ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
20,4 (1990), pp. 565–572

R. RÓŻAŃSKI (Wrocław)

MINIMAX SEQUENTIAL ESTIMATION OF PARAMETERS OF RANDOM FIELDS

1. Introduction. The paper is devoted to the problem of minimax sequential estimation of parameters of random fields. We prove a theorem (Theorem 3) which is related to the results of Dvoretzky et al. [2] and Rhiel [7]. The theorem proved is an improvement of the result obtained by the author in [10]. We prove that, under some assumptions, for a square loss connected with the error of estimation and for a cost function $c(|K|)$ of the observation of a random field X_s, $s \in R^2$, on the set K the simple plan is a minimax sequential plan among all sequential plans (τ, f), where τ is a Markov stopping set with respect to some family $\mathscr{G}$ of compact subsets of R^2. Examples of the application of this theorem to the Poisson, Wiener and Ornstein–Uhlenbeck fields are also given. Analogous problems of minimax sequential estimation for stochastic processes were considered in [7], [8], [12].

2. Preliminaries. Let X_s, $s \in R^2$, be a random field, and V a set of realizations of this field. Assume that this random field generates a probability measure μ_θ, defined on $(V, \mathscr{F})$, where $\mathscr{F}$ is a σ-algebra of subsets of V generated by cylindrical sets, and $\theta \in A \subset R$ is a parameter. Let $\mathscr{G}$ denote a family of compact subsets K of R^2, and $\delta(K)$ the diameter of K. Suppose that the family $\mathscr{G}$ satisfies the following condition ([9], [11]):

CONDITION 1. *There exists a countable family of compact sets $P_i(n)$, $n \in N$, $i \in I \subset N$, such that*

$$\sup\{\delta(P_i(n))\} \to 0 \quad \text{as } n \to \infty$$

and for each $K \in \mathscr{G}$ there exists a finite covering $C_n \in \mathscr{G}$ of K by some sets among $P_i(n)$, $i \in I$, for which

$$C_{n+1} \subset C_n, \quad \bigcup_{n=1}^{\infty} C_n = K.$$

The σ-algebra of subsets of V generated by cylindrical sets

$$\{v\colon (v(s_1), v(s_2), \ldots, v(s_n)) \in B\}, \quad B \in \mathscr{B}_{R^2},\ s_i \in K,\ i = 1, 2, \ldots, n,$$

is denoted by $\mathscr{F}_K$, and the restriction of μ_θ to the σ-algebra $\mathscr{F}_K$ is denoted by μ_θ^K.

Definition 1 (see [3] and [9]). A *Markov stopping set* τ is a mapping $\tau\colon V \to \mathscr{G}$ such that for every $K \in \mathscr{G}$

$$\{v\colon \tau(v) \subseteq K\} \in \mathscr{F}_K.$$

Definition 2. A σ-algebra $\mathscr{F}_\tau$ of sets $U \in \mathscr{F}$ such that for every $K \in \mathscr{G}$

$$U \cap \{v\colon \tau(v) \subseteq K\} \in F_K$$

is called a *pre-τ-σ-algebra corresponding to a Markov stopping set* τ.

Denote by μ_θ^τ the measure μ_θ restricted to the σ-algebra $\mathscr{F}_\tau$. The following is true:

Theorem 1 ([9], [11]). *If $\mathscr{G}$ satisfies Condition* 1 *and*

$$\frac{d\mu_\theta^K}{d\mu_{\theta_0}^K}(v) = g_{\theta_0}(K, v, \theta) \quad \textit{for each } K \in \mathscr{G},$$

where g_{θ_0} is such that for each $K_n \downarrow K$

$$g_{\theta_0}(K_n, v, \theta) \to g_{\theta_0}(K, v, \theta)$$

μ_θ-almost surely for each $\theta \in A$, then

$$\mu_\theta^\tau \ll \mu_{\theta_0}^\tau \quad \textit{and} \quad \frac{d\mu_\theta^\tau}{d\mu_{\theta_0}^\tau}(v) = g_{\theta_0}(\tau(v), v, \theta).$$

Definition 3. By a *sequential plan* we mean a pair $\delta = (\tau, f)$ that consists of a Markov stopping set τ and a mapping

$$f\colon (V, \mathscr{F}_\tau) \to (A, \mathscr{B}_A),$$

where $\mathscr{B}_A$ is a family of Borel subsets of A. The mapping f is called an *estimator of the parameter* θ.

Let the density function $d\mu_\theta^K / d\mu_{\theta_0}^K$ be given by the formula

$$\frac{d\mu_\theta^K}{d\mu_{\theta_0}^K}(v) = g_{\theta_0}(Q(K), S(K, v), \theta),$$

where g_{θ_0} is a Borel function on $R^2 \times A$; $S\colon \mathscr{G} \times V \to R$ is such that, for every $K \in \mathscr{G}$, $S(K, \cdot)$ is $\mathscr{F}_K$-measurable and $S(K_n, v) \to S(K, v)$ μ_θ-almost surely for each $\theta \in A$ whenever $K_n \downarrow K$, $K, K_n \in \mathscr{G}$; Q is a set function from $\mathscr{G}$ into R such that for each $K_n \downarrow K$, K_n, $K \in \mathscr{G}$, $Q(K_n) \to Q(K)$ as $n \to \infty$.

In this case we can infer by Theorem 1 that the statistic $(Q(\tau), S(\tau))$ is sufficient for the parameter θ, and therefore we can restrict ourselves to the estimators of the form $f(Q(\tau), S(\tau))$.

Let $L(f, \theta)$ denote the loss incurred by a statistician if θ is a true value of the parameter and f is an estimator of θ he uses. Let $c(|K|)$, where $|K|$ denotes Lebesgue measure of the set K, be the cost function representing the cost of the observation of the random field on the set K. The function c: $[0, \infty] \to [0, \infty]$ is assumed to be continuous, non-decreasing with $c(0) = 0$ and $c(\infty) = \infty$. Then the risk function is given by

$$R(\delta, \theta) = \mathrm{E}_\theta[L(f, \theta) + c(|\tau|)].$$

We assume that $R(\delta, \theta) < \infty$.

DEFINITION 4. A sequential plan $\hat{\delta} = (\hat{\tau}, \hat{f})$ is called *minimax* if

$$\sup_\theta R(\hat{\delta}, \theta) = \inf_\delta \sup_\theta R(\delta, \theta).$$

Let Φ be a prior distribution on the parameter space $(A, \mathscr{B}_A)$. If $R(\delta, \theta)$ is a $\mathscr{B}_A$-measurable function, then for each sequential plan δ the Bayes risk with respect to the prior distribution Φ is given by

$$r(\delta, \Phi) = \int_A R(\delta, \theta) \Phi(d\theta).$$

DEFINITION 5. A sequential plan $\hat{\delta} = (\hat{\tau}, \hat{f})$ is called a *Bayes plan* with respect ot Φ if

$$r(\hat{\delta}, \Phi) = \inf_\delta r(\delta, \Phi).$$

Let us define a probability measure π_Φ on $(V \times A, \mathscr{F} \times \mathscr{B}_A)$ by the formula

$$\pi_\Phi(U \times B) = \int_B \mu_\theta(U) \Phi(d\theta)$$

for each $U \in \mathscr{F}$ and $B \in \mathscr{B}_A$. Observe that

$$\pi_\Phi(V \times B) = \int_B \mu_\theta(V) \Phi(d\theta) = \Phi(B)$$

and

$$\pi_\Phi(U \times A) = \int_A \mu_\theta(U) \Phi(d\theta) \stackrel{\mathrm{df}}{=} \mu_\Phi(U)$$

for each $U \in \mathscr{F}$ and $B \in \mathscr{B}_A$.

From the general theorem ([1], p. 293) on existence of a transition probability we infer that for any Markov stopping set τ there exists a transition probability measure $\Psi_{\Phi,\tau}(v, \cdot)$ such that

$$\pi_\Phi(U \times B) = \int_U \Psi_{\Phi,\tau}(v, B) d\mu_\Phi(v)$$

for each $U \in \mathscr{F}_\tau$ and $B \in \mathscr{B}_A$. We can also write

$$\Psi_{\Phi,\tau}(v, B) = \big(\pi_\Phi(V \times B) | \mathscr{F}_\tau \times \{\emptyset, A\}\big)(v)$$

μ_Φ-almost everywhere. The measure $\Psi_{\Phi,\tau}$ is called the *posterior probability* of θ having observed the realization v on the set τ.

DEFINITION 6. The mapping $Y_{K,\Phi}\colon V \to R_+$, $K \in \mathcal{G}$,

$$Y_{K,\Phi}(v) = \inf_f \Big[\int_A L(f, \theta) \Psi_{\Phi,K}(v, d\theta) + c(|K|) \Big]$$

is called a *stochastic decision process.*

DEFINITION 7. A sequential plan $\delta = (\tau, f)$ is a *simple plan* if $\tau(v) = K$, $K \in \mathcal{G}$ for almost all $v \in V$.

3. Minimax sequential estimation in random fields.

THEOREM 2 (see also [7]). *If there exists an estimator $f'(Q(\tau), S(\tau))$ such that*

$$Y_{K,\Phi}(v) = \int_A L(f'(Q(K), S(K, v)), \theta) \Psi_{\Phi,K}(v, d\theta) + c(|K|)$$

μ_Φ-almost surely for each $K \in \mathcal{G}$, then for any Markov stopping set τ

$$E_{\mu_\Phi}(Y_{\tau,\Phi}) = r(\delta', \Phi) = \inf_\delta r(\delta, \Phi),$$

where $\delta' = (\tau, f'(Q(\tau), S(\tau)))$.

Proof. For every sequential plan $\delta = (\tau, f(Q(\tau), S(\tau)))$ we can write

$$\begin{aligned} r(\delta, \Phi) &= E_{\pi_\Phi}\big[L(f(Q(\tau), S(\tau)), \theta) + c(|\tau|)\big] \\ &= \int_V d\mu_\Phi(v) \Big(\int_A L(f(Q(\tau), S(\tau)), \theta) \Psi_{\Phi,\tau}(v, d\theta) \Big) + E_{\pi_\Phi}(c(|\tau|)) \\ &\geqslant \int_V d\mu_\Phi(v) \Big(\int_A L(f'(Q(\tau), S(\tau)), \theta) \Psi_{\Phi,\tau}(v, d\theta) \Big) + E_{\pi_\Phi}(c(|\tau|)) \\ &= E_{\pi_\Phi}\big(L(f'(Q(\tau), S(\tau)), \theta)\big) + E_{\pi_\Phi} c(|\tau|) = E_{\mu_\Phi} Y_{\tau,\Phi}. \end{aligned}$$

Remark 1. If the decision process $Y_{K,\Phi}$ is deterministic and there exists a simple plan $\tau_0 = K_0$ such that

$$Y_{K_0,\Phi} = \inf_{K \in \mathcal{G}} Y_{K,\Phi},$$

then the sequential plan $\delta_0' = (\tau_0, f'(Q(\tau_0), S(\tau_0)))$ is a Bayes plan among all sequential plans.

The following theorem that we prove is related to the well-known theorem of Dvoretzky et al. [2] and to the theorem of Rhiel [7]. This theorem improves the results obtained by Różański [10].

THEOREM 3. *Assume that for some sequence Φ_n, $n = 1, 2, \ldots$, of prior distributions of the parameter θ the corresponding stochastic decision processes Y_{K,Φ_n} are deterministic. Let*

$$Y_K^\infty = \lim_{n \to \infty} Y_{K,\Phi_n} \qquad \liminf_{n \to \infty} \inf_{K \in \mathcal{G}} Y_{K,\Phi_n} = \inf_{K \in \mathcal{G}} \lim_{n \to \infty} Y_{K,\Phi_n}.$$

If there exists a simple plan $\delta_0 = (\tau_0, f(Q(\tau_0), S(\tau_0)))$ such that $\tau_0 = K_0$ a.e., $K_0 \in \mathcal{G}$, and

$$\sup_\theta R(\delta_0, \theta) \leqslant Y_{K_0}^\infty = \inf_{K \in \mathcal{G}} Y_K^\infty,$$

then δ_0 is a minimax sequential plan among all sequential plans (τ, f), where τ is a Markov stopping set with respect to $\mathcal{G}$.

Proof. By Theorem 2 we get

$$\inf_{\delta} r(\delta, \Phi_n) = \inf_{K} Y_{K,\Phi_n},$$

$$\sup_{\theta} R(\delta_0, \theta) \leqslant Y_{K_0}^{\infty} = \lim_{n\to\infty} \inf_{K\in\mathcal{G}} Y_{K,\Phi_n} = \lim_{n\to\infty} r(\delta_n', \Phi_n).$$

By the well-known theorem (see, e.g., the monographs [4], p. 90, and [15], p. 374) we infer that the plan (τ_0, f_0) is minimax.

4. Examples.

1. *Poisson random field.*

DEFINITION 8. Let $\mathcal{B}_{R^2}^b$ be the family of bounded Borel subsets of R^2. Assume that the family $\{N(B), B\in\mathcal{B}_{R^2}^b\}$ of random variables has the following properties:

1° for an arbitrary set of disjoint bounded Borel subsets $B_1, B_2, \ldots, B_n$ of R^2 the random variables $N(B_1), N(B_2), \ldots, N(B_n)$ are independent;

2° $P(N(B_i) = k) = (\theta|B_i|)^k \exp(-\theta|B_i|)/k!$.

The random field

$$N_z = N(R_z), \quad R_z = [0, x]\times[0, y], \ (x, y)\in R_+^2,$$

is called a *Poisson random field.*

The unknown parameter θ is to be estimated. By [6] the measure $\mu_\theta^{R_z}$ corresponding to the random field N_s, $s\in R_z$, is absolutely continuous with respect to the measure $\mu_1^{R_z}$ corresponding to the Poisson random field with $\theta = 1$ and

$$\frac{d\mu_\theta^{R_z}}{d\mu_1^{R_z}} = \theta^{N_z}\exp(-\theta|R_z|).$$

Let $L(f, \theta) = \theta^{-1}(f-\theta)^2$ and let us choose a sequence of prior distributions of the parameter θ given by the density functions

$$\varphi_n(\theta) = n^{-1}\exp(-\theta/n).$$

The density of the posterior distribution of the parameter having observed the realization v on the set R_z takes the form

$$\Psi_{n,R_z} = \frac{(d\mu_\theta^{R_z}/d\mu_1^{R_z})\varphi_n(\theta)}{\int_0^\infty (d\mu_\theta^{R_z}/d\mu_1^{R_z})\varphi_n(\theta)d\theta}$$

$$= \left(|R_z| + \frac{1}{n}\right)^{N(R_z)+1} \frac{\theta^{N(R_z)}}{N(R_z)!}\exp\left(-\theta\left(|R_z| + \frac{1}{n}\right)\right).$$

We also have

$$Y_{R_z,n} = \frac{1}{|R_z|+n^{-1}} + c(|R_z|).$$

From Theorem 3 we infer that the simple plan $\delta_0 = (R_{z0}, f(N(R_{z0})))$ such that

$$f(N(R_{z0})) = \frac{N(R_{z0})}{|R_{z0}|}, \quad \frac{1}{|R_{z0}|} + c(|R_{z0}|) = \min_{R_z, z\in R_+^2} \left(\frac{1}{|R_z|} + c(|R_z|)\right)$$

is a minimax sequential plan among all sequential plans $\delta = (\tau, f(N(\tau)))$, where τ is a Markov stopping set with respect to $\mathscr{G} = \{R_z, z \in R_+^2\}$.

2. *Wiener random field.*

Definition 9. Assume that the family $\{W(B), B \in \mathscr{B}_{R^2}^b\}$ of random variables has the following properties:

1° for an arbitrary set of disjoint bounded Borel subsets $B_1, B_2, \ldots, B_n$ of R^2 the random variables $W(B_1), W(B_2), \ldots, W(B_n)$ are independent;

2° the random variable $W(B)$ is normally distributed with the mean value equal to zero and the variance $|B|$.

Then the family of random variables $W_z = W(R_z)$ is called a *Wiener random field.*

Let us consider the random field $X_z = \theta|R_z| + W_z$. By [13] the measure $\mu_\theta^{R_z}$ corresponding to the random field X_s, $s \in R_z$, is absolutely continuous with respect to the measure $\mu_0^{R_z}$ corresponding to the Wiener field W and

$$\frac{d\mu_\theta^{R_z}}{d\mu_0^{R_z}} = \exp\left[\theta X_z - \frac{\theta^2}{2}|R_z|\right].$$

Let $L(f, \theta) = (f-\theta)^2$ and let us choose a sequence of prior distributions of the parameter θ given by the following density functions:

$$\varphi_n(\theta) = \frac{1}{\sqrt{2\pi n}} \exp(-\theta^2/2n).$$

Then the density of the posterior distribution of the unknown parameter having observed the realization v on the set R_z takes the form

$$\Psi_{n,R_z} = \frac{\sqrt{n^{-1}+|R_z|}}{2} \exp\left(-\frac{(\theta - X_z(n^{-1}+|R_z|))^2}{2(n^{-1}+|R_z|)}\right).$$

So

$$Y_{R_z,n} = \frac{1}{|R_z|+n^{-1}} + c(|R_z|).$$

By Theorem 3 we conclude that the simple plan $\delta_0 = (R_{z0}, f(X_{z0}))$ such that

$$f(X_{z0}) = \frac{X_{z0}}{|R_{z0}|}, \quad \frac{1}{|R_{z0}|} + c(|R_{z0}|) = \min_{R_z, z\in R_+^2} \frac{1}{|R_z|} + c(|R_z|)$$

is a minimax sequential plan among all sequential plans $\delta = (\tau, f(X(\tau)))$, where τ is a Markov stopping set with respect to $\mathcal{G} = \{R_z, z \in R_+^2\}$.

3. *Ornstein–Uhlenbeck random field* (see also [10]).

By the *Ornstein–Uhlenbeck random field* we mean a homogeneous Gaussian random field X_s, $s \in R_+^2$, with the mean walue θ and the covariance function

$$R((h_1, h_2)) = \exp(-\alpha|h_1| - \beta|h_2|)$$

(see [5]). By [14] the measure $\mu_\theta^{R_z}$ corresponding to the Ornstein–Uhlenbeck field with mean θ is absolutely continuous with respect to the measure $\mu_0^{R_z}$ corresponding to the Ornstein–Uhlenbeck field with mean 0 and

$$\frac{d\mu_\theta^{R_z}}{d\mu_0^{R_z}} = \exp\left(\frac{\theta}{4} S(R_z, v) - \frac{\theta^2}{8} Q(R_z)\right),$$

where

$$S(R_z, v) = v(0, 0) + v(x, 0) + v(0, y) + v(x, y) + \alpha \int_0^x v(u, 0)du$$

$$+ \alpha \int_0^x v(u, y)du + \beta \int_0^y v(0, t)dt + \beta \int_0^y v(x, t)dt$$

$$+ \alpha\beta \int_0^x \int_0^y v(u, t)dudt,$$

$$Q(R_z) = (\alpha x + 2)(\beta y + 2).$$

Let $L(f, \theta) = (f - \theta)^2$. As in [10], let us consider a sequence of prior distributions of the parameter θ given by the density functions

$$\varphi_n(\theta) = \frac{1}{2\sqrt{2\pi n}} \exp\left(-\frac{\theta^2}{8n}\right).$$

The density of the posterior distribution of the parameter takes the form

$$\psi_{R_z, n} = \frac{1}{\sqrt{2\pi A}} \exp\left(-\frac{B^2}{2A}\right),$$

where

$$A = \frac{4}{Q(R_z) + n^{-1}} \quad \text{and} \quad B = \theta - \frac{S(R_z)}{Q(R_z) + n^{-1}}.$$

So

$$Y_{R_z, n} = \frac{4}{Q(R_z) + n^{-1}}.$$

By Theorem 3 we infer that the simple plan $\delta_0 \doteq (R_{z_0}, f(Q(R_{z_0}), S(R_{z_0})))$ such that

$$f(Q(R_{z_0}), S(R_{z_0})) = \frac{S(R_{z_0})}{Q(R_{z_0})}$$

and

$$\frac{4}{Q(R_{z_0})} + c(|R_{z_0}|) = \min_{R_z, z\in R_+^2} \left(\frac{4}{Q(R_z)} + c(|R_z|)\right)$$

is a minimax sequential plan among all sequential plans $\delta = (\tau, f(Q(\tau), S(\tau)))$, where τ is a Markov stopping set with respect to $\mathscr{G} = \{R_z, z \in R_+^2\}$.

References

[1] J. R. Barra, *Notions fondamentales de statistique mathématique*, Dunod, Paris 1971.

[2] A. Dvoretzky, J. Kiefer and J. Wolfowitz, *Sequential decision problems for processes with continuous time parameter. Problems of estimation*, Ann. Math. Statist. 24 (1953), pp. 403–415.

[3] I. V. Evstigneev, *Optional times for random fields*, Theory Probability Appl. 223 (1977), pp. 575–581.

[4] T. S. Ferguson, *Mathematical Statistics. A Decision Theoretic Approach*, Academic Press, New York 1967.

[5] C. X. Guyon and B. Prum, *Identification et estimation de semi-martingales représentables par rapport à un brownien à un indice double*, pp. 211–232 in: *Processus aléatoires à deux indices*, Lecture Notes in Math. 863, Springer-Verlag, Berlin 1981.

[6] K. Krickeberg, *Statistical problems on point processes*, pp. 197–224 in: *Mathematical Statistics*, Banach Center Publications 6, Warszawa 1980.

[7] R. Rhiel, *Sequential Bayesian and minimax decisions based on stochastic processes*, Sequential Analysis 4 (1985), pp. 213–245.

[8] R. Różański, *On minimax sequential estimation of the mean value of a stationary Gaussian Markov process*, Zastos. Mat. 17 (1982), pp. 401–408.

[9] — *Sequential estimation in random fields*, Probab. Math. Statist. 9 (1987), pp. 77–93.

[10] — *Minimax sequential estimation in random fields. Application to the Ornstein–Uhlenbeck case*, Statistics and Decisions 5 (1987), pp. 271–279.

[11] — *Markov stopping sets and stochastic integrals. Application in sequential estimation for a random diffusion field*, Stochastic Process. Appl. 32 (1989), 237–251.

[12] S. Trybuła, *Some investigations in minimax estimation theory*, Dissertationes Math. (Rozprawy Mat.) 240 (1985).

[13] E. Wong and M. Zakai, *Likelihood ratios and transformation of probability associated with two-parameter Wiener process*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 40 (1977), pp. 283–308.

[14] I. I. Yadrenko, *On the absolute continuity of probability measures corresponding to Gaussian homogeneous random fields*, Theory Probab. Math. Statist., Kiev, 9 (1972), pp. 152–161.

[15] S. Zacks, *The Theory of Statistical Inference*, J. Wiley, New York 1971.

ROMAN RÓŻAŃSKI
INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW

Received on 1987.11.24