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On vector fields coinduced on a differential space and their Lie product

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Abstract. Let $f: M \to N$ and D let be the differential structure f-coinduced from C to the set N. The condition for a vector field X defined on (M, C), necessary and sufficient in order that the vector field $f_* \circ X$ to be of the form $Z \circ f$, where Z is a vector field on (N, D), is given. Such a vector field Z being uniquely determined by X is called *f-coinduced from X*. The formula for the Lie product of the vector fields f-coinduced is derived.

0. Preliminaries. Let M be any set and let C be a set of real-valued functions defined on M. We denote by τ_C the weakest topology on M such that all functions of C are continuous. For any subset A of M we denote by C_A the set of all functions β defined on A and such that for every point p of A there exists a neighbourhood U of p open in τ_C and a function a of C such that $\beta | A \cap U = a | A \cap U$. The set C is said to be *closed with respect to the localization* iff $C_M = C$. We denote by $\operatorname{sc} C$ the set of all functions of the form $\varphi(a_1(\cdot), \ldots, a_s(\cdot))$, where φ is an arbitrary C^∞-function on $\boldsymbol{R}^s$, $a_1, \ldots, a_s$ belong to C and s is an arbitrary natural number. The set C is said to be *closed with respect to superposition* with real-valued C^∞-functions, if $\operatorname{sc} C = C$. The pair (M, C), where C is a set of real functions closed with respect to the localization and superposition with real-valued C^∞-functions, is called the ***differential structure*** of this space (cf. [1], [2] and [4]). Let f be a function defined on M having the values in N, i.e.,

$$f: M \to N. \tag{1}$$

We say that f maps smoothly the differential space (M, C) into the differential space (N, D), which we denote in the form

$$f: (M, C) \to (N, D), \tag{2}$$

iff for any β of D the function $\beta \circ f$ belongs to C. For any real-valued function defined on N we set $f^*(\beta) = \beta \circ f$. If C is a differential structure on M, then the set $f^{*-1}[C]$ (see [4]) is the differential structure on N, the so-called differential structure coinduced from C by the mapping (1), or equivalently, f-coinduced from C to the set N. In [4] we have proved that

0.1. *The differential structure D is f-coinduced from C to N if and only if for any function $g\colon N \to P$ and for any differential structure F on P in order that*

$$g\colon (N, D) \to (P, F) \tag{3}$$

be a smooth mapping it is necessary and sufficient that $g \circ f\colon (M, C) \to (P, F)$.

0.2. *If D is the differential structure f-coinduced from C to N, then the f-image $f[M]$ of the set M is open in τ_D and the topology induced in $N - f[M]$ by τ_D is discrete.*

The smooth mapping (2) is said to be *weakly coregular at the point p* iff there exist neighbourhoods U and V of points p and $f(p)$, respectively, open in τ_C and τ_D and a smooth mapping $\sigma\colon (V, D_V) \to (U, C_U)$ such that $\sigma(f(p)) = p$ and $f \circ \sigma = \mathrm{id}_V$. A mapping weakly coregular at every point of M is called *weakly coregular*. In [4] the following lemma is proved.

0.3. *If the mapping (2) is weakly coregular, then D is the differential structure f-coinduced from C to the set N and the mapping $f\colon (M, \tau_C) \to (N, \tau_D)$ is open.*

A mapping v which to any function a of C assigns the real number $v(a)$ in such a way that for any functions a and β of C and for any real number a the equalities

$$v(a+\beta) = v(a)+v(\beta), \quad v(aa) = av(a),$$
$$v(a\beta) = a(p)v(\beta)+\beta(p)v(a)$$

hold, is called a *vector tangent* to (M, C) at the point p. The vectors v and w tangent to (M, C) at p can be added and multiplied by reals as follows: $(v+w)(a) = v(a)+w(a)$ and $(a\cdot v)(a) = av(a)$ for any a of C. We then obtain the linear space $(M, C)_p$ of all vectors tangent to (M, C) at the point p, the so-called tangent space to (M, C) at p.

1. The f-coinduced vector fields. For any smooth mapping (2) we define a mapping f_* which to every vector v tangent to (M, C) at any point assigns a vector $f_* v$ tangent to (N, D) and defined by the formula

$$(f_* v)(\beta) = v(\beta \circ f) \quad \text{for } \beta \in D. \tag{4}$$

In particular, for any p of M we have the linear mapping

$$f_*\colon (M, C)_p \to (N, D)_{f(p)}. \tag{5}$$

From formula (4) it immediately follows that for every smooth mapping (2) and (3) we have $(g \circ f)_* = g_* \circ f_*$.

If C is a differential structure on M, then the set C_f of the form $(f^*[f^{*-1}[C]])_M$ is a differential structure (cf. [4]) on M. Evidently, $C_f \subset C$. For any vector v tangent to (M, C) at p we set $v_f(a) = v(a)$, when $a \in C_f$.

We then obtain the linear mapping

$$(6) \qquad v \mapsto v_f\colon (M, C)_p \to (M, C_f)_p.$$

We will prove the following lemma:

1.1. *If D is the differential structure f-coinduced from C to N, then for any point p of M there exists exactly one isomorphism $\hat{f}_p$ of the image of (6) onto the image of the linear mapping of the form (5) such that*

$$(7) \qquad \hat{f}_p(v_f) = f_* v \quad \text{for } v \text{ of } (M, C)_p.$$

Proof. Assume that the vectors v and w of $(M, C)_p$ fulfil the equality $f_* v = f_* w$. Then, for every β of D we have $v(\beta \circ f) = w(\beta \circ f)$. Let $\alpha \in (f^*[D])_M$. Then there exist a neighbourhood U of p open in $\tau_{f^*[D]}$ and a function $\alpha_1 \in f^*[D]$ such that $\alpha | U = \alpha_1 | U$. Then we have $\alpha | U = \beta_1 \circ f | U$ for some β_1 of D. Thus we get $v(\alpha) = v(\beta_1 \circ f) = w(\beta_1 \circ f) = w(\alpha)$. Hence it follows that $v_f = w_f$. Therefore, there exists a linear mapping l_p of the image of (5) such that $l_p(f_* v) = v_f$ for v of $(M, C)_p$. Now, if $v_f = w_f$, where v and w are of $(M, C)_p$, then $v(\alpha) = w(\alpha)$ for $\alpha \in f^*[D]$. Thus, $v(\beta \circ f) = w(\beta \circ f)$ for $\beta \in D$. In other words, $f_* v = f_* w$. Setting $\hat{f}_p = l_p^{-1}$, we obtain the isomorphism $\hat{f}_p$. It is easy to state that (7) characterizes this isomorphism.

The function V which assigns to every point p of M the vector $V(p)$ of the space $(N, D)_{f(p)}$ is said to be an *f-vector field* on (M, C) tangent to (N, D). The vector field is called a ***smooth*** one iff for every function β of D the function $\partial_V \beta$ defined for $p \in M$ by the formula $\partial_V \beta(p) = V(p)(\beta)$ belongs to C. If $(M, C) = (N, D)$ and $f = \mathrm{id}$, the f-vector field is called, shortly, a vector field on (M, C).

1.2. *If D is the differential structure f-coinduced from C to N, then every f-vector field V of the form $V_0 \circ f$, where V_0 is a vector field on (N, D), is smooth if and only if V_0 is smooth on (N, D). Therefore, V_0 is uniquely determined by V.*

Proof. Suppose that V is a smooth f-vector field on (N, D). For any $\beta \in D$ se have $\partial_V \beta = (\partial_{V_0} \beta) \circ f$. The function $\partial_V \beta$ belongs to D. From 0.1 it follows that $\partial_{V_0} \beta$ also belongs to D. In other words, V_0 is smooth on (N, D).

Let V_0 and V_1 be f-vector fields on (N, D) and $V_0 \circ f = V_1 \circ f$. Then $V_0 | f[M] = V_1 | f[M]$. The set $f[M]$ as well as one-point set $\{q\}$, where $q \in N - f[M]$, are open in τ_D. Hence it follows that $V_0 | N - f[M] = V_1 | N - f[M]$. Thus, $V_0 = V_1$.

1.3. *If D is the differential structure f-coinduced from C to the set N, X is a vector field on (M, C), then $f_* \circ X$ is a vector field of the form $Y \circ f$, where Y is a vector field on (N, D) if and only if the following condition is satisfied:*

(i) *for any $p, p' \in M$, if $f(p) = f(p')$, then $X(p)_f = X(p')_f$.*

If $f_* \circ X = Y \circ f$, then Y is uniquely determined by X. If, moreover, X is smooth, then Y is smooth.

Proof. Let us suppose

$$f_* \circ X = Y \circ f. \tag{8}$$

Then, for any $p, p' \in M$ such that $f(p) = f(p')$, we have $f_* X(p) = f_* X(p')$. Lemma 1.1 yields $\hat{f}_p(X(p)_f) = \hat{f}_p(X(p')_f)$. Hence $X(p)_f = X(p')_f$. Now assume (i). From 1.1 we obtain $f_* X(p) = f_* X(p')$ when $f(p) = f(p')$. Hence it follows that for any $q \in f[M]$ there exists a vector $Y(q)$ of $(N, D)_q$ such that for every $p \in M$ the equality $f_* X(p) = Y(f(p))$ holds. Setting $Y(q) = 0$ for $q \in N - f[M]$, we get the vector field Y satisfying (8). The second part of the statement is an immediate consequence of 1.2.

The vector field Y on (N, D), where D is f-coinduced from C, satisfying equality (8) will be denoted by fX and called the *vector field f-coinduced from X*. The vector field X satisfying (i) will be called *f-factorizable*.

1.4. *If D is f-coinduced from C to N, F is g-coinduced from D to P, then for any f-factorizable vector field X on (M, C) such that fX is g-factorizable, the equality*

$$gfX = (g \circ f) X \tag{9}$$

holds. Moreover, if X is smooth, then fX and $(g \circ f)X$ are smooth.

In particular, if (2) and (3) are coregular and onto, then (9) is satisfied.

Proof. Suppose that X is f-factorizable and fX is g-factorizable. Then we have

$$f_* \circ X = fX \circ f \quad \text{and} \quad g_* \circ fX = (gfX) \circ g.$$

Hence, for any $p \in M$ we get

$$(g \circ f)_* X(p) = g_*(f_* X(p)) = g_*(fX(f(p))) = (gfX)(g(f(p))).$$

Thus, $(g \circ f)_* \circ X = g(fX) \circ (g \circ f)$. This yields (9). The second part of the statement follows from Lemma 0.3.

2. Lie product of f-coinduced vector fields. Let X and Y be smooth vector fields on (M, C). For any $p \in M$ and $a \in C$ we set

$$[X, Y](p)(a) = X(p)(\partial_Y a) - Y(p)(\partial_X a). \tag{10}$$

We obtain (see [3]) the vector field $[X, Y]$, the so-called Lie product of X and Y. The correctness of the definition of the Lie product of X and Y given by formula (10) requires the vector fields to be tangent to (M, C). On the other hand, $f_* \circ [X, Y]$ is an f-vector field. It would be convenient to obtain a formula for this f-vector field. Therefore, it will be sufficient to obtain a formula for $f[X, Y]$.

2.1. *If D is the differential structure f-coinduced from C, then for every f-factorizable vector fields X and Y which are smooth on (M, C) the vector ield $[X, Y]$ is f-factorizable and the equality*

$$f[X, Y] = [fX, fY] \tag{11}$$

holds.

Proof. Suppose that X and Y are vector fields f-factorizable and smooth on (M, C). Then $f_*X = fX \circ f$ and $f_*Y = fY \circ f$. Thus, for any $p \in M$ and $\beta \in D$, we have

$$\partial_{fX}\beta\big(f(p)\big) = fX\big(f(p)\big)(\beta) = f_*X(p)(\beta) = X(p)(\beta \circ f) = \partial_X(\beta \circ f)(p).$$

Similarly for Y. In other words,

$$(\partial_{fX}\beta)\circ f = \partial_X(\beta \circ f) \quad \text{and} \quad (\partial_{fY}\beta)\circ f = \partial_Y(\beta \circ f).$$

Consequently,

$$\begin{aligned}[fX, fY]\big(f(p)\big)(\beta) &= fX\big(f(p)\big)(\partial_{fY}\beta) - fY\big(f(p)\big)(\partial_{fX}\beta)\\ &= f_*X(p)(\partial_{fY}\beta) - f_*Y(p)(\partial_{fX}\beta)\\ &= X(p)\big((\partial_{fY}\beta)\circ f\big) - Y(p)\big((\partial_{fX}\beta)\circ f\big)\\ &= X(p)\big(\partial_Y(\beta\circ f)\big) - Y(p)\big(\partial_X(\beta\circ f)\big)\\ &= [X, Y](p)(\beta\circ f) = f_*[X, Y](p)(\beta).\end{aligned}$$

Therefore, $f_*\circ[X, Y] = [fX, fY]\circ f$. Then $[X, Y]$ is f-factorizable and equality (11) is satisfied.

As an immediate corollary of 2.1 we obtain

2.2. *If the mapping* (2) *is weakly coregular and onto, then for every f-factorizable smooth vector fields X and Y on (M, C) the vector field $[X, Y]$ is f-factorizable and* (11) *holds.*

References

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