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NATURAL DEFINITION OF ENTROPY OF SEMIGROUPS

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Introduction. The natural definition of entropy of semigroup given in this paper is a generalization of the notion of the entropy of $\boldsymbol{Z}_+^N$. This definition is based on some special properties of semigroups in $\boldsymbol{Z}^N$ which are formulated and proved in this paper.

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Notation.

$\boldsymbol{N}$ — the set of positive integers.

$\boldsymbol{R}^N$ — the N-dimensional Euclidean space.

$\boldsymbol{Z}^N = \{(x^1, \ldots, x^N) \in \boldsymbol{R}^N\colon x^1, \ldots, x^N \text{ integers}\}$.

For $x, y \in \boldsymbol{R}^N$, $B(x, y)$ denotes a ball with the center x and the radius y.

For $A \subset \boldsymbol{R}^N$, A^n is the set $A \cap B(0, n)$.

1. Geometric structure of semigroups in $\boldsymbol{Z}^N$.

Definition 1. By a *convex cone* in $\boldsymbol{Z}^N$ we mean the set $\Lambda = \tilde{\Lambda} \cap \boldsymbol{Z}^N$, where $\tilde{\Lambda} \subset \boldsymbol{R}^N$ has the following properties:

(a) $\forall x \in \tilde{\Lambda}\ \forall t > 0\ \ tx \in \tilde{\Lambda}$;

(b) $\tilde{\Lambda}$ is convex;

(c) $\tilde{\Lambda}$ has positive Lebesgue measure.

If, in addition, $\tilde{\Lambda}$ is open, Λ will be called an *open convex cone* in $\boldsymbol{Z}^N$.

PROPOSITION 1. *A convex cone in $\boldsymbol{Z}^N$ is an additive semigroup.*

Definition 2. $\forall B \subset \boldsymbol{Z}^N\ \ \Omega(B) \overset{\text{df}}{=} \{z \in \boldsymbol{Z}^N\colon \exists n \in \boldsymbol{N}\ \ nz \in B\}$.

The next two propositions follow directly from Definitions 1 and 2.

PROPOSITION 2. *If $B \subset \boldsymbol{Z}^N$ is a semigroup, then so is $\Omega(B)$.*

PROPOSITION 3. *If Λ is a convex cone in $\boldsymbol{Z}^N$ and $B \subset \Lambda$, then $\Omega(B) \subset \Lambda$.*

In the sequel, G stands for a fixed additive semigroup in $\boldsymbol{Z}^N$.

LEMMA 1. *If Λ is a convex cone in $\boldsymbol{Z}^N$ such that $\Lambda + g_0 \subset G$ for some $g_0 \in G$, then for each $h \in G$ there is $g \in G$ such that*

$$\Omega\Big(\bigcup_{n=0}^{\infty}(\Lambda + nh)\Big) + g \subset G.$$

Proof. Let $h \in G$. We have $\Lambda = \tilde{\Lambda} \cap Z^N$, where $\tilde{\Lambda}$ has properties (a)-(c) of Definition 1. For $x \in \mathrm{Int}\tilde{\Lambda} \cap Z^N$ there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \mathrm{Int}\tilde{\Lambda}$. If $n_0 \in N$ is sufficiently large, then $h \in B(0, \varepsilon n_0)$. Hence $n_0 x + h \in B(n_0 x, n_0 \varepsilon)$. In view of (a) of Definition 1, $B(n_0 x, n_0 \varepsilon) = n_0 B(x, \varepsilon) \subset \tilde{\Lambda}$, so we have $n_0 x + h \in \tilde{\Lambda}$. But $n_0 x + h \in Z^N$, so $n_0 x + h \in \Lambda$. Putting $\lambda_0 \overset{\mathrm{df}}{=} n_0 x$ we have $h + \lambda_0 \in \Lambda$, and $\lambda_0 \in \Lambda$. We shall verify that the desired inclusion is satisfied for $g = g_0 + \lambda_0$. To this end it suffices to prove that

$$\Omega\Big(\bigcup_{n=0}^{\infty} (\Lambda + nh)\Big) + \lambda_0 \subset \bigcup_{n=0}^{\infty} (\Lambda + nh).$$

Let us fix $x \in \Omega\Big(\bigcup_{n=0}^{\infty} (\Lambda + nh)\Big)$. By Definition 2,

$$x = \frac{\lambda + lh}{m} \quad \text{for some } l, m \in N.$$

There exist $p, r \in N$ such that $l = pm + r$, $r < m$. Hence

$$x + \lambda_0 = \frac{\lambda + rh + m\lambda_0}{m} + ph = \frac{\lambda + (m-r)\lambda_0 + r(h + \lambda_0)}{m} + ph.$$

In virtue of Proposition 1 and Definition 1,

$$\frac{\lambda + (m-r)\lambda_0 + r(h+\lambda_0)}{m} \in \Lambda.$$

LEMMA 2. *If Λ is an open convex cone in Z^N and $h \in Z^N$, then $\Omega\big(\bigcup_{n=0}^{\infty} (\Lambda + nh)\big)$ is a convex cone in Z^N.*

Proof. By assumption $\Lambda = \tilde{\Lambda} \cap Z^N$, where $\tilde{\Lambda}$ satisfies the conditions of Definition 1. For all $A \subset R^N$ we define the set

$$\tilde{\Omega}(A) \overset{\mathrm{df}}{=} \{x \in R^N : \exists t > 0 \ tx \in A\}.$$

We shall prove that $\tilde{\Omega}\big(\bigcup_{n=0}^{\infty} (\tilde{\Lambda} + nh)\big)$ satisfies conditions (a)-(c) of Definition 1. It is easy to verify (a). To verify (b), let

$$x, y \in \tilde{\Omega}\Big(\bigcup_{n=0}^{\infty} (\tilde{\Lambda} + nh)\Big).$$

i.e. there exist $s, t > 0$ such that

$$sx, ty \in \bigcup_{n=0}^{\infty} (\tilde{\Lambda} + nh).$$

One can assume that $sx,\ ty \in \tilde{\Lambda}+mh$ for some $m \in N\cup\{0\}$. Let $\tau < 1$ be a fixed positive number. Obviously, there exist $r > 0$ and $0 \leqslant \sigma \leqslant 1$ such that

$$r\big(\tau x+(1-\tau)y\big) = \sigma sx+(1-\sigma)ty.$$

By the convexity of $\tilde{\Lambda}+mh$ we obtain

$$r\big(\tau x+(1-\tau)y\big) \in \tilde{\Lambda}+mh$$

and, consequently,

$$\tau x+(1-\tau)y \in \Omega\Big(\bigcup_{n=0}^{\infty}(\tilde{\Lambda}+nh)\Big).$$

The set $\tilde{\Omega}\Big(\bigcup_{n=0}^{\infty}(\tilde{\Lambda}+nh)\Big)$ is of positive measure because it contains $\tilde{\Lambda}$.

Finally, we prove the equality

$$\Omega\Big(\bigcup_{n=0}^{\infty}(\Lambda+nh)\Big) = \tilde{\Omega}\Big(\bigcup_{n=0}^{\infty}(\tilde{\Lambda}+nh)\Big)\cap \boldsymbol{Z}^N.$$

Let $x \in \tilde{\Omega}\Big(\bigcup_{n=0}^{\infty}(\tilde{\Lambda}+nh)\Big)\cap \boldsymbol{Z}^N$, i.e.

$$sx = y'+mh \quad \text{for some } s > 0,\ y' \in \tilde{\Lambda},\ m \in \boldsymbol{N}\cup\{0\}.$$

The set $\tilde{\Lambda}$ is assumed to be open, thus we can find $y \in \tilde{\Lambda}$ and positive integers p, q such that $(p/q)\,x = y+mh$. Therefore $px = qy+qmh$, where $qy \in \Lambda$, and this implies

$$x \in \Omega\Big(\bigcup_{n=0}^{\infty}(\Lambda+nh)\Big).$$

We have shown that

$$\tilde{\Omega}\Big(\bigcup_{n=0}^{\infty}(\tilde{\Lambda}+nh)\Big)\cap \boldsymbol{Z}^N \subset \Omega\Big(\bigcup_{n=0}^{\infty}(\Lambda+nh)\Big).$$

The opposite inclusion is obvious.

One can easily prove the following

PROPOSITION 4. *If* $\Lambda = \tilde{\Lambda}_1\cap \boldsymbol{Z}^N = \tilde{\Lambda}_2\cap \boldsymbol{Z}^N$ *is a convex cone in* $\boldsymbol{Z}^N$, *then* $\mathrm{Int}\tilde{\Lambda}_1\cap \boldsymbol{Z}^N = \mathrm{Int}\tilde{\Lambda}_2\cap \boldsymbol{Z}^N$ *and* $\bar{\tilde{\Lambda}}_1\cap \boldsymbol{Z}^N = \bar{\tilde{\Lambda}}_2\cap \boldsymbol{Z}^N$.

Definition 3. Let $\Lambda = \tilde{\Lambda}\cap \boldsymbol{Z}^N$ be a convex cone in $\boldsymbol{Z}^N$. We define

$$\mathrm{Int}\,\Lambda \overset{\mathrm{df}}{=} \mathrm{Int}\tilde{\Lambda}\cap \boldsymbol{Z}^N \quad \text{and} \quad \bar{\Lambda} \overset{\mathrm{df}}{=} \bar{\tilde{\Lambda}}\cap \boldsymbol{Z}^N.$$

THEOREM 1. *If G contains a convex cone in $\boldsymbol{Z}^N$, then there exist a sequence (Λ_m) of convex cones in $\boldsymbol{Z}^N$ and a sequence (g_m) of elements of G such that*

(a) $\Lambda_m \subset \Lambda_{m+1}$ *for* $m \in \boldsymbol{N}$;

(b) $\Lambda_m+g_m \subset G$ *for* $m \in \boldsymbol{N}$;

(c) $\bigcup_{m=1}^{\infty}\Lambda_m \subset \Omega(G) \subset \bigcup_{m=1}^{\infty}\bar{\Lambda}_m$.

Proof. By assumption G contains a convex cone Λ in $\boldsymbol{Z}^N$. Put

$$\Lambda_1 \overset{\text{df}}{=} \operatorname{Int}\Lambda$$

and choose an arbitrary $g_1 \in G$. Let us order the elements of G as follows: $e_1, e_2, \ldots, e_m, \ldots$ Put

$$\Lambda_2 \overset{\text{df}}{=} \operatorname{Int}\Big(\Omega\big(\bigcup_{n=0}^{\infty} \Lambda_1 + ne_1\big)\Big).$$

By Lemmas 1 and 2 the set Λ_2 is an open convex cone in $\boldsymbol{Z}^N$ and there exists $g_2 \in G$ such that $\Lambda_2 + g_2 \subset G$. Suppose that an open convex cone Λ_m in $\boldsymbol{Z}^N$ and an element $g_m \in G$ are constructed so that $\Lambda_m + g_m \subset G$. By Lemmas 1 and 2, we can define an open convex cone in $\boldsymbol{Z}^N$, namely

$$\Lambda_{m+1} \overset{\text{df}}{=} \operatorname{Int}\Big(\Omega\big(\bigcup_{n=0}^{\infty} (\Lambda_m + ne_m)\big)\Big),$$

and we can find $g_{m+1} \in G$ such that $\Lambda_{m+1} + g_{m+1} \subset G$. As a result of the above-described construction we get the sequences (Λ_m) and (g_m) which satisfy — as we will prove — conditions (a)-(c) of the theorem. To check conditions (a) and (b) is immediate. We will inductively prove the inclusion

$$\bigcup_{m=1}^{\infty} \Lambda_m \subset \Omega(G).$$

By assumption, $\Lambda_1 \subset G \subset \Omega(G)$. Assume $\Lambda_m \subset \Omega(G)$ for some $m \in \boldsymbol{N}$. Let $z \in \Lambda_{m+1}$, i.e. $pz = \lambda + ne_m$ for some $p \in \boldsymbol{N}$, $\lambda \in \Lambda_m$, $n \in \boldsymbol{N} \cup \{0\}$. By the inductive assumption, $\lambda \in \Omega(G)$. Consequently, $\Omega(G)$ being a semigroup (Proposition 2), we have $pz \in \Omega(G)$, so $z \in \Omega(G)$.

It is clear that $e_m \in \bar{\Lambda}_m$ for $m \in \boldsymbol{N}$, so

$$G \subset \bigcup_{m=1}^{\infty} \bar{\Lambda}_m .$$

By this inclusion and Proposition 3,

$$\Omega(G) \subset \bigcup_{m=1}^{\infty} \bar{\Lambda}_m .$$

COROLLARY 1. *If G contains a convex cone in $\boldsymbol{Z}^N$, then there exists a convex cone Λ in $\boldsymbol{Z}^N$ such that* $\operatorname{Int}\Lambda \subset \Omega(G) \subset \Lambda$.

Proof. We take

$$\Lambda \overset{\text{df}}{=} \bigcup_{m=0}^{\infty} \bar{\Lambda}_m,$$

where (Λ_m) is a sequence from Theorem 1

2. Natural definition of entropy. The semigroup G, as a subset of Z^N, generates a subgroup of Z^N isomorphic to $Z^{N'}$ for some $N' \in N$. Thus without loss of generality we can assume that G generates Z^N.

PROPOSITION 5. *A semigroup $H \subset Z^N$ generates Z^N iff H contains a convex cone in Z^N.*

Proof. Obviously, if H contains a convex cone in Z^N, then H generates Z^N.

Assume that $H \subset Z^N$ generates Z^N. This means, in particular, that there exist $h_1, \ldots, h_N, h_1', \ldots, h_N' \in H$ such that

$$(1, 0, \ldots, 0) = h_1' - h_1, \quad (0, 1, 0, \ldots, 0) = h_2' - h_2, \quad \ldots \quad (0, \ldots, 0, 1) = h_N' - h_N.$$

Set $h = \sum_{i=1}^{N} h_i$. Then

$$(1, 0, \ldots, 0) + h,\ (0, 1, 0, \ldots, 0) + h, \ldots, (0, \ldots, 0, 1) + h \in H.$$

For $n \in N$ we define

$$\Delta_n \overset{\text{df}}{=} \left\{nh + (k^1, \ldots, k^N) \colon k^1, \ldots, k^N \in N, \sum_{i=1}^{N} k^i \leqslant n\right\}$$

and

$$\Delta \overset{\text{df}}{=} \bigcup_{n=0}^{\infty} \Delta_n.$$

It is easily seen that $\Delta \subset H$ and Δ contains a convex cone in Z^N.

By Proposition 5, G contains a convex cone in Z^N which will be denoted by Δ_*.

Definition 4. A subset $I \subset Z^N$ is called a *rectangle* in Z^N if there exists a $(z^1, \ldots, z^N) \in Z^N$ such that

$$I = \{(x^1, \ldots, x^N) \in Z^N \colon 0 \leqslant x^i < z^i \text{ for } i = 1, \ldots, N\}.$$

LEMMA 3. *Let $\varepsilon > 0$ and let (n_l) be a sequence of positive integers such that*

$$\lim_l n_l = \infty.$$

Then there exist

(a) *a rectangle I in Z^N;*

(b) *positive integers $l_1, \ldots, l_k$ and $t_1, \ldots, t_k$;*

(c) *$z_{i,j} \in Z^N$, $j = 1, \ldots, t_i$, $i = 1, \ldots, k$, such that*

$$I = \bigcup_{j=1}^{t_1} (G^{n_{l_1}} + z_{1,j}) \cup \ldots \cup \bigcup_{j=1}^{t_k} (G^{n_{l_k}} + z_{k,j}) \cup I';$$

the sets in this union are pairwise disjoint and

$$\frac{\operatorname{card} I'}{\operatorname{card} I} < \varepsilon.$$

Proof. This fact is proved [1] for the sequence (Λ^n), where Λ is a cone in $\boldsymbol{Z}^N$. In particular, this lemma is valid for (Λ^n) when Λ is a convex cone in $\boldsymbol{Z}^N$ from Corollary 1. Therefore, Lemma 3 remains true for the sequence $(\Omega(G))^n$. To complete the proof it suffices to prove the following

PROPOSITION 6. *We have*

$$\lim_n \frac{\operatorname{card} G^n}{\operatorname{card}(\Omega(G))^n} = 1.$$

Proof. Fix $m \in \boldsymbol{N}$. Let $\Lambda_m = \tilde{\Lambda}_m \cap \boldsymbol{Z}^N$ be a convex cone from Theorem 1 and g_m an element of G such that $\Lambda_m + g_m \subset G$. Let $\Lambda = \tilde{\Lambda} \cap \boldsymbol{Z}^N$ be a convex cone from Corollary 1. Put $c \overset{\mathrm{df}}{=} [\|g_m\|]$, where $\|\cdot\|$ denotes the Euclidean norm in $\boldsymbol{R}^N$. Since

$$\operatorname{card} G^n \geqslant \operatorname{card}\big((\Lambda_m + g_m) \cap B(g_m, n-c-1)\big) = \operatorname{card}(\Lambda_m)^{n-c-1},$$

we get

$$1 \geqslant \frac{\operatorname{card} G^n}{\operatorname{card}(\Omega(G))^n} \geqslant \frac{\operatorname{card} G^n}{\operatorname{card} \Lambda^n} \geqslant \frac{\operatorname{card}(\Lambda_m)^{n-c-1}}{\operatorname{card}(\Lambda_m)^n} \, \frac{\operatorname{card}(\Lambda_m)^n}{\operatorname{card} \Lambda^n}.$$

We have

$$(1) \qquad \lim_n \frac{\operatorname{card}(nA \cap \boldsymbol{Z}^N)}{n^N} = |A|$$

for any convex bounded set $A \subset \boldsymbol{R}^N$, where $|\cdot|$ denotes the Lebesgue measure. This is a simple consequence of measurability of such a set A in the sense of Jordan.

It follows from (1) that

$$(2) \qquad \lim_n \frac{\operatorname{card}(\Lambda_m)^{n-c-1}}{\operatorname{card}(\Lambda_m)^n} = 1,$$

$$(3) \qquad \lim_n \frac{\operatorname{card}(\Lambda_m)^n}{\operatorname{card} \Lambda^n} = a_m,$$

where $a_m = |(\tilde{\Lambda}_m)^1| / |(\tilde{\Lambda})^1|$. Since $\Lambda_m \subset \Lambda_{m+1}$ and $\Lambda = \bigcup_{m=1}^{\infty} \Lambda_m$, we get

$$(4) \qquad \lim_m a_m = 1.$$

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[1] See K. Ziemian, *On topological and measure entropy of semigroups*, Société Mathématique de France, Astérisque 51 (1978), p. 457-472.

According to (2) and (3) we have

$$1 \geqslant \liminf_n \frac{\operatorname{card} G^n}{\operatorname{card}(\Omega(G))^n} \geqslant a_m \quad \text{and} \quad 1 \geqslant \limsup_n \frac{\operatorname{card} G^n}{\operatorname{card}(\Omega(G))^n} \geqslant a_m$$

for all $m \in N$. It follows from (4) that the limit

$$\lim_n \frac{\operatorname{card} G^n}{\operatorname{card}(\Omega(G))^n}$$

exists and equals 1.

LEMMA 4. *Let $\varepsilon > 0$ and let I be a rectangle in Z^N. For $n \in N$ large enough one can find $w_1, \ldots, w_s \in Z^N$ such that*

$$G^n = \bigcup_{i=1}^{s} (I + w_i) \cup (G^n)';$$

the sets in this sum are pairwise disjoint and

$$\frac{\operatorname{card}(G^n)'}{\operatorname{card} G^n} < \varepsilon.$$

Proof. Let I be a rectangle in Z^N. Fix δ ($0 < \delta < \frac{1}{2}$). Let $m \in N$ be so large that

$$a_m > 1 - \delta. \tag{5}$$

For $n \in N$ large enough we have

$$\frac{\operatorname{card}(A_m + g_m)^n}{\operatorname{card}(\Omega(G))^n} > a_m - \delta \tag{6}$$

and, by (1), there exist $w_1, \ldots, w_s \in Z^N$ such that

$$(A_m + g_m)^n \supset \bigcup_{i=1}^{s} (I + w_i), \tag{7}$$

the sets $I + w_i$, $i = 1, \ldots, s$, are pairwise disjoint, and

$$\frac{\operatorname{card} \bigcup_{i=1}^{s} (I + w_i)}{\operatorname{card}(A_m + g_m)^n} > 1 - \delta.$$

Since $(A_m + g_m)^n \subset G^n$, we have

$$G^n = \bigcup_{i=1}^{s} (I + w_i) \cup (G^n)'.$$

To complete the proof it is sufficient to estimate the expression $\operatorname{card}(G^n)'/\operatorname{card} G^n$. Using (5)-(7) we get

$$\frac{\operatorname{card}(G^n)'}{\operatorname{card} G^n} \leqslant 1 - \frac{\operatorname{card}\bigcup_{i=1}^{s}(I+w_i)}{\left(\Omega(G)\right)^n}$$

$$= 1 - \frac{\operatorname{card}\bigcup_{i=1}^{s}(I+w_i)}{\operatorname{card}(\Lambda_m+g_m)^n} \frac{\operatorname{card}(\Lambda_m+g_m)^n}{\operatorname{card}\left(\Omega(G)\right)^n} < 1-(1-\delta)(1-2\delta).$$

Let T denote an action of G on a topological compact (probabilistic) space X. Let A be an open cover (a measurable finite partition) of X. For each $B \subset G$ let

$$A_B \overset{\mathrm{df}}{=} \bigvee_{g \in B} (T^g)^{-1} A.$$

$H(A_B)$ stands for the topological (measure) entropy of the cover (partition) A_B.

We have shown (op. cit.) that

$$h(T, A) = \lim_n \frac{1}{\operatorname{card}(\Lambda_*)^n} H(A_{(\Lambda_*)^n})$$

is a well-defined entropy of A with respect to T.

THEOREM 2 (natural definition of entropy). *The limit*

$$\lim_n \frac{1}{\operatorname{card} G^n} H(A_{G^n})$$

exists and equals $h(T, A)$.

Proof. Fix $\varepsilon > 0$. For $n \in N$ large enough we have

$$(8) \qquad \frac{1}{\operatorname{card}(\Lambda_*)^n} H(A_{(\Lambda_*)^n}) \leqslant h(T, A) + \varepsilon.$$

Let I be a rectangle from Lemma 3 constructed for ε and $\left((\Lambda_*)^n\right)$. By Lemma 4, for sufficiently large $n \in N$ we obtain

$$(9) \qquad G^n = \bigcup_{i=1}^{s}(I+w_i) \cup (G^n)';$$

the sets appearing in this sum are pairwise disjoint and

$$\frac{\operatorname{card}(G^n)'}{\operatorname{card} G^n} < \varepsilon.$$

According to (9), (8), and Lemma 3, we get

$$(10)\qquad \frac{1}{\operatorname{card} G^n} H(A_{G^n}) \leqslant \frac{1}{\operatorname{card} G^n} \sum_{i=1}^{s} H(A_{I+w_i}) + H(A)\varepsilon$$

$$\leqslant \frac{1}{\operatorname{card} G^n} s\big(t_1 H(A_{(\Lambda_*)^{n_1}}) + \ldots + t_k H(A_{(\Lambda_*)^{n_k}})\big) + \frac{\operatorname{card} I'}{\operatorname{card} G^n} H(A) + \varepsilon H(A)$$

$$\leqslant h(T, A) + 2\varepsilon H(A).$$

Hence, since ε is arbitrary, we obtain

$$(11)\qquad \limsup_n \frac{1}{\operatorname{card} G^n} H(A_{G^n}) \leqslant h(T, A).$$

Now, let (n_l) be a sequence of positive integers such that

$$(12)\qquad H(A_{G^{n_l}}) \leqslant \operatorname{card} G^{n_l} \Big(\liminf_n \frac{1}{\operatorname{card} G^n} H(A_{G^n}) + \varepsilon\Big)$$

for all $l \in N$, and let I be a rectangle from Lemma 3 constructed for ε and the sequence (G^{n_l}). For $n \in N$ sufficiently large we have

$$(\Lambda_*)^n = \bigcup_{p=1}^{r} (I + z_p) \cup \big((\Lambda_*)^n\big)',$$

where $z_p \in G$, $p = 1, \ldots, r$, and the sets in the sum are pairwise disjoint. In the same way as in (10), we obtain from (11) and (12) the inequality

$$\frac{1}{\operatorname{card}(\Lambda_*)^n} H(A_{(\Lambda_*)^n}) \leqslant \liminf_n \frac{1}{\operatorname{card} G^n} H(A_{G^n}) + \varepsilon.$$

Since ε is arbitrary, we get

$$h(T, A) \leqslant \liminf_n \frac{1}{\operatorname{card} G^n} H(A_{G^n}),$$

which combined with (11) completes the proof.

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