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Extensions of Laplace transform spaces and their topologies

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A number of constructions, have appeared in the past two decades, of spaces of generalized functions, notably L. Schwartz, *Theorie des distributions*, I, II. Some of these obtain generalized functions as continuous linear functionals on an appropriate test function space. Others obtain distributions as boundary values of analytic functions and others proceeding from Van der Pol's operational calculus begin with Laplace transforms. Each approach has certain advantages of completeness, generality and simplicity. Beginning with the elementary premise of defining derivatives for functions not differentiable in the usual sense, Laplace transforms or operational calculus provides an easy way to formally extend the class of functions. Various authors have given ways of justifying this formal extension. In what follows we propose to compare certain aspects of such valid extensions. We will discuss only extensions beginning with functions possessing Laplace transforms, this is not to claim that extensions are inherently better but merely recognize that for some problems the extensions can be more easily described. Moreover, in a large class of applications Laplace transforms are an important tool.

Let $L^1[0, \infty)$ denote, as is usual, the space of measurable functions, with support in the right half line and which are Lebesgue integrable. The Laplace transform is then a mapping of L^1 into Θ $(Rz > 0)$. The usual topology for Θ is the compact open topology or uniform convergence on compact subsets. The domain, L^1, is a commutative Banach space and the mapping is multiplicative but the image is not a closed subset of Θ nor is it a normed space. Moreover, the image does not contain the most elementary analytic functions, i.e., polynomials. We shall attempt to show that this discrepancy is, in fact, one of the underlying motivations for beginning with Laplace transforms to construct generalized function spaces. L^1 is "complete" algebraically and topologically except for an identity and inverses for all elements. Although standard constructions are available for adjoining an identity and imbedding the ring in a field of quotients, this does not seem appropriate in terms of applications of Laplace transforms. Rather, the adjunction of derivatives is the most elementary and perhaps the most important extension. Adjunction of derivatives corresponds to adjunction of polynomials to the image space. While we will not concern ourselves with the construction of generalized function spaces, we will be interested in different topologies that can be utilized on a space of functions possessing Laplace transforms and how they relate to the extension problem. The details of such extensions may be found in Mikusiński [4], which is an algebraic construction, Miller [3], which is topological and this author [6], among others. One essential difference between [3], [4], [6] and what follows is that here one considers a space of functions possessing Laplace transforms without fixing the half-plane of convergence. This space will be a linear metric space and be closed under multiplication but the metric does not behave properly under multiplication. The motive was given by Mukherjee and Ganguly [5] and several of their results are quoted for reference.

A. ONE-SIDED TRANSFORMS

I. Notation. A function, F, defined and measurable (at least) on $[0, \infty)$ is said to be (absolutely) *Laplace transformable* if there exists σ such that

$$\int_0^\infty e^{-st}|F(t)|\,dt < \infty, \quad s > \sigma.$$

For such a function, let $\sigma_F = \inf\{\sigma\}$, σ_F is called the *abscissa* of convergence (see [1], among others). As has been pointed out in [5], the sum of any two functions is again transformable and

$$\sigma_{F+G} \leqslant \max(\sigma_F, \sigma_G).$$

It is also easily seen that with $*$ denoting convolution

$$\sigma_{F*G} \leqslant \max(\sigma_F, \sigma_G)$$

by an elementary application of Fubini's theorem. The space of transformable functions is denoted T^1.

We shall also be interested in the following subspaces

$$(1) \qquad T_B^1 = \left\{F \middle| F \in T', \sup_{s>\sigma_F} \int_0^\infty e^{-st}|F(t)|\,dt < \infty\right\};$$

T_B^1 may be normed by

$$\|F-G\| = \sup_{s>\max(\sigma_F,\sigma_G)} \int_0^\infty e^{-st}|F(t)-G(t)|\,dt$$

and it follows that

(i) $\|F+G\| \leqslant \|F\|+\|G\|$,

(ii) $\|aF\| = |a|\cdot\|F\|$,

(iii) $\|F*G\| \leqslant \|F\|\ \|G\|$.

LEMMA 1. *If* $F\in T_B^1$, *then*

$$\int_0^\infty e^{-\sigma_F t}|F(t)|\,dt = \|F\|.$$

Probably this is a known result but it does not appear to be stated explicitly anywhere, we therefore include it for the sake of completeness. A proof may be constructed as follows:

Let Q be the set $s>\sigma_F$ which is ordered by inverse inequality. The set of functions $\{e^{-st}|F(t)|s>\sigma_F\}$ converges, in the sense of limits with respect to directed sets, to $e^{-\sigma_F t}|F(t)|$ a.e. on $[0,\infty)$ note the monotonic property of the convergence. Since all the integrals are bounded by $\|F\|$, $\int_0^\infty e^{-\sigma_F t}|F(t)|\,dt$ exists and is the limit,

$$T^1(\sigma) = \{F|F\in T^1,\ \sigma_F=\sigma\}, \tag{2}$$

$$T_B^1(\sigma) = \{F|F\in T_B^1,\ \sigma_F=\sigma\}. \tag{3}$$

The latter two collections of supspaces are of interest because they can be used to characterize T^1, T_B^1 as inductive limit spaces.

II. Properties of T^1. As noted above T^1 is an algebra but as yet has no topology. For those functions in $T^1-T_B^1$

$$\sup_{s>\sigma_F}\int_0^\infty e^{-st}|F(t)|\,dt = \infty$$

but since the sup is taken over non-negative numbers we can construct a Fréchet type metric. This metric is given by Mukherjee and Ganguly in [5] and for elements in T_B^1 is equivalent to the norm-topology. For F, $G\in T^1$

(i) $m_r(F,G) = \int_0^r |e^{-\sigma_F t}F(t)-e^{-\sigma_G t}G(t)|\,dt$,

(ii) $M(F,G) = \lim\limits_{r\to\infty}\dfrac{m_r(F,G)}{1+m_r(F,G)}$,

(iii) $\varrho(F,G) = |\sigma_F-\sigma_G|+M(F,G)$.

LEMMA 2 (Mukherjee and Ganguly).

1. *For* F, G, H *in* T^1

   (a) $\varrho(F,G) = 0 \Leftrightarrow F = G$ (*a.e.*),

   (b) $0 \leqslant \varrho(F,G) = \varrho(G,F) \leqslant 1$,

   (c) $\varrho(F,H) \leqslant \varrho(F,G) + \varrho(G,H)$.

2. *The mapping* $F \to \sigma_F$ *is continuous from* $T^1 \to R$ *with the metric topology on* T^1.

3. T^1 *is complete with the metric topology.*

4. T^1 *is not connected.*

Consider the mapping S_h: $F \to e^{ht}F$, $h > 0$ which is an isometry of $T^1(\sigma)$ into $T^1(\sigma+h)$ and $T^1_B(\sigma)$ into $T^1_B(\sigma+h)$ so that

$$T^1 = \bigcup_{\sigma>0} T^1(\sigma)$$

as an inductive limit space, since T^1 is already known to be complete and the mappings $F \to \sigma_F$ is continuous, the representation is of no great importance. However, completeness remains to be shown for T^1_B and hence if it could be represented as an inductive limit of a countable number of complete spaces, then completeness of T^1_B would follow. For each integer n, let

$$T^1_B(n)^* = \bigcup_{n<\sigma\leqslant n} S_{n-\sigma} T^1_B(\sigma)$$

note that $F \to S_h F$ is norm preserving, i.e.

$$\sup_{s<\sigma_F} \int_0^\infty e^{-st}|F(t)|\,dt = \sup_{s>\sigma_F+h} \int_0^\infty e^{-st}e^{ht}|F(t)|\,dt.$$

The injections $T^1_B(n)^* \overset{S_1}{\to} T^1_B(n+1)^*$ are continuous, hence

$$T^1_B = \bigcup_{n=0}^\infty T^1_B(n)^*$$

is an inductive limit space. It remains then to be shown that each $T^1_B(n)^*$ is complete.

LEMMA 3. *For each* $n \geqslant 0$, $T^1_B(n)^*$ *is a commutative Banach algebra.*

It is obvious from the preceeding that completeness is the only property yet requiring proof. But by Lemma 1

$$T^1_B(n)^*$$

is just the L^1 space with respect to the measure $e^{-nt}dt$, such spaces are known to be complete.

III. Induced topologies. As we indicated in the introduction, it is sometimes useful to induce a topology from either the range or image space to the other. For $F \in T_B^1$

$$f(z) = \int_0^\infty e^{-st} F(t)\, dt$$

exists and is analytic for $x = R(z) > \sigma_F$ furthermore

$$|f(z)| \leqslant \|F\| \quad \text{for } R_z > \sigma_F.$$

Denote by A_B^1 the set of functions, f, such that for each there is a half-plane, $R(z) > \sigma_f$, where f is analytic and uniformly bounded. In particular, this means that all the functions in A_B^1 are of polynomial growth degree 0. As is well-known, all such functions are representable as Laplace transforms. A_B^1 is an algebra with pointwise addition and multiplication, and we norm it by

$$\|f\| = \sup_{R(z) > \sigma_f} f(z).$$

It is immediate that the norm topology for A_B^1 is equivalent to the compact-open topology, further, it is shown in [8] that the degree of polynomial growth is preserved under convergence with the compact open topologies, hence by the completeness of f, A_B^1 is a commutative Banach algebra which is the image of T_B^1, the topology of A_B^1 coincides with that induced from T_B^1.

Miller [3], however, has shown that if T_B^1 is given a topology induced by the compact-open topology on A_B^1, then T_B^1 is not complete since in the completion all members are infinitely differentiable. For other comparisons of topologies that can be utilized for A_B^1 and T_B^1, see [7], [8].

B. BILATERAL TRANSFORMS

In this part we will consider some extensions of the results in A to the space of functions possessing bilateral transforms, the particular construction we will use does not include as a subspace T^1, but it can be imbedded in the space obtained.

IV. The space T^2. By analogy with the notations in I, a function defined and measurable on $(-\infty, \infty)$ is said to be (absolutely) *Laplace transformable* if there exists (σ, η), $\sigma < \eta$, such that

$$\int_{-\infty}^{\infty} e^{-st} |F(t)|\, dt < \infty \quad \text{for } \sigma < s < \eta$$

in like manner $\eta_F = \sup\{\eta\}$, $\sigma_F = \inf\{\sigma\}$ and (σ_F, η_F) is called the *convergence pair for F*. In order to define the sum or product of two functions it is necessary to insure that the common region of existence is non-empty. Suppose θ is fixed and unless otherwise indicated we shall suppose that $-\infty < \sigma_F < \theta < \eta_F < \infty$ for all F. The collection of such functions will be denoted T^2, T^2 is given an algebraic structure as follows

$$\max(\sigma_F, \sigma_G) \geqslant \begin{cases} \sigma_{F+G}, \\ \sigma_{F*G}, \end{cases} \qquad \min(\eta_F, \eta_G) \leqslant \begin{cases} \eta_{F+G}, \\ \eta_{F*G}, \end{cases}$$

and addition of F, G is pointwise multiplication being convolution. The conditions $\sigma_F < \theta < \eta_F$, $\sigma_G < \theta < \eta_G$ insure $\sigma_{F+G} < \theta < \eta_{F+G}$.

T^2 can be given a metric topology in a manner similar to T^1

(i) $m_r(F, G) = \int\limits_0^r |e^{-\sigma_F t}F(t) - e^{-\sigma_G t}G(t)|\,dt + \int\limits_{-r}^0 |e^{-\eta_F t}F(t) - e^{-\eta_G t}G(t)|\,dt$,

(ii) $M(F, G) = \lim\limits_{r\to\infty} \dfrac{m_r(F, G)}{1+m_r(F, G)}$,

(iii) $\varrho(F, G) = |\sigma_F - \sigma_G| + |\eta_F - \eta_G| + M(F, G)$.

Lemma 4.

1. *For any F, G, H in T^2*
   (a) $\varrho(F, G) = 0 \Leftrightarrow F = G$,
   (b) $0 \leqslant \varrho(F, G) = \varrho(G, F) \leqslant 1$,
   (c) $\varrho(F, H) \leqslant \varrho(F, G) + \varrho(G, H)$.

2. *The mapping $F \to (\sigma_F, \eta_F)$ of $T^2 \to R^2$ is continuous, where T^2 has the metric topology.*

3. *T^2 is complete with respect to the metric topology.*

Proof. 1 is a straightforward consequence of the definition so we will not give the proof. 2 follows immediately by noting that the topology on R^2 given by $\varrho^*(p_1, p_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$ is equivalent to the Euclidean metric. To establish 3 write each function as the sum of its restrictions to the left-half and right-half line respectively. From the definition of ϱ it follows that a sequence $\{F_n\}$ is Cauchy if and only if the sequences of restrictions $\{F_n^+\}$, $\{F_n^-\}$ are Cauchy. The proof given in [5] suitably modified completes the proof.

We remark at this point that $F \to (\sigma_F, \eta_F)$ is not linear (nor is $F \to \sigma_F$ which contradicts Note 4, [5]).

V. Subspaces. Let

$$T_B^2 = \Big\{F \,|\, \sup_{\sigma_F \leqslant s \leqslant \eta_F} \int\limits_{-\infty}^{\infty} e^{-st}|F(t)|\,dt < \infty\Big\};$$

then $T_B^2 \subset T^2$ and T_B^2 has an obvious norm topology which is equivalent to the metric topology of T^2.

The following is contained in more general results in Taylor [9] and we state it for completeness.

LEMMA 5. *$T_B^2(\sigma,\eta)$ is a commutative Banach space and its maximal ideal space is identifiable with $\sigma < R(z) < \eta$. It is necessary to change the norm for the bilateral case since in general one can only expect*

$$\|F\| \leqslant \int_0^\infty e^{-\sigma_F t}|F(t)|\,dt + \int_{-\infty}^0 e^{-\eta_F t}|F(t)|\,dt$$

and one of the terms on the right might be infinite.

The classes of subspaces $T^1(\sigma)$, $T_B^1(\sigma)$ have as their counterparts

$$T^2(\sigma,\eta) = \{F|F\in T^2,\ \sigma_F = \sigma < \eta = \eta_F\},$$
$$T_B^2(\sigma,\eta) = \{F|F\in T_B^2,\ \sigma_F = \sigma < \eta = \eta_F\}.$$

Although one can define inductive limit spaces for T^2, T_B^2 it is less interesting because of the restriction $\sigma < \theta < \eta$. In both cases the inductive limit will be the class of functions, F, such that $e^{-\theta t}F \in L^1(-\infty,\infty)$, i.e., has a Fourier transform. The imbedding map is also different

$$T^2(\sigma,\eta) \xrightarrow{S_{h,k}} T^2(\sigma+h,\eta-k),$$

where

$$F \to \begin{cases} e^{ht}F(t), & t>0,\\ e^{-kt}F(t), & t\leqslant 0,\end{cases}$$

and

$$\sigma+h < \theta < \eta-k.$$

VI. Induced topologies. For $F\in T_B^2$

$$f(z) = \int_{-\infty}^{\infty} e^{-zt}F(t)\,dt$$

exists and is analytic for $\sigma_F < x = R(z) < \eta_F$. Also

$$|f(z)| \leqslant \|F\|.$$

Let A_B^2 denote the class of functions analytic in some strip $\sigma_f < R(z) < \eta_f$, $\sigma_f < \theta < \eta_f$ and uniformly bounded in the strip. In exactly the same way as for A_B^1,

$$\sup_{\sigma_f < R(z) < \eta_f} |f(z)| = \|f\|$$

provides a norm topology and $F \to \int_{-\infty}^{\infty} e^{-zt}F(t)\,dt$ is an isometric isomorphism of T_B^2 onto A_B^2.

Although the result is not stated in [3], it is possible to show that if A_B^2 were completed by the compact-open topology as a subset of Θ, then T_B^2 with the induced topology is not complete; as before the lack of completeness is a consequence of the infinite differentiability of all elements in the completion.

VII. Misc. comments. There are some intrinsic differences between T_B^2 and T_B^1 which could be better seen if one defines T_B^2 by weight functions. Let $w(t)$ be a continuous function on $(-\infty, \infty)$ such that

$$w(t_1+t_2) \leqslant w(t_1)\,w(t_2)$$

for all t_1, t_2. Further suppose that

$$w(t) \geqslant e^{-st}, \quad -\infty < \sigma < s < \eta < \infty.$$

Using the weight function, one constructs a measure, $w(t)\,dt$, and then the L^1 space for that measure. The space

$$T_B^2 = \bigcup_{\substack{w \\ -\infty<\sigma<0<\eta<\infty}} L^1\big(w(t)\,dt\big).$$

If, however, $\eta = \infty = \theta$, then the sub multiplicative character is lost so that T_B^1 is not obtained from such spaces. Taylor [9] has used such a construction in a more general setting to study ideals in the measure algebra of a locally compact group.

We note also that if absolute convergence of the integrals for T^1, T^2 is replaced by conditional convergence that very little of this will apply.

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