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An inverse function theorem in Fréchet spaces without smoothing operators

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Abstract. Inverse function theorems in Fréchet spaces are proved without use of smoothing operators as in the classical theory of Nash and Moser.

1. We consider Fréchet spaces with increasing sequences of norms; for instance $C^{\infty}(M)$, the space of C^{∞}-functions on a compact manifold M, $S(\boldsymbol{R}^n)$, the space of rapidly decreasing functions and $H(D)$, the space of holomorphic functions on D with the topology of uniform convergence on compact. The Nash–Moser inverse function theorems are designed to overcome the difficulty caused by a "loss of differentiability" in solving the linearized problem. This is usually done by means of smoothing operators. If a space admits such operators, the method of Nash–Moser works (see [2]–[4]), but one can consider spaces without this property. We avoid the need of smoothing operators, but our assumptions about differentiability are very strong. The theorems obtained are similar to those of Bourbaki [1] and they are well applicable to spaces with norms not expressed by derivatives.

Let E and F be Fréchet spaces with topologies given by increasing sequences of norms; i.e., E and F are equipped with norms $\|\cdot\|_n$, $n \in N$, and for each n and each vector x, $\|x\|_n \leqslant \|x\|_{n+1}$; both E and F are complete. The same topology is induced by the F-norm

$$\|x\|_F = \sum_{n \in N} 2^{-n} \|x\|_n/(1+\|x\|_n).$$

We shall consider only continuous mappings between these spaces. Let U and V be open subsets of E and F, respectively, and let p be an integer.

DEFINITION. A mapping $f\colon U \to F$ continuous with respect to the topologies given by the $(n+p)$-th norm in E and the n-th norm in F (for each $n \in N$) is called a *mapping of order p*. A bijection $f\colon U \to V$ such that f is of order p and f^{-1} is of order $-p$ is called a *p-homeomorphism.*

It is obvious that a p-homeomorphism is a homeomorphism and a mapping of order p is continuous. For linear operators these definitions admit a simple interpretation. A linear operator $T\colon E \to F$ is of order p if and only if, for each $n \in N$, there exists $c_n \geqslant 0$ such that

$$\|Tx\|_n \leqslant c_n \|x\|_{n+p}, \qquad x \in E.$$

For a linear operator T of order p we can define the sequence of its norms, $\|T\|_n = \sup_{x \neq 0} \|Tx\|_n / \|x\|_{n+p}$. It is not increasing in general.

2. Definition. Let $x_0 \in U \subset E$, where U is open, and let $f\colon U \to F$. We say that f is *uniformly strictly p-differentiable* (*u.s.p-dif.*) at x_0 if there exists a linear operator $T\colon E \to F$ of order p such that

$$\lim_{\substack{\|\cdot\|_{n+p} \\ x,y \to x_0}} \|\alpha_{n+p}(x, y)\|_n = 0$$

uniformly with respect to $n \in N$, where

$$\alpha_{n+p}(x, y) = \big(f(x) - f(y) - T(x-y)\big)/\|x-y\|_{n+p}.$$

For example, linear operators of order p and constant maps are u.s.p-dif. at any point. The strength of the condition imposed in the definition consists in the demand of uniform convergence. Observe that bilinear operators of order p, i.e., bilinear maps with the property

$$\|f(x_1, x_2)\|_n \leqslant c_n \|x_1\|_{n+p} \|x_2\|_{n+p}$$

for any n, are u.s.p-dif. only if the sequence (c_n) is bounded.

Example. In the space $C^\infty \langle a, b\rangle$ consider the norms

$$\|x\|_n = \max_{k \leqslant n} \sup_{t \in \langle a,b\rangle} |x^{(k)}(t)|,$$

and the map $f\colon C^\infty \langle a, b\rangle \to C^\infty \langle a, b\rangle$ given by

$$f(x)(t) = (x'(t))^2.$$

It is easy to see that

$$\lim_{x,y \xrightarrow{n+1} 0} \|f(x) - f(y)\|_n / \|x-y\|_{n+1} = 0$$

for any $n \in N$, so the uniform strict derivative at 0, if it exists, is equal to 0. But if we take $x_\lambda(t) = \lambda \exp t$, $y(t) = 0$, then

$$\|f(x_\lambda) - f(y)\|_n / \|x_\lambda - y\|_{n+1} = 2^n \lambda \exp b,$$

which proves that the convergence is not uniform.

We shall give below examples of spaces which better suit our definition.

We assume that the following condition is satisfied in E:

$$(*) \qquad \bigcap_{n \in N} K_n(0, \delta) \neq \{0\} \quad \text{for every } \delta > 0,$$

where $K_n(0, \delta) = \{x\colon \|x\|_n < \delta\}$. Condition $(*)$ means that there exists $x \in E$

such that $\sup\|x\|_n < \infty$. Hence, the spaces

$$C^\infty(M) \quad \text{with} \quad \|x\|_n = \max_{|\alpha| \leqslant n} \sup_{t\in M} |D^\alpha x(t)|,$$

$$H(D) \quad \text{with} \quad \|x\|_n = \sup_{t\in K_n} |x(t)|,$$

where (K_n) is an increasing sequence of compact sets whose union is a complex domain D, satisfy condition $(*)$ (they include constant functions). On the other hand, the space of rapidly decreasing functions with norms:

$$\|x\|_n = \max_{|\alpha| \leqslant n} \sup_t (1+|t|^2)^{n/2} |D^\alpha x(t)|$$

does not possess this property.

EXAMPLE. Let $A = [a_{lk}]_{l,k\in N}$ be an infinite matrix of real (or complex) numbers and suppose that $a_{1k} \neq 0$ for $k \in N$. Let S_A denote the space of sequences $(t_k)_{k\in N}$ such that $(a_{lk} t_k)_{k\in N}$ is bounded for each $l \in N$, with the norms

$$\|(t_k)\|_n = \max_{l \leqslant n} \sup_{k\in N} |a_{lk} t_k|.$$

Then S_A is complete and does not satisfy condition $(*)$ if and only if $a_{lk} \to \infty$ as $l \to \infty$ for any $k \in N$.

LEMMA 1. *If $f\colon U \to F$ is u.s.p-dif. at x_0, then*

$$\lim_{x,y\to x_0} \|f(x)-f(y)-T(x-y)\|_F/\|x-y\|_F = 0$$

(convergence with respect to the topology in E).

Recall that we have assumed the continuity of f. We use this assumption only in the proof of Lemma 1.

Proof. Let $\varepsilon > 0$. There exists $\delta \in (0, 1)$ such that, if $\|x-x_0\|_{n+p} \leqslant \delta$, $\|y-x_0\|_{n+p} \leqslant \delta$ and $n \in N$, then

$$\|\alpha_{n+p}(x, y)\|_n < \varepsilon/3 \cdot 2^p,$$

where $\alpha_{n+p}(x, y)$ is defined in Section 2. Owing to condition $(*)$, the set of all such pairs (x, y) is nonempty. We have:

$$\|f(x)-f(y)-T(x-y)\|_F \leqslant 2^p \cdot \sup_n \{\|\alpha_{n+p}(x, y)\|_n (1+\|x-y\|_{n+p})\} \|x-y\|_F$$

$$< 2^p(\varepsilon/3 \cdot 2^p)(1+2\delta)\|x-y\|_F$$

and therefore

$$\|f(x)-f(y)-T(x-y)\|_F/\|x-y\|_F < \varepsilon.$$

By the continuity of the function on the left on $U \times U \setminus \Delta$, where Δ is the diagonal, the set of pairs $(x, y) \in U \times U \setminus \Delta$ for which the above inequality holds is open. Projecting this set onto the product factors and taking the intersection, we get a neighbourhood of x_0, on which the inequality is true. Q.E.D.

3. Let us assume that a linear operator $T\colon E \to F$ is of order p and surjective. T induces $\tilde{T}\colon E/\ker T \to F$ and, from the Closed Graph Theorem, $\tilde{T}$ is a linear homeomorphism. In $E/\ker T$ we have the sequence of induced norms

$$\|[x]\|_n = \inf_{y \in \ker T} \|x-y\|_n.$$

It is easy to check that $\tilde{T}$ is of order p, $\|\tilde{T}\|_n \leqslant \|T\|_n$ for each $n \in N$ and that the condition

$$(**) \qquad \sup_{n \in N} \sup_{x \notin \ker T} \inf_{y \in \ker T} \|x-y\|_{n+p}/\|Tx\|_n = M < \infty$$

implies that $\tilde{T}^{-1}$ is of order $-p$ and, moreover,

$$\sup_n \|\tilde{T}^{-1}\|_{n+p} \leqslant M.$$

Consider the following condition for $\ker T$:

$(*{*}*)$ for each $x \in E$, there exists $\bar{x} \in \ker T$ such that $\|x-\bar{x}\|_n \leqslant 2 \inf\limits_{y \in \ker T} \|x-y\|_n$ for all $n \in N$.

It is satisfied, for instance, when $\ker T = \{0\}$.

THEOREM 1. *Let $f\colon U \to F$ be u.s.p-dif. at $x_0 \in U$. If the derivative T is surjective and conditions $(**)$ and $(*{*}*)$ are satisfied, then there exists a neighbourhood V of x_0 such that $f|V$ is open.*

Proof. Let us consider $g\colon U - x_0 \to E/\ker T$ given by

$$g(x) = \tilde{T}^{-1}\big(f(x+x_0) - f(x_0)\big).$$

It is obvious from $(**)$ that g is u.s.0-dif. at $0 \in E$ with the canonical projection pr: $E \ni x \to [x] \in E/\ker T$ as the derivative. By Lemma 1

$$\lim_{x,y \to 0} \|g(x) - g(y) - [x-y]\|_F / \|x-y\|_F = 0.$$

Using $(*{*}*)$, we can define a selector-function $S\colon E/\ker T \to E$ by taking for $S[y]$ an element of E such that pr $S[y] = [y]$ and $\|S[y]\|_F \leqslant 2\|[y]\|_F$.

One can choose $\delta_0 > 0$ such that $\|x\|_F \leqslant \delta_0$ and $\|y\|_F \leqslant \delta_0$ imply $\|g(x)-g(y)-[x-y]\|_F \leqslant 4^{-1}\|x-y\|_F$. We shall show that $K_F(g(x), \varrho) \subset g\big(K_F(x, 4\varrho)\big)$ for $x \in U - x_0$, $\|x\|_F < 5^{-1}\delta_0$ and $\varrho < 5^{-1}\delta_0$, where $K_F(y, r) = \{y\colon \|y\|_F < r\}$. Let $\|[y] - g(x)\|_F < \varrho$. We shall find $z \in E$ such that $\|z-x\|_F \leqslant 4\varrho$ and $g(z) = [y]$. We construct z as the limit of a sequence (z_k), which we define inductively.

Taking $z_0 = x+h_0$, where $h_0 = S([y]-g(x))$, we have

$$\|h_0\|_F < 2\varrho, \quad \|z_0-x\|_F < 2\varrho, \quad \|z_0\|_F < 2\varrho+5^{-1}\delta_0,$$

$$\|g(z_0)-[y]\|_F = \|g(z_0)-g(x)-\operatorname{pr} S([y]-g(x))\|_F \leqslant 4^{-1}\|z_0-x\|_F < 2^{-1}\varrho.$$

Assume that we have already defined $z_k = z_{k-1}+h_k$ such that

$$\|h_k\|_F < 2^{-k+1}\varrho, \tag{1}$$

$$\|z_k-x\|_F < \sum_{i=0}^{k} 2^{-i+1}\varrho, \tag{2}$$

$$\|z_k\|_F < \sum_{i=0}^{k} 2^{-i+1}\varrho+5^{-1}\delta_0, \tag{3}$$

$$\|g(z_k)-[y]\|_F < 2^{-k+1}\varrho. \tag{4}$$

Putting $z_{k+1} = z_k+h_{k+1}$, where $h_{k+1} = S([y]-g(z_k))$, we get

$$\|h_{k+1}\|_F \leqslant 2\|[y]-g(z_k)\|_F < 2^{-k}\varrho,$$

$$\|z_{k+1}-x\|_F \leqslant \|z_k-x\|_F+\|h_{k+1}\|_F < \sum_{i=0}^{k+1} 2^{-i+1}\varrho,$$

$$\|z_{k+1}\|_F \leqslant \|x\|_F+\|z_{k+1}-x\|_F < \sum_{i=0}^{k+1} 2^{-i+1}\varrho+5^{-1}\delta_0,$$

$$\begin{aligned}\|g(z_{k+1})-[y]\|_F &= \|g(z_{k+1})-g(z_k)-\operatorname{pr} S([y]-g(z_k))\|_F\\ &= \|g(z_{k+1})-g(z_k)-\operatorname{pr} h_{k+1}\|_F \leqslant 4^{-1}\|h_{k+1}\|_F < 2^{-k-2}\varrho.\end{aligned}$$

By induction, we obtain a sequence (z_k) satisfying (1)–(4). Condition (1) shows that (z_k) is a Cauchy sequence, so we can write $z = \lim z_k$. Moreover, by (2), $\|z-x\|_F \leqslant \sum_{i=0}^{\infty} 2^{-i+1}\varrho = 4\varrho$, that is, $z \in \overline{K_F(x, 4\varrho)}$. In view of (3) all z_k belong to $K_F(0, \delta_0)$, which was necessary for the calculations. Inequality (4) shows that $g(z) = [y]$, and this ends the proof. Q.E.D.

4. Suppose, from now on, that also F satisfies condition $(*)$.

THEOREM 2. *Let $f\colon E \supset U \to F$ be u.s.p-dif. at $x_0 \in U$ and let the derivative T be a p-homeomorphism satisfying condition $(**)$ (here this means simply that T^{-1} is of order $-p$ and $\sup\|T^{-1}\|_{n+p} < \infty$). Then there exist neighbourhoods V of x_0 and W of $f(x_0)$ such that $f|V$ is a homeomorphism of V onto W, $(f|V)^{-1}$ is u.s.$-p$-dif. at $f(x_0)$ and its derivative is T^{-1}.*

Proof. By the previous theorem, there exists an open set V_1 such that $x_0 \in V_1$ and $f|V_1$ is open. We denote by E_n the completion of E in the n-th norm, and similarly for F. By differentiability, $f\colon (U, \|\cdot\|_p) \to (F, \|\cdot\|_0)$ satisfies the Lipschitz condition on a neighbourhood of x_0. Hence, f has an extension to $f_0\colon E_p \supset \tilde{U} \to F_0$, where $\tilde{U}$ is open in E_p, $x_0 \in \tilde{U}$. Moreover, T is extendable to $T_0\colon E_p \to F_0$, a linear homeomorphism, and T_0 is the strict derivative of f_0 at x_0 in the Banach space sense, i.e.,

$$\lim_{x,y \xrightarrow{E_p} x_0} \|f_0(x)-f_0(y)-T_0(x-y)\|_0/\|x-y\|_p \doteq 0.$$

Therefore, by the Inverse Function Theorem (see [1]), f_0 is injective on a certain neighbourhood $\tilde{U}'$ of x_0 in E_p. Taking $V_2 = \tilde{U}' \cap E$, we see that $f|V_2$ is injective. Thus f maps $V = V_1 \cap V_2$ homeomorphically onto an open set $W \subset F$.

Let us consider $h = (f|V)^{-1}\colon W \to V$. We shall show that h is u.s.$-p$-dif. at $f(x_0)$ with T^{-1} as the derivative. Let $M = \sup\|T^{-1}\|_{n+p} < \infty$. We choose $\varepsilon \in (0, (2M)^{-1})$, write α_{n+p} as in Section 2 and take $\delta > 0$ such that $\|\alpha_{n+p}(x, y)\|_n \leqslant \varepsilon$ for $\|x-x_0\|_{n+p} \leqslant \delta$, $\|y-x_0\|_{n+p} \leqslant \delta$ and $n \in N$. Then

$$T^{-1}(f(x)-f(y))-(x-y) = \|x-y\|_{n+p}T^{-1}\alpha_{n+p}(x, y).$$

Hence,

$$\|x-y\|_{n+p} \leqslant M(1-M\varepsilon)^{-1}\|f(x)-f(y)\|_n \leqslant 2M\|f(x)-f(y)\|_n.$$

Now, let $u, v \in W$, $f(x) = u$, $f(y) = v$. We get

$$\begin{aligned}
&\|h(u)-h(v)-T^{-1}(u-v)\|_{n+p}/\|u-v\|_n \\
&\qquad = \|x-y-T^{-1}(f(x)-f(y))\|_{n+p}/\|f(x)-f(y)\|_n \\
&\qquad \leqslant 2M\|T^{-1}\|_{n+p}\|f(x)-f(y)-T(x-y)\|_n/\|x-y\|_{n+p} \\
&\qquad \leqslant 2M^2\|\alpha_{n+p}(x, y)\|_n,
\end{aligned}$$

so the left-hand side tends to 0 uniformly with respect to $n \in N$. Q.E.D.

5. Definition. Let $f\colon U \to F$, $x_0 \in U$, and let p be an integer. We shall say that f is *strictly p-differentiable* (resp. *p-differentiable*) at x_0 if there exists a linear operator $T\colon E \to F$ of order p such that

$$\|f(x)-f(y)-T(x-y)\|_n/\|x-y\|_{n+p} \to 0 \quad \text{when } \|x-x_0\|_{n+p} \to 0,\ \|y-x_0\|_{n+p} \to 0$$

(resp. $\|f(x)-f(x_0)-T(x-x_0)\|_n/\|x-x_0\|_{n+p} \to 0$ as $\|x-x_0\|_{n+p} \to 0$) for all $n \in N$.

We denote by $L_p(E, F)$ the space of linear operators of order p from E into F, with the topology given by the norms: $\|T\|_n = \sup\{\|Tx\|_n/\|x\|_{n+p}\colon x \neq 0\}$. Although this is not an increasing sequence of norms, we shall use the same definition of the order of a mapping for functions taking values in $L_p(E, F)$.

We shall need a special result on differentiability in Banach spaces.

Lemma 2. *Let X, Y be Banach spaces, U an open subset of X, and let*

$f\colon U \to Y$. For f to be of class C^1 on U, it is sufficient and necessary that f be strictly differentiable at any point $x_0 \in U$.

Proof. Suppose that $f \in C^1(U)$, $x_0 \in U$ and $\varepsilon > 0$. Take $\delta > 0$ such that $\|Df(x) - Df(x_0)\| \leqslant \varepsilon$ for $\|x - x_0\| \leqslant \delta$ and consider the mapping $g\colon U \to Y$

$$g(x) = f(x) - f(x_0) - Df(x_0)(x - x_0).$$

Then $Dg(x) = Df(x) - Df(x_0)$ for $x \in U$. By the Mean Value Theorem,

$$\begin{aligned} \|f(x) - f(y) - Df(x_0)(x-y)\| &= \|g(x) - g(y)\| \\ &\leqslant \|x-y\| \sup\{\|Dg(z)\|\colon \|z - x_0\| \leqslant \delta\} \leqslant \varepsilon\|x-y\|, \end{aligned}$$

which shows the strict differentiability at x_0.

Assume that f is strictly differentiable at any point, $x_0 \in U$ and $\varepsilon > 0$. There exists $\delta > 0$ such that, for $\|x - x_0\| \leqslant \delta$ and $\|y - x_0\| \leqslant \delta$,

$$\begin{aligned} \tfrac{1}{2}\varepsilon &\geqslant \|f(x) - f(y) - Df(x_0)(x-y)\|/\|x-y\| \\ &\geqslant \big(\|(Df(x_0) - Df(x))(x-y)\|/\|x-y\|\big) - \\ &\qquad - \big(\|f(y) - f(x) - Df(x)(y-x)\|/\|y-x\|\big). \end{aligned}$$

Now, take $\delta_1(x) \in (0, \delta - \|x - x_0\|)$ such that

$$\|f(y) - f(x) - Df(x)(y-x)\|/\|y-x\| \leqslant \tfrac{1}{2}\varepsilon$$

for $\|y - x\| \leqslant \delta_1(x)$. Hence

$$\|(Df(x_0) - Df(x))(x-y)\|/\|x-y\| \leqslant \varepsilon$$

for $\|x - x_0\| \leqslant \delta$, $\|y - x\| \leqslant \delta_1(x)$. The set of all such $(x-y)$'s contains a neighbourhood of 0, so $\|Df(x_0) - Df(x)\| \leqslant \varepsilon$ for $\|x - x_0\| \leqslant \delta$, which proves the continuity of Df at x_0. Q.E.D.

LEMMA 3. *Let $f\colon E \supset U \to F$ and let $Df(x)$ stand for the derivative of f at x (the sense of the term differentiability must be always made clear). The following conditions are equivalent:*

(a) *f is strictly p-differentiable at each point $x \in U$;*

(b) *f is p-differentiable at each $x \in U$ and the map $Df\colon U \to L_p(E, F)$ is of order p.*

Proof. Suppose f to be strictly p-dif. on U. Since f, considered as a map of $(U, \|\cdot\|_{n+p})$ into $(F, \|\cdot\|_n)$, satisfies locally the Lipschitz condition, f has an extension $f_n\colon U_{n+p} \to F_n$, where U_{n+p} is open in the completion E_{n+p} of $(E, \|\cdot\|_{n+p})$ and U is dense in U_{n+p}. Moreover, f_n has a strict derivative at every point of U in the Banach space sense, so, by Lemma 2, $f_n \in C^1(U)$. For an arbitrary $x \in U$, $Df_n(x)$ is the extension of $Df(x)$ to a linear bounded operator of E_{n+p} into F_n. Since $\|Df_n(x)\| = \|Df(x)\|_n$, we see that $\|x_k - x\|_{n+p} \to 0$ implies $\|Df(x_k) - Df(x)\|_n \to 0$, which proves (b).

The converse is obvious because f is of class C^1 on U as a map of

$(U, \|\cdot\|_{n+p})$ into $(F, \|\cdot\|_n)$ for all $n \in N$. In view of Lemma 2, this map is strictly differentiable. Q.E.D.

We shall write $f \in C_p^1(U)$ if f satisfies one of the conditions of Lemma 3. Notice that all such mappings are of order p.

THEOREM 3. *Let $f \in C_p^1(U)$ and let both E and F satisfy condition $(*)$. If f is u.s.p-dif. at $x_0 \in U$ and the derivative $Df(x_0)$ is a p-homeomorphism satisfying $(**)$, then there exist neighbourhoods V of x_0 and W of $f(x_0)$ such that $f|V$ is a C_p^1-diffeomorphism of V onto W, i.e., $f|V \in C_p^1(V)$ and $(f|V)^{-1} \in C_{-p}^1(W)$.*

Proof. Using Theorem 2, we find V and W such that $f|V$ is a homeomorphism of V onto W. By the same arguments as in the proof of Theorem 2 we obtain that $(f|V)^{-1}$ has a strict derivative of order $-p$ at each point $f(x) \in W$, equal to $(Df(x))^{-1}$. Thus $(f|V)^{-1} \in C_{-p}^1(W)$. Q.E.D.

6. The strength of the assumption of uniform strict p-differentiability is the reason for which typical nonlinear operators are not good for the above scheme. However, if $f = T+h$, where T is a linear operator of order p and h maps a neighbourhood of x_0 into $F^{(m)} = \{y \in F: \|y\|_m = \sup_n \|y\|_n\}$ and $\|h(x)-h(y)\|_n/\|x-y\|_{n+p} \to 0$ for $n \leqslant m$ when $x, y \to x_0$ in the $(n+p)$-th norm, then f is u.s.p-dif. at x_0 with the derivative T.

EXAMPLE. Let $E = F = C^\infty(K)$, where K is a compact subset of $\boldsymbol{R}^N$, $\|x\|_n = \max_{|\alpha| \leqslant n} \sup_{t \in K} |D^\alpha x(t)|$. Taking g_k: $C^\infty(K) \to C^\infty(K)$, $k = (k_1, k_2, \ldots, k_N)$, $|k| \leqslant m$, such that $\|g_k(x)-g_k(y)\|_0/\|x-y\|_p \to 0$ when $x, y \to x_0$ in the p-th norm, we define a nonlinear integral operator h: $C^\infty(K) \to C^\infty(K)$ by the formula

$$h(x)(t) = \sum_{|k| \leqslant m} t^k \int_K g_k(x)(s)\,ds.$$

If T is a linear operator of order p (differential, for instance), then $f = T+h$ behaves locally as T.

EXAMPLE. Let X denote the space of sequences $(t_k)_{k=1}^\infty$ such that $([(k+l)!/k!]\,t_{k+l})_{k=1}^\infty$ is bounded for any $l \in N$. The n-th norm in X is defined by the formula

$$\|(t_k)\|_n = \max_{0 \leqslant l \leqslant n} \sup_k [(k+l)!/k!]\,|t_{k+l}|.$$

It can be shown that X with the sequence of norms $\|\cdot\|_n$, $n \in N$, is complete, and that condition $(*)$ is satisfied.

Fix $p \geqslant 1$ and define a linear operator T: $X \to X$ by

$$T((t_k)) = ([(k+p)!/k!]\,t_{k+p}).$$

It is easily seen that $\|T((t_k))\|_n \leqslant \|(t_k)\|_{n+p}$ for any $n \in N$; thus T is of order p.

Moreover, $\ker T = \{(s_k)\in X:\ s_k = 0,\ k > p\}$. Let $(t_k)\notin \ker T$. Then

$$\inf_{(s_k)\in\ker T} \frac{\|(t_k)-(s_k)\|_{n+p}}{\|T((t_k))\|_n} = \frac{\max\limits_{p\leqslant l\leqslant n+p}\ \sup\limits_k [(k+l)!/k!]\,|t_{k+l}|}{\max\limits_{0\leqslant l\leqslant n}\ \sup\limits_k [(k+l+p)!/k!]\,|t_{k+l+p}|} = 1,$$

which proves condition (∗∗). Condition (∗∗∗) can be obtained immediately by using the natural projection onto $\ker T$.

Take $h\colon X\to X$ given by $h((t_k)) = (t_{k+p}^2)$. It can be shown that

$$\|h(x)-h(y)\|_n/\|x-y\|_{n+p} \leqslant \|x\|_{n+p}+\|y\|_{n+p}.$$

Hence $f = T+h$ is u.s.p-dif. at $x_0 = 0$ with the derivative T. The assumptions of Theorem 1 are satisfied and therefore f is open on a neighbourhood of x_0.

Example. Let $H(D)$ be the space of all holomorphic functions on a complex domain D with the norms introduced in Section 2. Let $x_0\in H(D)$ and let g be a holomorphic function on a neighbourhood V of $\overline{x_0(D)}$; if $x_0(D)$ is unbounded, g is assumed to be holomorphic also at infinity. Let U be an open subset of $H(D)$ containing all functions $x\in H(D)$ for which $x(D)\subset V$. We define a mapping $f\colon U\to H(D)$ by the formula

$$f(x)(t) = g(x(t)).$$

We shall show that f is u.s.0-dif. at x_0 with the derivative $(Th)(t) = g'(x_0(t))\cdot h(t)$. Let W be an open set in the complex plane such that $\overline{x_0(D)}\subset W\subset \overline{W}\subset V$. If $x_0(D)$ is unbounded, we can choose W with a compact boundary. By using the Maximum Principle and the Cauchy Inequalities, one can verify that the power series with coefficients $M_k/k!$, where

$$M_k = \sup_{z\in W}|g^{(k)}(z)|,$$

has the radius of convergence $\delta > 0$. Take $x, y\in U$ such that $\|x-x_0\|_n < \frac{1}{2}\delta$, $\|y-x_0\|_n < \frac{1}{2}\delta$ and $x(D)\cup y(D)\subset W$. Then

$$\begin{aligned}
&\|f(x)-f(y)-T(x-y)\|_n/\|x-y\|_n \\
&\quad\leqslant \sup_{t\in K_n}\left\{|g'(y(t))-g'(x_0(t))| + \left|\sum_{k=1}^{\infty}\frac{1}{(k+1)!}g^{(k+1)}(y(t))[x(t)-y(t)]^k\right|\right\} \\
&\quad\leqslant \|g'\circ y-g'\circ x_0\|_n + \sum_{k=1}^{\infty}\frac{M_{k+1}}{(k+1)!}\|x-y\|_n^k.
\end{aligned}$$

This inequality and the continuity of g' on $\overline{W}$ give the assertion.

If we assume additionally that

$$\inf_{z \in x_0(D)} |g'(z)| > 0,$$

then the derivative T is invertible and $\sup \|T^{-1}\|_n < \infty$. Hence, by Theorem 3, f is a C_0^1-diffeomorphism on a neighbourhood of x_0.

7. Remark. We can replace everywhere the words "for all $n \in N$" by "for all n belonging to a subsequence of N"; but then, in Theorem 3, $f|V$ will be only a diffeomorphism.

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