Gamma_2}A^{-1}) = 0,$$

and this proves the equivalence in question.

Now we shall prove a very important proposition for our consideration.

PROPOSITION 6.4. *Let $\varphi\colon G \to G$ be an endomorphism and*

$$H^{\varphi} = \{\xi \in G\colon \varphi(\xi) = \xi\}.$$

For each reduced bundle $P_0(M, H^{\varphi})$ of $P(M, G)$ there is one and only one covariant object A of the type φ on $P(M, G)$ such that $A(p) = 1$ for $p \in P_0$. Furthermore, the correspondence which with $P_0(M, H^{\varphi})$ associates the object A is a one-to-one correspondence between the set of reduced bundles $P_0(M, H^{\varphi})$ and the set of covariant objects A of the type φ on $P(M, G)$ such that $A(P) \subset \mathcal{O}^{\varphi}$, where $\mathcal{O}^{\varphi}$ denotes the orbit through 1 of the action $L^{(\varphi, \mathrm{id})}$.

Proof. Let $P_0(M, H^{\varphi})$ be a reduced bundle of $P(M, G)$ and let A be a covariant object on $P(M, G)$ of the type φ such that $A(p_0) = 1$ for $p_0 \in P_0$. Now, if $p \in P$, then there are $p_0 \in P_0$ and $\xi \in G$ such that $p = p_0 \cdot \xi$, and hence

$$A(p) = A(p_0 \cdot \xi) = (A \circ R_{\xi})(p_0) = (L^{(\varphi, \mathrm{id})}_{\xi^{-1}} \circ A)(p_0) = \varphi(\xi^{-1})\xi.$$

This proves the uniqueness of A.

To prove the existence of A associated with a reduced bundle $P_0(M, H)$ we define

$$A(p) = \varphi(\xi^{-1})\xi, \tag{6.4.1}$$

where $p = p_0 \cdot \xi$ and $p_0 \in P_0 \cdot$ $A(p)$ does not depend on the choice of p_0 in P_0. Exactly, if $p = p_0 \cdot \xi = p_0' \cdot \xi'$, then $p_0' = p_0 \cdot \lambda$ for some element $\lambda \in H^{\varphi}$. From $p_0 \cdot \xi = p_0 \cdot \lambda \xi'$ follows $\xi = \lambda \xi'$, and hence

$$\varphi(\xi^{-1})\xi = \varphi(\xi'^{-1})\varphi(\lambda^{-1})\lambda\xi' = \varphi(\xi'^{-1})\xi'.$$

Now it is easy to verify that A is a covariant object on $P(M, G)$ of the type φ such that $A(p) = 1$ for $p \in P_0$ and $A(P) \subset \mathcal{O}^{\varphi}$.

In order to prove the second part of our proposition, let $A\colon P \to G$ be a covariant object of the type φ such that $A(P) \subset \mathcal{O}^{\varphi}$. We define

$$P_0 = \{p \in P\colon A(p) = 1\}.$$

We must verify that P_0 is a reduced bundle of $P(M, G)$ with H^{φ} as a structural group, that is, we must verify two conditions:

(i) if $p_0 \in P_0$, then $p_0 \cdot \xi \in P_0 \Leftrightarrow \xi \in H^{\varphi}$,

(ii) in some neighbourhood of each point of M there is a section $\sigma\colon U \to P$ such that $\sigma(U) \subset P_0$.

The first condition is trivial. To prove the second one, let $\tilde{\sigma}\colon U \to P$ be any section defined in some neighbourhood U of a point $x_0 \in M$, and write $\eta_0 = (A \circ \sigma)(x_0) \in \mathcal{O}^{\varphi}$. We consider the mapping

$$\varkappa\colon G \ni \xi \to \varphi(\xi^{-1})\xi \in \mathcal{O}^{\varphi}.$$

It is easy to see that

$$\varkappa(\xi_1) = \varkappa(\xi_2) \Leftrightarrow \xi_1 \xi^{-1} \in H^{\varphi},$$

and hence $\varkappa$ induces a diffeomorphism ψ for which the diagram commutes

$$\begin{tikzcd} G \arrow[rr, "\varkappa"] \arrow[dr, "\pi"'] & & \mathcal{O}^{\varphi} \\ & G/H^{\varphi} \arrow[ur, "\psi"'] & \end{tikzcd}$$

Since π: $G \to G/H^{\varphi}$ is a principal fibre bundle (see [6], p. 55, example 5.1), in some neighbourhood of the point $\psi^{-1}(\eta_0)$ there is a section $\tilde{\varrho}$ of π. Now $\tilde{\varrho} = \tilde{\varrho} \circ \psi^{-1}$ is defined in some neighbourhood of η_0 and $\varkappa \circ \varrho = \mathrm{id}$, that is,

$$\varkappa\big((\varrho(\eta))^{-1}\big)\varrho(\eta) = \eta. \tag{6.4.2}$$

Next we define a section

$$\sigma\colon\ U \to P, \qquad \sigma(x) = \tilde{\sigma}(x)\cdot \varrho\Big(\big(A(\tilde{\sigma}(x))\big)^{-1}\Big).$$

$\sigma(x)$ belongs to P_0 because

$$\begin{aligned} (A\circ\sigma)(x) &= A\Big(\tilde{\sigma}(x)\cdot\varrho\big((A\circ\tilde{\sigma})(x)\big)^{-1}\Big) \\ &= \varphi\Big[\varrho\Big(A\big(\sigma(x)\big)\Big)\Big]A\big(\tilde{\sigma}(x)\big)\Big[\varrho\Big(A\big(\tilde{\sigma}(x)\big)\Big)\Big]^{-1} \\ &= \varphi\big(\varrho(\eta)\big)\,\eta\,\big(\varrho(\eta)\big)^{-1}, \qquad \text{where } \eta = A\big(\tilde{\sigma}(x)\big) \\ &\overset{(6.4.2)}{=} 1, \end{aligned}$$

and this completes the proof.

To end this section we state the following notation: for the action $L_a^{(\varphi,\psi)}$ of G on G and $\eta \in G$, we denote by

$$\mathcal{O}_{\eta}^{(\varphi,\psi)} = \{\varphi(\xi)\eta\psi(\xi^{-1})\colon\ \xi \in G\} \tag{6.5}$$

the orbit of G through η.

7. Definition of conjugate connections. Suppose we are given a principal fibre bundle $P(M, G)$, two connections Γ_1 and Γ_2 in $P(M, G)$ and an endomorphism Φ: $G \to G$ of the Lie group G. Let ω_1 and ω_2 denote connection forms of Γ_1 and Γ_2, respectively.

DEFINITION 7.1. Γ_1 is called *Φ-conjugate* with Γ_2 if there is a reduced fibre bundle $P_0(M, H)$ of $P(M, G)$, where

$$H = \{\xi \in G\colon\ \Phi(\xi) = \xi\}, \tag{7.1.1}$$

such that for every local section σ: $U \to P_0$ of $P_0(M, H)$ we have

$$\sigma^*\omega_2 = \mathscr{L}\Phi\cdot\sigma^*\omega_1, \tag{7.1.2}$$

where $\sigma^*\omega_i$ is an inverse image of ω_i by σ, $\mathscr{L}\Phi$ is an endomorphism of the Lie algebra $\mathscr{L}(G)$ induced by Φ and the operation $\mathscr{L}\Phi\cdot\sigma^*\omega_1$ is defined by the formula $(L\Phi\cdot\sigma^*\omega_1)_x = \mathscr{L}\Phi\circ(\sigma^*\omega_1)_x$ for x in U.

In [3] we prove that if Φ is an involutive automorphism, then this definition coincides with the definition of W. Wiediernikov [11].

Now we prove the following lemma.

LEMMA 7.2. *Γ_1 is Φ-conjugate with Γ_2 if and only if there is a reduced fibre bundle $P_0(M, H)$, where H is defined by* (7.1.1) *such that for any point p_0 in P_0 and for any trivialization of P in some nieghbourhood U of $x_0 = \pi(p_0)$ such that $P|U = U\times G$, $P_0|U = U\times H$, $p_0 = (x_0, e)$ (precisely, the above formulas give some identifications), and for any vector field v: $M\to TM$, we have*

$$H_v^{\Gamma_1}(p_0) = v_{x_0}\oplus w,\qquad H_v^{\Gamma_2}(p_0) = v_{x_0}\oplus(d_e\Phi)(w), \tag{7.2.1}$$

for some vector $w\in T_eG$.

Proof. To begin with, we suppose that Γ_1 is Φ-conjugate with Γ_2 and let $P_0(M, H)$ be a reduced bundle of $P(M, G)$ satisfying condition (7.1.2). Let we fix a point $p_0\in P_0$, a trivialization of $P(M, G)$ such that $P_0|U = U\times H$, $p_0 = (x_0, e)$, and a vector field v: $M\to TM$. There is an h: $U\to G$ such that $h(x_0) = e$ and

$$H_v^{\Gamma_1}(p_0) = d_e\sigma(v_{x_0}) = v_{x_0}\oplus(d_eh)(v_{x_0}) = v_{x_0}\oplus w, \tag{7.2.2}$$

where σ: $U\to P$, $\sigma(x) = \big(x, h(x)\big)$. If we denote by $\sigma_0(x) = (x, e)$ a section of P_0, then $\sigma = \sigma_0\cdot h$ and from Proposition 1.4 in [6], p. 66, identifying $\mathscr{L}(G)$ with T_eG, we obtain

$$\begin{aligned} 0 = \omega_1\big(H_v^{\Gamma_1}(p_0)\big) &= \omega_1\big(d_{x_0}\sigma(v_{x_0})\big) = (\sigma^*\omega_1)_{x_0}(v_{x_0}) \\ &= [ad_{h^{-1}(x_0)}\circ(\sigma^*\omega_1)_{x_0}](v_{x_0}) + d_{x_0}h(v_{x_0}) \\ &= (\sigma_0^*\omega_1)_{x_0}(v_{x_0}) + w \end{aligned}$$

or

$$(\sigma_0^*\omega_1)_{x_0}(v_{x_0}) = -w. \tag{7.2.3}$$

Now, from (7.2.2) and (7.2.3) we obtain

$$\begin{aligned} \omega_2\big(H_v^{\Gamma_2}(p_0)\big) &= (\sigma^*\omega_2)_{x_0}(v_{x_0}) = (\sigma_0^*\omega_2)_{x_0}(v_{x_0}) + d_{x_0}h(v_{x_0}) \\ &= \big(\mathscr{L}\Phi\circ(\sigma_0^*\omega_1)_{x_0}(v_{x_0}) + w\big) \\ &= -d_e\Phi(w) + w, \end{aligned}$$

because if we identify $\mathscr{L}(G)$ with T_eG, then $\mathscr{L}\Phi$ is identifies with $d_e\Phi$. This last formula implies (7.2.1), that is, $H_v^{\Gamma_2}(p_0) = v_{x_0}\oplus d_e\Phi(w)$, because

$$\begin{aligned} \omega_2\big(v_{x_0}\oplus d_e\Phi(v_{x_0})\big) &= \omega_2\big(H_{v_0}^{\Gamma_1}(p_0) + 0\oplus\big(d_e\Phi(w) - w\big)\big) \\ &= -d_e\Phi(w) + w + d_e\Phi(w) - w = 0. \end{aligned}$$

(Let us remark that $\omega_2(0\oplus u) = u$.)

Secondly, we shall prove the sufficient condition. We suppose that there is a reduced bundle $P_0(M, H)$ of $P(M, G)$ satisfying (7.2.1). We must show that for each section $\sigma\colon U \to P_0$ of P_0 formula (7.1.2) is true. In order to do this, let us fix a section $\sigma\colon U \to P_0$. It defines a trivialization of $P(M, G)|U$ such that $P_0|U = U\times H$ and $\sigma(x) = (x, e)$ for $x \in U$. Let us fix an arbitrary point x_0 of U and an arbitrary vector u_0 of $T_{x_0}M$. We can choose a vector field v on M such that $v_{x_0} = u_0$. Since for $u \in T_pP$

(7.2.5) $$\omega_i(u) = \omega_i\big(u - H_v^{\Gamma_i}(p)\big),$$

we have

$$\begin{aligned}
(\sigma^*\omega_2)_{x_0}(u_0) &= \omega_2\big(d_{x_0}\sigma(u_0)\big) = \omega_2(u_0+0) && \text{from (7.2.5),}\\
&= \omega_2\big(u_0\oplus 0 - H_v^{\Gamma_2}(\sigma(x_0))\big) && \text{from (7.2.1),}\\
&= \omega_2\big(0\oplus(-d_e\Phi)(w)\big) && \\
&= -d_e\Phi(w) && \text{because } \omega_i(0+u) = u \\
& && \text{for } u\in T_eG,\\
&= -(d_e\Phi\circ\omega_1)_{x_0}(0\oplus w) && \text{from (7.2.5),}\\
&= -(d_e\Phi\circ\omega_1)_{x_0}\big(0\oplus w - H_v^{\Gamma_1}(\sigma(x_0))\big) && \text{from (7.2.1),}\\
&= -(d_e\Phi\circ\omega_1)_{x_0}(u_0\oplus 0) && \text{from } (d_{x_0}\sigma)(v_{x_0}) = u_0\oplus 0,\\
&= (d_e\Phi\circ\omega_1)\big(d_{x_0}\sigma(u_0)\big) && \\
&= (\mathscr{L}\Phi\cdot\sigma^*\omega_1)_{x_0}(u_0). &&
\end{aligned}$$

Since x_0 and u_0 have been arbitrary, formula (7.1.2) is proved.

8. Geometric interpretation of conjugate connections — the main theorem. Now we prove our main theorem:

THEOREM 8.1. *Let Γ_1 and Γ_2 be two connections in a principal fibre bundle $P(M, G)$, and let Φ be an endomorphism of the group G. We denote by ω_1 and ω_2 connection forms of Γ_1 and Γ_2, respectively. The following conditions are equivalent:*

(8.1.1) *Γ_1 is Φ-conjugate with Γ_2.*

(8.1.2) *There is a covariant object A on $P(M, G)$ of the type Φ such that $A(P) \subset \mathcal{O}_1^{(\varphi,\mathrm{id})}$, and for each vector field v on M*

$$\nabla_v^{\Gamma_1\Gamma_2}A = 0.$$

(8.1.3) *There is a contravariant object B on $P(M, G)$ of the type Φ such that $B(P) \subset \mathcal{O}_1^{(\mathrm{id},\varphi)}$, and for each vector field v on M*

$$\nabla_v^{\Gamma_2\Gamma_1}B = 0.$$

(*The sets $\mathcal{O}_1^{(\varphi,\mathrm{id})}$ and $\mathcal{O}_1^{(\mathrm{id},\varphi)}$ are defined by* (7.5.).)

Proof. If $A\colon P\to G$ is a covariant object of the type Φ, then $B = A^{-1}$ is a contravariant object of the type Φ and the equivalence of (8.1.2) and (8.1.3) follows immediately from Proposition 6.3. Thus, we need only prove the equivalence of conditions (8.1.1) and (8.1.2).

According to Lemma 7.2 condition (8.1.1) is equivalent to the existence of a reduced fibre bundle $P_0(M, H)$ of $P(M, G)$ satisfying (7.2.1). Let A be a covariant object on $P(M, G)$ of the type Φ associated with $P_0(M, H)$ (see Proposition 6.4). If we fix a point $p_0 \in P_0$ and a trivialization of $P(M, G)|U$ in some neighbourhood U of $x_0 = \pi(p_0)$ such that $P_0|U = U \times H$ and $p_0 = (x_0, e)$, then the object A is given by the formula

$$(8.1.4) \qquad A(x, \xi) = (A \circ R_\xi)(x, e) = L^{(\Phi, \mathrm{id})}_{\xi^{-1}}\big(A(x, e)\big) = \Phi(\xi^{-1})\xi$$

for $(x, \xi) \in P|U = U \times G$. The trivialization of $P|U$ defines also a trivialization of $(P+P)|U = U \times G \times G$ and the object $\bar{A}$ (see (2.5.1)) is given by

$$(8.1.5) \qquad \bar{A}(x, \xi, \eta) = A^{(\Phi)}_{\eta^{-1}\xi}\big(A(x, \xi)\big) = \Phi(\xi^{-1})\eta = A^0(\xi, \eta).$$

If we introduce the notation

$$(8.1.6) \qquad H^{\Gamma_1}_v(p_0) = v_{x_0} \oplus w, \quad H^{\Gamma_2}_v(p_0) = v_{x_0} \oplus \bar{w},$$

then we have

$$\begin{aligned}(\nabla^{\Gamma_1\Gamma_2}_v A)(p_0) &= d_{(p_0, p_0)}\bar{A}\big(H^{\Gamma_1}_v(p_0) \oplus H^{\Gamma_2}_v(p_0)\big) \\ &= d_{(e,e)}A^0(w \otimes w) \quad \text{from Proposition 0}, \\ &= d_e{}'A^0(w) + d_e{}''A^0(w),\end{aligned}$$

where

$$('A^0)(\xi) = A^0(\xi, e) = \Phi(\xi^{-1}) = (\Phi \circ k)(\xi), \quad k(\xi) = \xi^{-1},$$
$$(''A^0)(\xi) = A^0(e, \xi) = \xi = \mathrm{id}(\xi).$$

Since $d_e k = -\mathrm{id}$, we obtain

$$(8.1.7) \qquad (\nabla^{\Gamma_1\Gamma_2}_v A)(p_0) = -d_e\Phi(w) + w.$$

This last formula implies that $P_0(M, H)$ satisfies (7.2.1) if and only if $\nabla^{\Gamma_1\Gamma_2}_v A = 0$, and hence conditions (8.1.1) and (8.1.2) are equivalent.

We can also prove other forms of the main theorem.

THEOREM 8.2. *Let $\Phi = ad_\eta \circ \psi$, where ψ is an endomorphism of G and $\eta \in G$, and let Γ_1 and Γ_2 be two connections in $P(M, G)$. The following conditions are equivalent:*

(8.2.1) *Γ_1 is Φ-conjugate with Γ_2.*

(8.2.2) *There is a covariant object C on $P(M, G)$ of the type ψ such that $C(P) = \mathcal{O}^{(\psi, \mathrm{id})}_\eta$, and for each vector v on M*

$$\nabla^{\Gamma_1\Gamma_2}_v C = 0.$$

(8.2.3) *There is a contravariant object D on $P(M, G)$ of the type ψ such that $D(P) = \mathcal{O}_{\eta^{-1}}^{(\mathrm{id},\psi)}$, and for each vector field v on M*

$$\nabla_v^{\Gamma_2\Gamma_1} D = 0.$$

Proof. Let C be a covariant object of the type ψ and write $D = C^{-1}$. Now

$$\begin{aligned} C(P) &= \{\psi(\xi^{-1})\eta\xi\colon\ \xi\in G\} = \mathcal{O}_{\eta}^{(\psi,\mathrm{id})},\\ &\Updownarrow\\ D(P) &= \{\xi^{-1}\eta^{-1}\psi(\xi)\colon\ \xi\in G\} = \mathcal{O}_{\eta^{-1}}^{(\mathrm{id},\psi)}\end{aligned}$$

and hence Proposition 6.3 implies the equivalence of (8.2.2) and (8.2.3). Thus, in order to prove our theorem we only need to show the equivalence of conditions (8.1.2) and (8.2.2).

Let $A\colon P\to G$ be a covariant object of the type Φ, and set

(8.2.4) $$C\colon P\to G,\quad C(p) = \eta^{-1}A(p) = (L_{\eta^{-1}}\circ A)(p),$$

where $L_{\eta^{-1}}$ is the standard left translation on G. C is a covariant object of the type ψ because

$$\begin{aligned} C(p\cdot\xi) &= \eta^{-1}A(p\cdot\xi) = \eta^{-1}\Phi(\xi^{-1})A(p) = \psi(\xi^{-1})\eta^{-1}A(p)\\ &= (L_{\xi^{-1}}^{(\psi,\mathrm{id})}\circ C)(p).\end{aligned}$$

Next it is easy to see that

(8.2.5) $$\begin{aligned} A(P) &= \{\Phi(\xi^{-1})\xi\colon\ \xi\in G\} = \mathcal{O}_{1}^{(\psi,\mathrm{id})},\\ &\Updownarrow\\ C(P) &= \{\psi(\xi^{-1})\eta\xi\colon\ \xi\in G\} = \mathcal{O}_{\eta}^{(\psi,\mathrm{id})}.\end{aligned}$$

Let us remark that (see (2.5.1))

$$\begin{aligned}\bar{C}(p, q) &= \Lambda_{g(p,q)}^{(\psi)}\big(C(p)\big) = \psi\big(g(p, q)\big)\eta^{-1}A(p) = \eta^{-1}\Phi\big(g(p, q)\big)A(p)\\ &= (L_{\eta^{-1}}\circ\bar{A})(p, q),\end{aligned}$$

and hence,

$$\nabla_v^{\Gamma_1\Gamma_2}C = d\bar{C}\cdot H_v^{\Gamma_1+\Gamma_2}\circ\Delta = dL_{\eta^{-1}}\circ d\bar{A}\circ H_v^{\Gamma_1+\Gamma_2}\circ\Delta = dL_{\eta^{-1}}\circ\nabla_v^{\Gamma_1\Gamma_2}A.$$

Since $dL_{\eta^{-1}}$ is an isomorphism we have

(8.2.6) $$\nabla_v^{\Gamma_1\Gamma_2}C = 0 \Leftrightarrow \nabla_v^{\Gamma_1\Gamma_2}A = 0.$$

Formulas (8.2.5) and (8.2.6) prove the equivalence of (8.2.2) and (8.2.3).

Now follows

THEOREM 8.2. *The following conditions are equivalent:*

(8.3.1) *Γ_1 is ad_η-conjugate with Γ_2.*

(8.3.2) *There is a G-afinor C on $P(M, G)$ such that $C(P) \subset \mathcal{O}_{\eta}^{(\mathrm{id}, \mathrm{id})}$, and*

$$\nabla_v^{\Gamma_1 \Gamma_2} C = 0$$

for each vector v on M.

(8.3.3) *There is a G-affinor D on $P(M, G)$ such that $D(P) \subset \mathcal{O}_{\eta^{-1}}^{(\mathrm{id}, \mathrm{id})}$, and*

$$\nabla_v^{\Gamma_1 \Gamma_2} D = 0$$

for each vector field v on M.

9. Image of conjugate connections. Let $P(M, G)$ and $P'(M, G')$ be two principal fibre bundles over the same base M, and let $f = (f_P, f_G)$ be a homomorphism of $P(M, G)$ into $P'(M, G')$; i.e., $f_G\colon G \to G'$ is a homomorphism of Lie groups and $f_P\colon P \to P'$ is a differentiable mapping which makes the diagram

$$\begin{tikzcd} P \arrow[rr, "f_P"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutative, and, furthermore, $f_P(p \cdot \xi) = f_P(p) \cdot f_G(\xi)$ for $p \in P$ and $\xi \in G$.

Let Γ be a connection in $P(M, G)$. From Proposition 6.1 in [6], p. 79, there is one and only one connection Γ' in $P'(M, G')$ such that df_P transforms horizontal vectors for Γ into horizontal vectors for Γ'. If ω and ω' are the connection forms of Γ and Γ', respectively, then

$$f_P^* \omega' = \mathscr{L}(f_G) \cdot \omega \tag{9.1}$$

(see Proposition 6.1 in [6], p. 79). We write $\Gamma' = f(\Gamma)$.

Next, suppose we are given two manifolds F, F' and families of left translations

$$\begin{aligned} &\lambda_\xi, \Lambda_\xi\colon F \to F \quad (\xi \in G), \\ &\lambda'_{\xi'}, \Lambda'_{\xi'}\colon F' \to F' \quad (\xi' \in G') \end{aligned} \tag{9.2}$$

such that $\lambda_\xi \circ \Lambda_\eta = \Lambda_\eta \circ \lambda_\xi$ and $\lambda'_{\xi'} \circ \Lambda'_{\eta'} = \Lambda'_{\eta'} \circ \lambda'_{\xi'}$. Also, let $l\colon F \to F'$ be a mapping such that for all $\xi \in G$

$$l \circ \lambda_\xi = \lambda'_{f_G(\xi)} \circ l, \quad l \circ \Lambda_\xi = \Lambda_{f_G(\xi)} \circ l. \tag{9.3}$$

For a homomorphism $f = (f_P, f_G)$ of $P(M, G)$ into $P'(M, G')$ we write

$$f + f = (f_P + f_P, f_G \times f_G)\colon P(M, G) + P(M, G) \to P'(M, G') + P'(M, G'),$$

where $f_P + f_P$ is the restriction of $f_P \times f_P$ to $P + P \subset P \times P$. It is clear that $f + f$ is a homomorphism.

Now we show two lemmas.

LEMMA 9.4. *If Γ_1 and Γ_2 are two connections in $P(M, G)$, then*

$$d(f_P+f_P)\circ H_v^{\Gamma_1+\Gamma_2} = H_v^{f(\Gamma_1)+f(\Gamma_2)}\circ(f_P+f_P).$$

Proof. If ω_i and ω_j' denote the connection forms of Γ_i and $f(\Gamma_i)$, respectively, then, by (9.1), we have

$$f_P^*\omega_i' = \mathscr{L}(f_G)\cdot\omega_i,$$

and hence, for each $p\in P$, the condition $(\omega_i)_p\big(H_v^{\Gamma_i}(p)\big) = 0$ implies

$$0 = \big(\mathscr{L}(f_G)\cdot\omega_i\big)_p\big(H_v^{\Gamma_i}(v)\big) = f_P^*\omega_i'\big(H_v^{\Gamma_i}(p)\big) = (\omega_i)_{f_P(p)}\Big(d_pf_P\big(H_v^{\Gamma_i}(p)\big)\Big).$$

Thus $H_v^{f(\Gamma_i)}\big(f_P(p)\big) = d_pf_P\big(H_v^{\Gamma_i}(p)\big)$ or $H_v^{f(\Gamma_i)}\circ f_P = df_P\circ H_v^{\Gamma_i}$. By applying (2.7) this formula implies our lemma.

LEMMA 9.5. *If $A\colon P\to F$ and $A'\colon P'\to F'$ are an $(F, \lambda_\xi\circ\Lambda_\xi)$-object and an $(F', \lambda'_{\xi'}\circ\Lambda'_{\xi'})$-object on $P(M, G)$ and $P'(M, G')$, respectively, such that*

(9.5.1) $$l\circ A = A'\circ l,$$

then

$$dl\circ\nabla_v^{\Gamma_1\Gamma_2}A = \nabla_v^{f(\Gamma_1)f(\Gamma_2)}A'\circ f_P.$$

Proof. As usual, we define $\bar{A}$ and $\bar{A}'$

$$\bar{A}(p, q) = \Lambda_{g(p,q)}\big(A(p)\big),\qquad \bar{A}'(p, q) = \Lambda'_{g'(p,q)}\big(A'(p)\big),$$

where $g\colon P+P\to G$, $g'\colon P'+P'\to G'$ are defined by (2.4). Let us remark that

$$\begin{aligned}
[\bar{A}'\circ(f_P+f_P)](p, q) &= A'\big(f_P(p), f_P(q)\big)\\
&= \Lambda'_{g'(f(p),f(q))}\Big(A'\big(f_P(p)\big)\Big) && \text{from (9.5.1)},\\
&= \Lambda'_{g'(f_P(p),f_P(q))}\big((l\circ A)(p)\big) && \text{from (9.3)},\\
&= (l\circ\Lambda_\xi\circ A)(p),
\end{aligned}$$

where ξ is any element of G such that $f_G(\xi) = g'\big(f_P(p), f_P(q)\big)$. Since the equality $q = p\cdot g(p, q)$ implies

$$f_P(q) = f_P(p)\cdot f_G\big(g(p, q)\big),$$

and hence $g'\big(f_P(p), f_P(q)\big) = f_G\big(g(p, q)\big)$, we can put $\xi = g(p, q)$ in the last formula. Now we obtain

$$\bar{A}'\circ(f_P+f_P) = l\circ\bar{A},$$

and hence

$$\begin{aligned}
\nabla_v^{f(\Gamma_1)f(\Gamma_2)}A'\circ f_P &= d\bar{A}'\circ H_v^{f(\Gamma_1)+f(\Gamma_2)}\circ\Delta'\circ f_P\\
&= d\bar{A}\circ H_v^{f(\Gamma_1)+f(\Gamma_2)}\circ(f_P+f_P)\circ\Delta\\
&= dl\circ d\bar{A}\circ H_v^{\Gamma_1+\Gamma_2}\circ\Delta = dl\circ\nabla_v^{\Gamma_1\Gamma_2}A.
\end{aligned}$$

Now we can prove

PROPOSITION 9.6. *Let $P = (f_P, f_G)$ be an epimorphism of $P(M, G)$ onto $P'(M, G')$ and $\varphi\colon G \to G$, $\varphi'\colon G' \to G'$ be endomorphisms of G and G' respectively such that $f_G \circ \varphi = \varphi' \circ f_G$. If Γ_1 is φ-conjugate with Γ_2, then $f(\Gamma_1)$ is φ'-conjugate with $f(\Gamma_2)$.*

Proof. By Theorem 8.1, there is a covariant object A on $P(M, G)$ of the type φ such that $A(P) \subset \mathcal{O}^{\varphi}$ and $\nabla_v^{\Gamma_1\Gamma_2} A = 0$. Let $P_0(M, H)$ be a reduced bundle of $P(M, G)$ associated with A (see Proposition 6.4), where

$$H = \{\xi \in G\colon \varphi(\xi) = \xi\}.$$

Now $\tilde{P}_0 = f_P(P_0)$ is a reduct bundle of $P'(M, G')$ with structural group $\tilde{H}' = f_G(H)$. Let us remark that

$$\begin{aligned} \xi \in H' &\Leftrightarrow \xi \in H\colon\ \xi' = f_G(\xi) \\ &\Leftrightarrow \xi \in G\colon\ \xi' = f_G(\xi) \quad \text{and} \quad \varphi(\xi) = \xi \\ &\Leftrightarrow \xi' = (f_G \circ \varphi)(\xi) = (\varphi' \circ f_G)(\xi) = \varphi'(\xi'), \end{aligned}$$

and thus

$$\tilde{H}' \subset \{\xi' \in G'\colon \varphi'(\xi') = \xi'\} = H'.$$

By Proposition 5.3 in [6], p. 53, there is one and only one reduced bundle $P_0'(M, H')$ of $P'(M, G')$ such that $\tilde{P}_0' \subset P_0'$, and, by Proposition 6.4, $P_0'(M, H')$ determines a covariant object A' of the type φ' on $P'(M, G')$ such that

$$P_0' = \{p \in P'\colon A'(p') = 1\}.$$

Now it is clear that $A'(P') \subset \mathcal{O}_1^{(\varphi', \mathrm{id})}$ and, for $p = p_0 \cdot \xi$, $p_0 \in P_0$

$$\begin{aligned} (A' \circ f_P)(p) &= A'\big(f_P(p_0) \cdot f_G(\xi)\big) && \text{since } A' \text{ is of type } \varphi', \text{ and thus} \\ &= \varphi'\big(f_G(\xi^{-1})\big) A'\big(f_P(p_0)\big) f_G(\xi) && \text{from } f_G \circ \varphi = \varphi' \circ f_G, \text{ and} \\ & && f_P(p_0) \in P_0', \\ &= f_G\big(\varphi(\xi^{-1})\big) f_G(\xi) \\ &= f_G\big(\varphi(\xi^{-1})\,\xi\big) \\ &= (f_G \circ A)(p) && \text{because } A(p) = \varphi(\xi^{-1}) A(p_0)\,\xi \\ & && \qquad\qquad = \varphi(\xi^{-1})1 \end{aligned}$$

or

$$A' \circ f_G = f_G \circ A.$$

Thus Lemma 9.5 implies (for $l = f_G$) $df_G \circ \nabla_v^{\Gamma_1\Gamma_2} A = \nabla_v^{f(\Gamma_1)f(\Gamma_2)} A' \circ f_G = 0$. Since f_P is surjective, $\nabla_v^{f(\Gamma_1)f(\Gamma_2)} A' = 0$, i.e. $f(\Gamma_1)$ is φ'-conjugate with $f(\Gamma_2)$.

10. Conjugate connection of the order r, $r \geqslant 2$. For investigations of couples of conjugate connections defined in $P(M, G)$ it is important to know a classification of endomorphisms of the group G. Such a classification is due to M. Kucharzewski and A. Zajtz [7] for the linear group $GL(n, R) = L_n^1$ and to A. Zajtz [12], [13] for the group L_n^r, $r \geqslant 2$.

Conjugate linear connections were considered by the author in [3]. Now we explain the case of connections of the order r, $r \geqslant 2$. First of all we recall the classification of endomorphisms of the group L_n^r, $r \geqslant 2$.

Let us remark that $L_n^2 = GL(n, R) \times \Omega^1$, where

$$\Omega^1 = \{(a_{jk}^i) \in R^{n^3}\colon\ a_{jk}^i = a_{kj}^i\}$$

and for $\alpha = (A, X)$, $\beta = (B, Y) \in L_n^2$

$$\alpha\beta = (AB, X \cdot B + A \cdot Y),$$

where $A = [a_j^i]$, $B = [b_j^i]$, $X = [x_{jk}^i]$, $Y = [y_{jk}^i]$ and

$$X \cdot B = [x_{rs}^i b_j^r b_k^a], \qquad A \cdot Y = [a_s^i y_{jk}^s].$$

It is clear that $(A, X)^{-1} = (A^{-1}, -A^{-1} \cdot X \cdot A^{-1})$.

For $r > 2$, we have

$$L_n^r = L_n^{r-1} \times \Omega^{r-1} = GL(n, R) \times \Omega^1 \times \ldots \times \Omega^{r-1},$$

where

$$\Omega^{r-1} = \{(a_{i_1 \ldots i_r}^i) \in R^{n^{r+1}}\colon\ a_{i_1 \ldots i_r}^i \text{ is symmetric with regard to } i_1, \ldots, i\},$$

and, furthermore, the canonical projections $L_n^r \to L_n^{r-1}$ and the canonical inclusions $GL(n, R) \to L_n^r$ are homomorophisms.

Now we can formulate (see [13])

PROPOSITION 10.1. (a) *If $\varphi\colon L_n^2 \to L_n^2$ is a homomorphism, then either*

(10.1.1) *φ is an inner automorphisms, or*

(10.1.2) *φ is a composition of an inner automorphism and a homomorphism $\Phi_{(\varkappa,\nu)}$ for some constants $\varkappa$, $\nu \in R$, where*

$$\Phi_{(\varkappa,\nu)}(A, X) = \left(A, \varkappa\left(x_{jk}^i - \frac{2}{n+1} a_{(j}^i x_{k)s}^r b_r^s\right) + \frac{2\nu}{n+1} a_{(j}^i x_{k)s}^r b_r^s\right)^{(1)}$$

and $A = [a_j^i]$, $A^{-1} = [b_j^i]$, $X = [x_{jk}^i]$, or

(10.1.3) *φ is a composition of an inner authomorpism and a homorphism*

$$(A, X) \to (\psi(A), 0),$$

(1) (i, k) denotes the symmetrization.

where ψ is an endomorphism of the linear group $GL(n, R)$,

(b) *If φ: $L_n^r \to L_n^r$ is an endomorphism, $r \geqslant 3$, then either φ is an inner automorphism or φ is a composition of an inner automorphism and a homomorphism*

$$(A, X_1, \ldots, X_{r-1}) \to (\psi(A), 0, \ldots, 0),$$

where ψ is an endomorphism of the linear group $GL(n, R)$.

Hence follows immediately (see [12])

PROPOSITION 10.2. (a) *If φ: $L_n^r \to L_n^r$, $r \geqslant 3$, is an automorphism, then φ is an inner automorphism.*

(b) *If φ: $L_n^2 \to L_n^2$ is an automorphism, then φ is a composition of an inner automorphism and a homomorphism $\Phi_{(1,\nu)}$, $\nu \in R$ (let us remark that $\Phi_{(1,0)} = \mathrm{id}$).*

Connections in $L^r M$ are called *connections* of order r on M. The above propositions mean that the most interesting case for investigations of conjugate connections of order r is the case $r = 2$ and the endomorphism $\Phi_{(\varkappa,\nu)}$. We shall consider this case; we begin with some general remarks.

First, Propositions 8.3 and 6.11 imply

PROPOSITION 10.3. *Let Γ_1 and Γ_2 be two connections of order r, $r \geqslant 1$, on M and $\eta \in L_n^r$. Γ_1 is ad_η-conjugate with Γ_2 if and only if there is an r-affinor A on M such that*

$$A(L^r M) \subset \mathcal{O}_\eta^{(\mathrm{id},\mathrm{id})} = \{\xi\eta\xi^{-1}:\ \xi \in G\}$$

and for each vector field v on M, each integral curve γ of v and each r-jet t on M we have

$$\nabla_v^{\Gamma_1} t = 0 \ \text{along}\ \gamma \Leftrightarrow \nabla_v^{\Gamma_2}(t \cdot A) = 0 \ \text{along}\ \gamma.$$

Secondly, for $r \geqslant 2$, let

$$\pi_r: L_n^r \to GL(n, R), \qquad \pi_r: L^r M \to L^1 M = LM$$

denote the canonical projections. It is clear that $\pi_r(p\cdot\xi) = \pi_r(p)\cdot\pi_r(\xi)$. By Proposition 10.1, for each homomorphism Φ: $L_n^r \to L_n^r$ there is a homomorphism φ: $GL(n, R) \to GL(n, R)$ such that the diagram

$$\begin{array}{ccc} L_n^r & \xrightarrow{\quad\Phi\quad} & L_n^r \\ {\scriptstyle\pi_r}\downarrow & & \downarrow{\scriptstyle\pi_r} \\ GL(n, R) & \xrightarrow{\varphi} & GL(n, R) \end{array}$$

commutes. Now, Proposition 9.6 implies

PROPOSITION 10.4. *Let Γ_1 and Γ_2 be connections of oreder r on M, and let Φ: $L_n^r \to L_n^r$, φ: $GL(n, R) \to GL(n, R)$ be endomorphisms such that $\pi_r \circ \Phi = \varphi \circ \pi_r$. If Γ_1 is Φ-conjugate with Γ_2, then $\pi_r(\Gamma_1)$ is φ-conjugate with $\pi_r(\Gamma_2)$.*

Now we shall consider the case $r=2$ and the endomorphism $\Phi_{(\varkappa,\nu)}$ of the group L^2_n for some $\varkappa$, ν.

If we introduce the mapping

$$|\ |\colon L^2_n \to \Omega^1$$

setting for $a = (a^i_j, x^i_{jk})$

$$|a| = (a^i_{(j} x^r_{k)s} b^s_r), \tag{10.5.1}$$

where $[b^i_j] = [a^i_j]^{-1}$, we can write

$$\Phi_{(\varkappa,\nu)}(A, X) = \left(A, \varkappa X + \frac{2(\nu-\varkappa)}{2}|(A, X)|\right). \tag{10.5.2}$$

We define an action of the group L^2_n on R^{n^3}, setting for $a = (A, X) \in L^2_r$

$$\lambda^{(\varrho,\omega)}_a\colon R^{n^3} \to R^{n^3} \quad (\varrho, \omega \in R),$$

$$\begin{aligned} \lambda^{(\varrho,\omega)}_a(u) &= A\cdot u\cdot A^{-1} - \varrho X A^{-1} + \frac{2\omega}{n+1}|a|\cdot A^{-1} \\ &= \left(u^s_{rp} b^r_j b^p_k a^i_s - \varrho x^i_{rs} b^r_j b^s_k + \frac{2\omega}{n+1}\delta^i_{(j} b^p_{k)} x^r_{ps} b^s_r\right), \end{aligned} \tag{10.6}$$

where $u = [u^i_{jk}]$, $X = [x^i_{jk}]$, $A = [a^i_j]$ and $A^{-1} = [b^i_j]$. It is easy to see that

$$\lambda^{(\varrho,\omega)}_a \circ \lambda^{(\varrho,\omega)}_b = \lambda^{(\varrho,\omega)}_{ab}$$

for $a, b \in L^2_n$.

DEFINITION 10.7. An $(R^{n^3}, \lambda^{(\varrho,\omega)}_a)$-object on M, where $\lambda^{(\varrho,\omega)}_a$ is given by (10.6), is called an *object of* (ϱ, ω)*-connection* on M.

We propose the above terminology because:

(10.7.1) If $\varrho = 1$ and $\omega = 0$, we obtain an object of linear connection [8], [5].

(10.7.2) If $\varrho = \omega = 1$, we obtain an object of projective connection [8].

(10.7.3) If $\varrho = 0$ and $\omega = 1$, we obtain an object similar to an object of contractible connection on M, [8], [5].

In our terminology, an object of contractible connection in an (R, h_a)-object on L^2M, where

$$h_a(u) = (u_s b^s_i - b^s_i x^p_{sq} b^q_p) \tag{10.7.3.a}$$

and $a = (A, X)$, $A = [a^i_j]$, $A^{-1} = [b^i_j]$, $X = [x^i_{jk}]$, $u = [u_i]$. In order to verify that an $(R^{n^3}, \lambda_a^{(0,1)})$-object in similar to an (R, h_a)-object, we consider two mappings:

$$(10.7.3.b)\qquad \begin{aligned} &j\colon R^{n^3}\to R, \quad j([x^i_{jk}]) = [x^k_{jk}],\\ &j\colon R\to R^{n^3}, \quad J([x_i]) = \left[\frac{2}{n+1}\,\delta^i_{(j}x_{k)}\right],\end{aligned}$$

and it is easy to see that

$$j\circ\lambda_a^{(0,1)} = h_a\circ j, \qquad J\circ h_a = \lambda_a^{(0,1)}\circ J.$$

These formulas mean that each of the above two objects is a concomitant of the second one [1], [5].

We introduce the following definitions (see [8]).

DEFINITION 10.8.1. An $(\Omega, \lambda_a^{(1,0)})$ object on L^2M is called an *object of symmetric linear connection on M*, where

$$\Omega = \{(a^i_{jk})\in R^{n^3}\colon\ a^i_{jk} = a^i_{kj}\}$$

and $\lambda_a^{(1,0)}$ denotes the restriction (we use the same symbol for this restriction) of $\lambda_a^{(1,0)}$ defined by (10.6) to Ω. It is possible because $\lambda_a^{(1,0)}(\Omega)\subset\Omega$.

DEFINITION 10.8.2. A $(K, \lambda_a^{(1,1)})$-object on L^2M is called an *object of projective connection* on M, where

$$K = \{(x^i_{jk})\in R^{n^3}\colon\ x^k_{ik} = 0 \text{ for } i = 1,\ldots,n\}.$$

It is clear that $\lambda_a^{(1,1)}(K)\subset K$.

DEFINITION 10.8.3. An (R, h_a)-object on L^2M is called an *object of contractible connection* on M, where h_a is given by (10.7.3.a).

Before we formulate our propositions we find some properties of the mapping

$$(10.9.1)\qquad f_{(\varrho,\omega)}\colon L^2_n\to R^{n^3}, \quad f_{(\varrho,\omega)}(a) = \lambda^{(\varrho,\omega)}_{a^{-1}}(0).$$

From the formula $(A, X)^{-1} = (A^{-1}, A^{-1}\cdot X\cdot A^{-1})$ and from (10.6) we have

$$(10.9.2)\qquad f_{(\varrho,\omega)}(a^i_j, x^i_{jk}) = \left(\varrho b^i_s x^s_{jk} - \frac{2\omega}{n+1}\,\delta^i_{(j}x^r_{k)s}b^s_r\right),$$

where $b^i_s a^s_j = \delta^i_j$ [2]. Now we calculate $f_{(\varrho,\omega)}\circ\Phi_{(\varkappa,\nu)}$. Namely we have

$$\begin{aligned}(f_{(\varrho,\omega)}\circ\Phi_{(\varkappa,\nu)})(a^i_j, x^i_{jk}) &= f_{(\varrho,\omega)}\left(a^i_j,\ \varkappa x^i_{jk} + \frac{2(\nu-\varkappa)}{n+1}\,a^i_{(j}x^r_{k)s}b^s_r\right)\\ &= \left(\varrho\varkappa b^i_s x^s_{jk} + \frac{2(\nu-\varkappa)\omega}{n+1}\,\delta^i_{(j}x^s_{k)r}b^r_s + \frac{2\omega}{n+1}\left[\varkappa\delta^i_{(j}x^r_{k)s}b^s_r +\right.\right.\\ &\qquad\left.\left. + \frac{2(\nu-\varkappa)}{n+1}\,\delta^i_{(j}y^q_{k)p}b^p_q\right]\right),\end{aligned}$$

---

(2) In the whole of this section we shall use the following notation: if $A = [a^i_j]$, then $A^{-1} = [b^i_j]$.

where $y^q_{kp} = \delta^q_{(k} x^r_{p)s} b^s_r$. Since $y^q_{pk} b^p_q = \frac{n+1}{2} x^r_{ks} b^s_r$, we have

$$(f_{(\varrho,\omega)} \circ \Phi_{(\varkappa,\nu)})(a^i_j, x^i_{jk}) = \left(\varrho\varkappa b^i_s x^s_{jk} + \frac{2(\nu\varrho - \varkappa\varrho - \omega\nu)}{n+1} \delta^i_{(j} y^q_{k)p} b^p_q\right),$$

that is,

$$f_{(\varrho,\omega)} \circ \Phi_{(\varkappa,\nu)} = f_{(\varrho\varkappa, \omega\nu + \varkappa\varrho - \nu\varrho)}. \tag{10.10}$$

Now we can prove

PROPOSITION 10.11. *Let Γ_1 and Γ_2 be two connections of order 2 on M, and let π: $L^2M \to LM$ be the canonical projection. Γ_1 is Φ_x-conjugate with Γ_2, where $\Phi_\varkappa = \Phi_{(\varkappa,\varkappa)}$, $\varkappa \neq 0, 1$, if and only if $\pi(\Gamma_1) = \pi(\Gamma_2)$ and there is an object of symmetric linear connection Γ on M such that for all vector fields v on M*

$$\nabla^{\Gamma_2}_v \Gamma = \varkappa \nabla^{\Gamma_1}_v \Gamma. \tag{10.11.1}$$

Proof. First of all, we suppose that Γ_1 is $\Phi_\varkappa$-conjugate with Γ_2. According to Lemma 7.2, this means that there is a reduced bundle $P_0(M, H)$ of L^2M, where

$$H = \{a \in L^2_n\colon\ \Phi_\varkappa(a) = a\},$$

such that for each point p_0 in P_0 and for any trivialization of P_0 in some neighbourhood U of the point $x_0 = \pi(p_0)$, such that $L^2M \mid U = U \times L^2_n$, $P_0 \mid U = U \times H$, $p_0 = (x_0, e)$, we have

$$H^{\Gamma_2}_v(p_0) = v_{x_0} \oplus d_e\Phi_\varkappa(w), \qquad H^{\Gamma_1}_v(p_0) = v_{x_0} \oplus w, \tag{10.11.2}$$

for all vectors field v on M.

Let us remark that

$$\begin{aligned} a \in H \Leftrightarrow \Phi_\varkappa(a) &= (a^i_j, \varkappa x^i_{jk}) = a \\ \Leftrightarrow x^i_{jk} &= 0 \quad (\text{because } \varkappa \neq 1). \end{aligned} \tag{10.11.3}$$

Thus $P_0(M, H)$ defines the mapping

$$\Gamma\colon L^2M \to \Omega, \qquad \Gamma(p) = \lambda^{(1,0)}_{a^{-1}}(0) = f_{(1,0)}(a) = (b^i_s x^s_{jk}), \tag{10.11.4}$$

where $p = p_0 \cdot a$ and $p_0 \in P_0$. (10.11.3) implies that Γ is independent of the choice of p_0 in P_0, and it is not difficult to show that Γ is an object of symmetric linear connection on M.

Next, let us fix a point $p_0 \in P_0$ and a trivialization of P_0 for which condition (10.11.2) is satisfied. Now

$$\Gamma(x, a) = f_{(1,0)}(a)$$

for $(x, a) \in L^2M \mid U$. From (10.11.2) we have

$$(\nabla_v^{\Gamma_1}\Gamma)(x_0, e) = d_{(x_0,e)}\Gamma(v_{x_0} \oplus w) = d_e f_{(0,1)}(w)$$

$$(\nabla_v^{\Gamma_2}\Gamma)(x_0, e) = d_{(x_0,e)}\Gamma\big(v_{x_0} + d_e\Phi_\chi(w)\big) = d_e(f_{(0,1)} \circ \Phi_\varkappa)(w).$$

From (10.10) we obtain

$$f_{(1,0)} \circ \Phi_\chi = f_{(1,0)} \circ \Phi_{(\varkappa,\varkappa)} = f_{(\varkappa,0)} = \varkappa f_{(1,0)}, \tag{10.11.5}$$

and hence

$$(\nabla_v^{\Gamma_2}\Gamma)(p_0) = \varkappa(\nabla_v^{\Gamma_1}\Gamma)(p_0)$$

for all p_0 in P_0, and by (1.5) this implies

$$\nabla_v^{\Gamma_2}\Gamma = \varkappa\nabla_v^{\Gamma_1}\Gamma.$$

The condition $\pi(\Gamma_1) = \pi(\Gamma_2)$ follows immediately from Proposition 9.6.

Now we suppose inversely that $\pi(\Gamma_1) = \pi(\Gamma_2)$ and there is an object of symmetric linear connection Γ on M such that condition (10.11.1) holds.

We shall first prove that

$$P_0 = \{p \in L^2M\colon\ \Gamma(p) = 0\}$$

is a reduced bundle of L^2M. Since for each point p_0 of P_0 we have

$$p_0 \cdot a \in P_0 \Leftrightarrow a \in H,$$

where H is given by (10.11.3), in order to verify that P_0 is a reduced bundle we need only to prove that in some neighbourhood U of any point of M there is a section $\sigma\colon U \to L^2M$ with values in P_0. If $\tilde{\sigma}\colon U \to L^2M$ is any section, we define

$$g\colon\ U \to L_n^2, \quad g(x) = \big(I, \tilde{\sigma}(\Gamma(x))\big), \quad I = [\delta_j^i]$$

(g is well defined because Γ is symmetric), and $\sigma = \tilde{\sigma} \cdot g$ is a section of L^2M such that $\sigma(x) \in P_0$ for $x \in U$, because

$$\Gamma\big(\sigma(x)\big) = \Gamma\big(\tilde{\sigma}(x) \cdot g(x)\big) = \lambda^{(1,0)}_{(g(x))^{-1}}\big(\Gamma(\tilde{\sigma}(x))\big) = 0$$

$$\big(g(x)^{-1} = \big(I, -\Gamma(\tilde{\sigma}(x))\big)\big).$$

It is clear that H is a structural group of P_0.

Next we fix a point p_0 in P_0 and a trivialization of P_0 in some neighbourhood U of $x_0 = \pi(p_0)$ such that $L^2M \mid U = U \times L_n^2$, $P_0 \mid U = U \times H$, $p_0 = (h_0, e)$. In order to prove that Γ_1 is $\Phi_\varkappa$-conjugate with Γ_2, by Lemma 7.2, we need only to verify condition (10.11.2).

Let us remark that in our trivialization Γ is defined by formula (10.11.4), and hence, by (10.11.5), condition (10.11.1) means

$$d_e(f_{(1,0)}\circ \Phi_\varkappa)(w) = d_e f_{(1,0)}(\overline{w}),$$

where $H_v^{\Gamma_1}(p_0) = v_{x_0}\oplus w$, $H_v^{\Gamma_2}(p_0) = v_{x_0}\oplus\overline{w}$, that is

(10.11.6) $$\overline{w} - d_e\Phi_\varkappa(w)\in \ker d_e f_{(1,0)}.$$

It is easy to see that

$$\ker d_e f_{(1,0)} = T_I GL(n, R)\oplus 0 \subset T_I GL(n, R) + T_0\Omega = T_e L_r^2,$$

and hence, if $w = w_1\oplus w_2$, $\overline{w} = \overline{w}_1\oplus\overline{w}_2$ are decomposition of w and $\overline{w}$, respectively, such that w_1, $\overline{w}_1\in T_I GL(n, R)$ and w_2, $\overline{w}_2\in T_0\Omega$, then (10.11.6) implies

$$\overline{w}_2 = d_e\Phi_\varkappa(w_2).$$

On the other hand, the condition $\pi(\Gamma_1) = \pi(\Gamma_2)$ implies $w_1 = \overline{w}_1$. Thus we have

$$\overline{w} = d_e\Phi_\varkappa(w),$$

because $\Phi_\varkappa | GL(n, R) = \mathrm{id}$, and this completes the proof of our proposition.

We also show the following

PROPOSITION 10.12. *Γ_1 is $\Phi_\varkappa$-conjugate with Γ_2, where $\psi_\varkappa = \Phi_{(\varkappa,1)}$, $\varkappa \neq 0, 1$, if and only if $\pi(\Gamma_1) = \pi(\Gamma_2)$ and there is an object of projective connection π on M (see Definition 10.8.2) such that*

(10.12.1) $$\nabla_v^{\Gamma_2}\Pi = \varkappa\nabla_v^{\Gamma_1}\Gamma$$

for all vectors field v on M.

Proof. The proof of this proposition is similar to the proof of Proposition 10.11. For this reason we shall not prove the above proposition in detail, but we shall only explain some points of its proof – namely the points it which the proofs of the above two propositions are different.

First, the structural group H of P_0 is given by

$$H = \{a\in L_n^2 \colon \psi_\varkappa(a) = a\},$$

and hence, instead of (10.11.4), we obtain

$$a\in H \Leftrightarrow \psi_\varkappa(a) = \left(A, \varkappa X + \frac{2(1-\varkappa)}{n+1}|a|\right) = (A, X) = a$$

$$\Leftrightarrow \frac{2(1-\varkappa)}{n+1}|a| = (1-\varkappa)X$$

$$\Leftrightarrow \frac{2}{n+1}|a| = X \quad \text{(because } \varkappa \neq 1).$$

Secondly, instead of (10.11.4) we consider the mapping

$$\Pi: L^2M \to K, \qquad \Pi(p) = f_{(1,1)}(a) = \left(b^i_s x^s_{jk} - \frac{2}{n+1}\,\delta^i_{(j} x^r_{k)s} b^s_r\right),$$

where $p = p_0 \cdot a$, $p_0 \in P_0$, which is independent of the choice of p_0 in P_0. Π is an object of projective connection on M.

Thirdly, since (10.10) implies

$$f_{(1,1)} \circ \psi_\varkappa = f_{(1,1)} \circ \Phi_{(\varkappa,1)} = f_{(\varkappa,\varkappa)} = \varkappa f_{(1,1)},$$

instead of (10.11.1) we obtain (10.12.1).

Using the same method we can prove

PROPOSITION 10.13. *Γ_1 is χ_ν-conjugate with Γ_2, where $\chi_\nu = \Phi_{(1,\nu)}$, $\nu \neq 0$, if and only if $\pi(\Gamma_1) = \pi(\Gamma_2)$ and there is an object of $(0,1)$-connection A on M such that*

$$\nabla^{\Gamma_2}_v A = \nu \nabla^{\Gamma_1}_v A \tag{10.13.1}$$

for all vectors field v on M.

Since an object of $(0,1)$-connection is similar to an object of contractible connection, we can prove

PROPOSITION 10.14. *Γ_1 is χ_ν-conjugate with Γ_2, where $\chi_\nu = \Phi_{(1,\nu)}$, $\nu \neq 0$, if and only if $\pi(\Gamma_1) = \pi(\Gamma_2)$ and there is an object of contractible connection γ on M such that*

$$\nabla^{\Gamma_2}_v \gamma = \nu \nabla^{\Gamma_1}_v \gamma \tag{10.14.1}$$

for all vector fields v on M.

Proof. We need only to prove that the existence of an object of $(0,1)$-connection A satisfying (10.13.1) is equivalent to the existence of an object of contractible connection γ satisfying (10.14.1). But if A is given, then $\gamma = j \circ A$, where j is defined by (10.7.3.b), is an object of contractible connection on M, and by applying the formula

$$\nabla^{\Gamma_i}_v \gamma = dj \circ \nabla^{\Gamma_i}_v A, \quad i = 1, 2$$

(which follows immediately from the definition of $\nabla^{\Gamma_i}_v$), condition (10.13.1) implies (10.14.1).

Inversely, if γ is given, then $A = J \circ \gamma$, where J is defined by (10.7.3.b), is an object of $(0,1)$-connection on M, and by

$$\nabla^{\Gamma_i}_v A = dJ \circ \nabla^{\Gamma_i}_v \gamma$$

condition (10.14.1) implies (10.13.1).

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