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# Asymptotic behavior of non-linear, inhomogeneous differential equations via non-standard analysis

## Part II. Some applications to higher order equations

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**Abstract.** This paper is continuing and generalizing the results of part I which dealt exclusively with second order equations. Again the non-standard arguments of this paper could be "translated" into standard proofs but only at the cost of considerable complexity.

**1. Introduction.** To make this paper self contained we shall repeat the introductory comments of part I. It is assummed that the reader is familiar with Robinson's theory as outlined in [6] and [7]. $^*\boldsymbol{R}$ will denote the non-standard extension of the real line $\boldsymbol{R}$, which has the property that sentences formulated in language $\mathscr{L}$ are true in $^*\boldsymbol{R}$ if and only if they are true in $\boldsymbol{R}$ (see [7] for an exact statement). The higher order structure $^*\boldsymbol{R}$ is an ordered field with the binary (and $k$-nary) relations in $\boldsymbol{R}$, and the usual algebraic operations in $\boldsymbol{R}$ being expressible in $\mathscr{L}$ and therefore remaining true in $^*\boldsymbol{R}$. $\boldsymbol{R}$ is a proper subset of $^*\boldsymbol{R}$, while $^*\boldsymbol{R}$ contains positive elements which are smaller then any positive real number, and negative elements larger then any negative real number. They will be called *infinitesimals*. Their reciprocals will be called *infinite numbers.*

If $x, y$ are elements of $\boldsymbol{R}^*$ such that $|x-y|$ is an infinitesimal, we shall say that $x$ is *close* (or *infinitesimaly close*) to $y$. We denote this by writing $x \simeq y$. If $x$ is a real number, then we shall call $x$ a *standard number*, or a *standard element* of $^*\boldsymbol{R}$, otherwise it is called *non-standard.* $^*\boldsymbol{R}_{+\infty}$, $^*\boldsymbol{R}_{-\infty}$ will denote the infinite positive, or respectively the infinite negative elements of $\boldsymbol{R}^*$. $^*\boldsymbol{R}_{bd}$ denotes the elements of $^*\boldsymbol{R}$ which are bounded in absolute value by a standard number. The notation and terminology will follow [4] and [7]. The following lemmas are well-known and have been given in introductory remarks to part 1 of this paper.

LEMMA 1.1. *A standard function $x(t)$, $t \in [t_0, \infty)$ is oscillatory if and only if $x(t)$, $t \in {}^*\boldsymbol{R}$ vanishes for some $t \in {}^*\boldsymbol{R}_{+\infty}$.*

LEMMA 1.2. *$x(t)$ is unbounded if and only if $|x(t)| \in {}^*\boldsymbol{R}_{+\infty}$ for some $t \in {}^*\boldsymbol{R}_{+\infty}$.*

LEMMA 1.3. *$\int_{t_0}^{\infty} g(t)dt$ converges if and only if given any $\xi, \eta \in {}^*\boldsymbol{R}_{+\infty}$ it is true that $\int_{\xi}^{\eta} g(t)dt \approx 0$* ([6], p. 75).

LEMMA 1.4. *If $\lim_{T\to\infty} \int_{t_0}^{T} g(t)dt = +\infty$, then given any $A \in {}^*\boldsymbol{R}$, $A > 0$ $(A < 0)$ and any $\xi > t_0$ $(\in {}^*\boldsymbol{R})$, there exists $\eta > \xi$ $(\eta \in {}^*\boldsymbol{R})$ such that $\int_{\xi}^{\eta} g(t)dt > A$ $(< A)$, and for any $\xi \in {}^*\boldsymbol{R}_{bd}$, $\eta \in {}^*\boldsymbol{R}_{\infty}$ it is true that $\int_{\xi}^{\eta} g(t)dt \in {}^*\boldsymbol{R}_{+\infty}$ $({}^*\boldsymbol{R}_{-\infty})$.*

LEMMA 1.5 (STANDARD). *If for some $n < 1$, $x^{(n)}(t) < 0$ $(> 0)$, and is bounded away from zero on some ray $[t_0, \infty)$, then $\lim x(t) = -\infty$ $(+\infty)$ for any $n$ times continuously differentiable function of $t$, $t \in (t_0, \infty)$.*

**2. Some generalizations of oscillation theorems to higher order equations.** We shall consider first equations of the form

$$x^{(n)} + f(x, x', \ldots, x^{(n-1)}, t) = g(t), \tag{1}$$

$$x^{n} + c(t)f(x, x', \ldots, x^{n-1}) = g(t), \tag{2}$$

where $g(t) \in C[t_0, \infty)$, $f(\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1})$ is a continuous function of $\xi_1, \xi_2, \ldots, \xi_{n+1} \in \boldsymbol{R}^{n+1}$.

THEOREM 2.1. *We assume that $\lim_{b\to\infty} \int_{t}^{b} g(t)dt$ exists, and that for sufficiently large values of $\xi_{n+1}$ $\xi_1 f(\xi_1, \xi_2, \ldots, \xi_n, \xi_{n+1}) > 0$ if $\xi_1 \neq 0$. Then on any infinite ray $[\tau, +\infty)$ $(\tau \geqslant t_0)$, any (classical) solution of* (1) *can not be bounded away from zero.*

Proof. Let us assume to the contrary that there exists such a solution $\hat{x}(t)$ which is bounded away from zero on $[\tau, +\infty)$ for some $\tau \geqslant t_0$. Therefore it is easy to show that there exists a *standard number* $K > 0$, such that $|f(\hat{x}, \hat{x}^1, \ldots, \hat{x}^{n-1}, t)| > K$ for all $t > \tau$, and particularly for all $t \in {}^*\boldsymbol{R}_{+\infty}$. To show that this is true consider the sentence "$\exists \tau \in {}^*\boldsymbol{R}, \exists K \in {}^*\boldsymbol{R}_+ [t \in {}^*\boldsymbol{R} \& t > \tau \Rightarrow |f(x, x^1, \ldots, x^{n-1}, t)| > K]$". This sentence is in our language $\mathscr{L}$, and is true in ${}^*\boldsymbol{R}$, hence it is true in $\boldsymbol{R}$. We make use of Lemma 1.3. Given any $\xi, \eta \in {}^*\boldsymbol{R}_{+\infty}$ it is true that

$$\int_{\xi}^{\eta} g(t)dt \approx 0.$$

By the fundamental theorem of calculus

$$\hat{x}^{(n-1)}(\eta) = \hat{x}^{(n-1)}(\xi) + \int_{\xi}^{\eta} [g(t) - f(\hat{x}, \hat{x}', \ldots, \hat{x}^{n-1}, t)]dt$$

(integrating along the trajectory $\hat{x}(t; t_0, x_0)$). Let us assume without any loss of generality that $\hat{x}(t)$ is positive for all $t \in {}^*\boldsymbol{R}_{+\infty}$, and therefore for some (standard) $K$, $f(\hat{x}, \hat{x}', \ldots, \hat{x}^{(n-1)}, t) > K > 0$ for all $t \in {}^*\boldsymbol{R}_{+\infty}$ (a symmetric argument follows from the assumption $\hat{x}(t) < 0 \ \forall t \in {}^*\boldsymbol{R}_{+\infty}$). Hence

$$\hat{x}^{(n-1)}(\eta) \leqslant \int_{\xi}^{\eta} g(t)\,dt - K(\eta - \xi) + \hat{x}^{(n-1)}(\xi).$$

Regarding $\xi$ as fixed, choose $\eta$ so that

$$(*) \qquad \eta > \xi + \frac{x^{(n-1)}(\xi)}{K} + \frac{2}{K}.$$

Since the first term is infinitesimal and therefore smaller than one, we have

$$\hat{x}^{(n-1)}(\eta) < -1 \qquad \text{for all } \eta \text{ satisfying inequality } (*).$$

Since $\hat{x}(t)$ is positive, the negative sign of the $(n-1)$ derivative combined with the fact that $\hat{x}^{(n-1)}(t)$ is bounded away from zero on some ray $[\tau, \infty)$ implies that $\hat{x}(t)$ will change sign for some $t \in {}^*\boldsymbol{R}_{+\infty}$. (See part 1 of this paper.) Therefore by Lemma (1.1) $\hat{x}(t)$ is oscillatory contradicting our claim that it is bounded away from zero on some infinite ray.

It is easily shown that the hypothesis of this theorem do not necessarily imply oscillation. Consider as an example the equation $x^{(\mathrm{IV})} + x = 17e^{-2t}$ which satisfies all hypothesis of Theorem 2.1. The solution $x = e^{-2t}$ is non-oscillatory.

The next theorem is a generalization of results of H. Onose and J. S. W. Wong (see [5], [8], [9]).

We shall consider equations of the form:

$$(2) \qquad x^{(n)} + c(t)f(x, x', \ldots, x^{m-1}) = g(t)$$

or

$$(3) \qquad \big(a(t)x^{(m)}\big)^{(m)} + c(t)f(x, x', \ldots, x^{2m-1}) = g(t)$$

with $a(t) > 0$ on $[t_0, \infty)$, and

$$a(t) \in C^m[t_0, \infty), \qquad c(t) \in C[t_0, \infty),$$

$$g(t) \in C[t_0, \infty), \qquad f(\xi_1, \xi_2, \ldots, \xi_{2m}) \in C(\boldsymbol{R}^{2m}).$$

Instead of the usual definition of ess lim, we shall give an equivalent definition. Let $f(x)$ be a (standard) continuous function. We say that $f(x)$ has *essential limit* $\tilde{c}$, and denote it by $\operatorname{ess}\lim\limits_{x\to\infty} f(x) = \tilde{c}$, if given any $t_1 > 0$, $\varepsilon_1, \varepsilon_2 > 0$, there exists $N > 0$ such that the set of all $x > N$, for which $|f(x) - \tilde{c}| < \varepsilon_1$ is of Borel measure smaller than $\varepsilon_2$. An easier non-standard definition is obvious. (Measure of all $x \in {}^*\boldsymbol{R}_{+\infty}$, such that $|f(x) - c| \not\approx 0$, is infinitessimal.)

DEFINITION. A function $f(t)$ is *strongly bounded of order* $n$, if $f(t), f'(t), \ldots$ $\ldots, f^{(n)}(t)$ are bounded functions of $t$, for all $t \in [t_0, \infty)$, or equivalently, if

$$\sum_{\alpha=0}^{n} |D^{\alpha} f(t)|$$

is uniformly bounded for all $t \in [t_0, \infty)$.

THEOREM 2.2. *Suppose* $\liminf_{t\to\infty} c(t) = \gamma > 0$, *and* $\limsup_{T\to\infty} \left|\int^{T} g(t)\,dt\right| < \infty$. *Then for any strongly bounded solution of order* $n-1$ *it is true that either* $f(x(t), x'(t), \ldots, x^{(n-1)}(t))$ *is oscillatory, or else* $\operatorname{ess\,lim}_{t\to\infty} f(x(t), x'(t), \ldots$ $\ldots, x^{(n-1)}(t)) = 0$.

Proof. Let us assume that $f(x, x', \ldots, x^{n-1}(t))$ is non-oscillatory. Then $f(x(t), \ldots, x^{(n-1)}(t)) \neq 0$ for all $t \in {}^*\boldsymbol{R}_{+\infty}$ (see Lemma 1.1). Let us choose $t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty}$ and apply the fundamental theorem of calculus

$$x^{(n-1)}(t_2) - x^{(n-1)}(t_1) = -\int_{t_1}^{t_2} c(t) f(x, x', \ldots, x^{(n-1)}(t))\,dt + \int_{t_1}^{t_2} g(t)\,dt.$$

We have

$$x^{(n-1)} \in {}^*\boldsymbol{R}_{bd} \ \forall t \in {}^*\boldsymbol{R}_{+\infty}$$

by assumptions of strong boundedness of order $n-1$, while

$$\int_{t_1}^{t_2} g(t)\,dt \in {}^*\boldsymbol{R}_{bd}.$$

Hence

$$-\int_{t_1}^{t_2} c(t) f(x'(t), \ldots, x^{(n-1)}(t))\,dt \in {}^*\boldsymbol{R}_{bd}.$$

Since $c(t)$ and $f(x, \ldots, x^{n-1}(t))$ are of constant sign on ${}^*\boldsymbol{R}_{+\infty}$, this is possible only if

$$\int_{t_1}^{t_2} c(t) f(x', \ldots, x^{(n-1)})\,dt \approx 0.$$

Since $c(t)$ is bounded away from zero, this implies

$$\int_{t_1}^{t_2} f(x, \ldots, x^{n-1}(t))\,dt \approx 0 \quad \text{for any } t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty},$$

which implies that $\operatorname{ess\,lim}_{t\to\infty} f(x, x', \ldots, x^{(n-1)}(t)) = 0$, completing the proof.

COROLLARY. *If* $\limsup_{t\to\infty} c(t) < \infty$ *and if* $f(\xi_1, \xi_2, \ldots, \xi_n)$ *is a continuously differentiable function of* $\xi = [\xi_1, \xi_2, \ldots, \xi_n]$, $\xi \in \boldsymbol{R}^n$, *then* "$\operatorname{ess\,lim}_{t\to\infty}$" *may be replaced by* "$\lim_{t\to\infty}$" *in the statement of Theorem* 1.

Proof. Assume to the contrary that $\lim_{t\to\infty} f(x_1, \ldots, x^{(n-1)}(t)) \neq 0$ that the limit does not exist, but $\operatorname{ess\,lim}_{t\to\infty} f(x, \ldots, x^{n-1}(t)) = 0$. Then it is easy to see that there must exist some interval $[t_1, t_2] \epsilon\, {}^*\boldsymbol{R}_{+\infty}$ such that

$$\left|\frac{df(x, x^1, \ldots, x^{n-1}(t))}{dt}\right| \epsilon\, {}^*\boldsymbol{R}_{+\infty} \quad \text{for all } t \epsilon [t_1, t_2].$$

However,

$$\frac{df}{dt} = \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial x'} x'' + \ldots + \frac{\partial f}{\partial x^{(n-1)}} x^n(t)$$

$$= \sum_{\alpha=0}^{n-2} \frac{\partial f}{\partial x^\alpha} x^{(\alpha+1)} + \frac{\partial f}{\partial x^{(n-1)}} \big(-c(t)f - g(t)\big).$$

Since $f(\xi_1, \xi_2, \ldots, \xi_n)$ is a continuously differentiable function of $\xi_i$ which in turn are bounded (by some standard number) on ${}^*\boldsymbol{R}_{+\infty}$, the first term of the above expression is bounded (i. e. is an element of ${}^*\boldsymbol{R}_{bd}$ for all $t \epsilon [t_1, t_2]$). Since $c(t)$ and $f(x)$ are bounded, and $g(t)$ is also bounded (by a standard number), we conclude that $|df/dt|$ is bounded above by some standard number on $[t_1, t_2]$, which is a contradiction. We remark that with some obvious assumptions made a bout $f(x, \ldots, x^{(n-1)})$ the converse of this theorem is trivial.

THEOREM 2.3. *We shall consider the differential equation* (1) *subject to the following assumptions*:

(i) $\lim_{t\to\infty} g(t) = 0$.

(ii) $f(\xi_1, \xi_2, \ldots, \xi_n, t)$ *is a continuous function of* $\xi = [\xi_1, \xi_2, \ldots, \xi_n]$, *and of* $t$, *and* $\xi_1 f(\xi_1, \xi_2, \ldots, \xi_n, t) > 0$ *if* $\xi_1 \neq 0$ *for all sufficiently large values of* $t$. *Then* $\boldsymbol{0} \equiv [0, 0, \ldots, 0]$ *is a limit point of* $\xi = [\xi_1(t), \xi_2(t), \ldots, \xi_n(t)]$.

Outline of proof. Assume that $\hat{x}(t)$ is a non-oscillatory solution of (1). Without any loss of gererality we assume $\hat{x}(t) > 0$ for all $t \epsilon\, {}^*\boldsymbol{R}_{+\infty}$. If for some $\tilde{t} \epsilon\, {}^*\boldsymbol{R}_\infty$ it is true that $\hat{x}(\tilde{t}) \not\approx 0$, then by continuity $f(\hat{x}, \hat{x}', \ldots, \hat{x}^{(n-1)}, \tilde{t}) \not\approx 0$. It follows from equation (1) that $\hat{x}^{(n)}(\tilde{t}) < 0$ and $\hat{x}^{(n)}(\tilde{t}) \not\approx 0$. We claim that there must exist $t_1 > \tilde{t}$ such that $\hat{x}^{(n)}(t) < 0$ on $(t, t_1)$ and $\hat{x}^{(n)}(t) \approx 0$. Otherwise $\hat{x}^{(n)}(t)$ is negative and bounded away from zero on the infinite ray $[\tilde{t}, \infty)$. It follows from Lemma 1.5 that $\hat{x}(t)$ must eventually become negative, which is a contradiction. But $\hat{x}^{(n)}(t_1) \approx 0$ implies $f(\hat{x}, \ldots, \hat{x}^{n-1}, t_1) \approx 0$ which implies that $\hat{x}(t_1) \approx 0$. However, we shall prove that these statements imply that $\hat{x}'(t_1) \approx \hat{x}''(t_1) \approx \ldots \ldots \approx \hat{x}^{(n)}(t_1) \approx 0$.

Take first the possibility $x'(t_1) \not\approx 0$, and $x'(t_1) > 0$. Clearly there exists a point $\bar{t}_1 < t_1$, and $\bar{t}_1 \approx t_1$ such that $x'(\bar{t}_1) \approx 0$. We can show that there is no loss of generality if we take some $\hat{t} \approx \bar{t}_1$, $x'(\hat{t}) \approx 0$, as our $t_1$. (Similarly if $x'(t_1) < 0$, $\exists \bar{t}_1 > t_1$ and $\bar{t}_1 \approx t_1$ such that $x'(\bar{t}_1) \approx 0$.) Hence (by continuity and using the fact that $x(t) > 0$) we can select $t_1$ so that: $x(t_1) \approx x'(t_1) \approx 0$. An identical argument shows that we can select $t_1$ so that $x(t_1) \approx x'(t_1) \approx x''(t_1) \approx \ldots \approx x^{(n)}(t_1) \approx 0$. Hence $(0, 0, \ldots, 0)$ is a limit point of $\xi_1, \xi_2, \ldots, \xi_n$, as required.

THEOREM 2.4. *Suppose that in addition to hypothesis* (i) *and* (ii) *of Theorem* 2.3, *it is also true that*

$$\int^{\infty} g(t)dt, \quad \int^{\infty} t\cdot g(t)dt, \quad \ldots, \quad \int^{\infty} t^{n-1}g(t)dt$$

*all converge. Then any solution $\hat{x}(t)$ of* (2) *is strongly bounded of order* $(n-1)$ *if and only if $c(t)\cdot f(\hat{x}, \ldots, \hat{x}^{(n-1)}(t))$ is strongly bounded on the trajectory $\hat{x}(t)$. However, if $f(\hat{x}, \ldots, \hat{x}^{(n-1)}(t))$ is bounded away from zero on some infinite ray $[t_0, \infty)$, then $\hat{x}(t)$ is strongly bounded of order $(n-1)$ if and only if $\int^{\infty} t^{n-1}c(t)dt$ exists.*

Proof. Let $\hat{x}(t)$ be a bounded non-oscillatory solution of (2). Without any loss of generality we assume that $\hat{x}(t) > 0$ for all $t \in {}^*\boldsymbol{R}_{+\infty}$. It follows from our assumptions concerning $f(\hat{x}, \ldots, \hat{x}^{n-1})$ that $\hat{x}^{(n)}$ is also bounded, and that $\hat{x}^{(n)}$ is negative whenever $x^{(n)} \not\approx 0$. We shall consider first the possibility: $\hat{x}(t) \approx 0$ for all $t \in {}^*\boldsymbol{R}_{+\infty}$ (and therefore $x^{(n)}(t) \approx 0$ if $t \in {}^*\boldsymbol{R}_{+\infty}$). Let us choose $\tau \in {}^*\boldsymbol{R}_{bd}$, $t \in {}^*\boldsymbol{R}_{+\infty}$. Then

$$\hat{x}^{(n-i)}(t) = -D_\tau^{-i}\big[f(\hat{x}, \ldots, x^{n-1}(t))\cdot c(t)\big] + \sum_{j=0}^{i-1} a_j \hat{x}^j(\tau) + D_\tau^{-i}(g(t)), \qquad i = 1, 2, \ldots, n-1,$$

where

$$D_\tau^{-i}(\Phi(t)) = \frac{1}{(i-1)!}\int_\tau^t (t-\xi)^{i-1}\Phi(\xi)d\xi.$$

$a_j$ are initial values of the appropriate derivatives at $\tau$. Hence $\sum a_j x^j(\tau)$ is some standard number. $D_\tau^{-i}({}^*g(t)) \in {}^*\boldsymbol{R}_{bd}$ by hypothesis.

Hence $\hat{x}^{(n-i)}(t) \in {}^*\boldsymbol{R}_{bd}$ if and only if

$$D_\tau^{-i}\big[(f(\hat{x}, \ldots, \hat{x}^{n-1}))\cdot c(t)\big] \in {}^*\boldsymbol{R}_{bd}.$$

COROLLARY 2a. *Suppose that $c(t)$ is positive and $\hat{x}\, f(\hat{x}, \ldots, \hat{x}^{n-1})$ is positive for sufficiently large values of $t$, $\int^{\infty} t^{n-1}g(t)dt$ exists, and $n$ is even integer. Then we obtain a following generalization of* [8]. *A necessary and* *sufficient condition for the existence of a bounded non-oscillatory solution of* (1) *is*

$$\int^{\infty} t^{n-1} c(t)\,dt < \infty.$$

We shall next consider equations of the form:

$$(3) \qquad \big(a(t)\Phi(x)\big)^{(n)} + c(t) f(x, x', \ldots, x^{n-1}) = g(t).$$

THEOREM 2.5. *Suppose that* $\liminf_{t\to\infty} a(t)/c(t) > 0$, *that* $\limsup_{t\to\infty} \left|\int_C^T g(s) s^{n-1} ds\right| < \infty$, *and that* $f\big(\hat{x}(t), \ldots, \hat{x}^{n-1}(t)\big)\Phi\big(x(t)\big) > 0$ *if* $\hat{x}(t) \neq 0$, *and* $f\big(\hat{x}(t), \ldots, \hat{x}^{n-1}(t)\big) = 0$ *if* $\hat{x}(t) = 0$. *Then any classical solution of* (3) *will have zero as a limit point.*

Proof. Assume there exists a solution $\hat{x}(t)$ such that for some (standard number) $T > 0$, and for some (standard) $m > 0$, $m_1 > 0$, it is true that $\hat{x}(t) > m\,|f(\hat{x}, \hat{x}', \ldots, \hat{x}^{n-1})| > m_1$ for all $t > T$, and that $|a(t)/c(t)| > 0$ for all $t > T$. We choose an arbitrary point $t_2 \in {}^*\boldsymbol{R}_{+\infty}$. Then

$$\int_T^{t_2} (s-t)^{n-1}\Big[\big[\Phi\big(x(s)\big)a(s)\big] + c(s) f\big(\hat{x}(s), \ldots, \hat{x}^{(n-1)}(s)\big)\Big] ds$$
$$= \varphi(T) + \int_T^{t_2} g(s)(s-t)^{n-1} ds, \qquad \varphi(T) \in {}^*\boldsymbol{R}_{bd}$$

or

$$\int_T^{t_2} c(s)(s-t)^{n-1}\left[\Phi\big(x(s)\big)\frac{a(s)}{c(s)} + f\big(\hat{x}(s), \ldots, x^{(n-1)}(s)\big)\right]$$
$$= \varphi(T) + \int_T^{t_2} g(s)(s-t)^{n-1} ds.$$

Since $\left|\int_T^{t_2} g(s)(s-t)^{n-1} ds\right| \in {}^*\boldsymbol{R}_{bd}$ by assumption, the right-hand side is an element of ${}^*\boldsymbol{R}_{bd}$. Since $\left|\int^{\infty} c(s)(s-t)^{n-1} ds\right| > 0$ and diverges, and consequently

$$\int_T^{t_2} c(s)(s-t)^{n-1} ds \in {}^*\boldsymbol{R}_{+\infty} \quad \text{for any } t_2 \in {}^*\boldsymbol{R}_{+\infty}$$

it follows that

$$\Phi\big(x(t_2)\big)\frac{a(t_2)}{c(t_2)} + f\big(\hat{x}(t_2), \ldots, x^{(n-1)}(t_2)\big) \approx 0$$

for some $t_2 \in {}^*\boldsymbol{R}_{+\infty}$. Since $\liminf a(t)/c(t) > 0$, and $\Phi$ and $f$ have the same sign, it follows that $f\big(x(t_2), \ldots, x^{(n-1)}(t_2)\big) \approx 0$, hence that $x(t_2) \approx 0$, which is a contradiction. Therefore $\liminf_{t\to\infty} |\hat{x}(t)| = 0$, completing the proof.

We are going to give yet another definition of an essential limit, and also define essential liminf, essential limsup. The non-standard definition of $\operatorname{ess}\lim_{t\to\infty}$, or ess limsup, ess liminf, is as follows:

$$\operatorname{ess}\lim_{x\to\infty} f(x) = C \quad (= +\infty, \text{ or } = -\infty)$$

if $f(x) \approx C$ ($f(x) \in \boldsymbol{R}_{+\infty}$, or $f(x) \in \boldsymbol{R}_{-\infty}$) for all $x \in {}^*\boldsymbol{R}_\infty$, except possibly for some $x$ on a subset $S \subset {}^*\boldsymbol{R}_\infty$ such that * measure $(S) \approx 0$. Similarly $\operatorname{ess}\limsup f(x) = C$ ($= +\infty$, or $= -\infty$) if $\limsup f(x) = C_i$ on ${}^*\boldsymbol{R}_{+\infty} \setminus S_i$, where $S_i$ is a subset of ${}^*\boldsymbol{R}_{+\infty}$ of infinitesimal measure, and $C = \liminf C_i$. Clearly esslimsup is unique. ess liminf is defined similarly. The above definitions can be generalized to describe an important class of assymptotic phenomena. We are also going to use the following terminology. Let $I$ be a class of intervals of ${}^*\boldsymbol{R}_\infty$ whose measure is infinite. If $\xi(t)$ is some function defined on some ray $[t_0, \infty)$ such that given a property p, and given an infinitesimal $\varepsilon > 0$ it is possible to choose $t_1 \in {}^*\boldsymbol{R}_\infty$ such that $\xi(t)$ has the property p on any interval $(t_i, t_{i+1}) \in I$, $t_i \geqslant t_1$ except possibly on a subset $S$ of $(t_i, t_{i+1})$ of measure $\mu(S) < \varepsilon$, then we shall say that $\xi(t)$ restricted to $I$ has essentially the asymptotic property p. It reduces to our previous definition if $I$ is itself a ray $I = [T, \infty) \subset {}^*\boldsymbol{R}_\infty$.

In the following section of this paper we shall study the structure of equations of the form (2), whose solutions $\hat{x}(t)$ have the property $\liminf_{t\to\infty} |\hat{x}(t)| < \infty$, $\limsup_{t\to\infty} |\hat{x}(t)| = +\infty$.

We shall first consider some implications of this behavior and some necessary conditions on functions $c(t)$, $g(t)$, subject to following assumptions concerning $f(x)$:

(i) $f(x) \in C^1(-\infty, +\infty)$,

(ii) $\limsup_{x\to\infty} \dfrac{f'(x)}{f(x)} < \infty$,

(iii) $\lim_{|\xi|\to\infty} f(\xi) = \infty$.

First we shall prove a lemma concerning such behavior of solutions of second order equations, then attempt to generalize these results to $n$-th order equations.

This lemma does not appear to be very important, but it has interesting corollaries, including Theorem 2.5.

Lemma 2b (standard). *Consider the equation*

$$x'' + c(t)f(x) = g(t), \tag{4}$$
$$c(t) \in C[t_0, \infty), \quad f(\xi) \in C(-\infty, +\infty), \quad g(t) \in C[t_0, \infty).$$

*Let the following assumptions be true:*
*There exists a function* $u(t) \in C^1[t_0, \infty)$, *such that*

(a) $\operatorname{ess}\liminf_{t\to\infty} \big(u(t)c(t)\big) > 0$,

(b) $\limsup |u(t)g(t)| < \infty$.

*Then* ess $\limsup |\hat{x}(t)| = \infty$, $\liminf |\hat{x}(t)| < \infty$, $\limsup |\hat{x}'(t)| < \infty$ *along any trajectory* $\hat{x}(t)$ *of* (4) *implies that it is true that* $\limsup \left| \dfrac{u'(t)-u(t)}{f\big(\hat{x}(t)\big)} \right| > 0$.

Proof. Assume to the contrary that

(c) $\lim \dfrac{u'(t)-u(t)}{f\big(\hat{x}(t)\big)} = 0$.

It follows from our assumptions that given any number $M \in {}^*\boldsymbol{R}_{+\infty}$ there exists $\tilde{t}_1 \in {}^*\boldsymbol{R}_\infty$ such that $\hat{x}(\tilde{t}_1) = M$, but for some $t > \tilde{t}_1$ $\hat{x}(t) \in {}^*\boldsymbol{R}_{bd}$. Similarly there exists $\tilde{t}_2 \in {}^*\boldsymbol{R}_\infty$, $\tilde{t}_2 > t$, such that $\hat{x}(\tilde{t}_2) = M$.

Let us choose $\tilde{t}_2 > \tilde{t}_1$ so that $\hat{x}(t) > M$ on $[\tilde{t}_1, \tilde{t}_2]$. If $\tilde{t}_2 - \tilde{t}_1 \approx 0$, then there exist following alternatives. Either for any $M \in {}^*\boldsymbol{R}_\infty$ there exists only an infinitesimal corresponding interval $[t_1, t_2] \supset [\tilde{t}_1, \tilde{t}_2]$ such that $\hat{x}(t) > M$ on $[t_1, t_2]$ in which case it follows that $t_1, t_2$ can be chosen so that $x'[t_1] \in {}^*\boldsymbol{R}_\infty$ and $x[t_2] \in {}^*\boldsymbol{R}_\infty$ which is a contradiction; or else there exists $M \in {}^*\boldsymbol{R}_\infty$ and an interval $[\tilde{t}_1, \tilde{t}_2] \supset [t_1, t_2]$, $t_2 - t_1 \not\approx 0$, such that $\hat{x}(t) > M$ for all $t \in (t_1, t_2)$.

Consider such an interval $[t_1, t_2] \in {}^*\boldsymbol{R}_{+\infty}$.

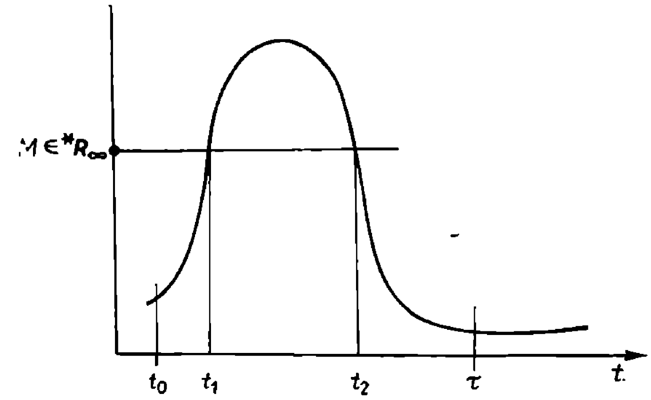


Fig. 1

Consider the behavior of the function $\psi(t) = u(t)\hat{x}'(t)/f\big(\hat{x}(t)\big)$ on the trajectory of equation (1) between the points $t_1, t_2$.

$$\int_{t_1}^{t_2} \psi'(t)\,dt = \int_{t_1}^{t_2} \left( \frac{u(t)\hat{x}''}{f(\hat{x})} + \frac{u'(t)\hat{x}'}{f(\hat{x})} - \frac{u(t)f'(\hat{x})\hat{x}'(t)}{f^2(\hat{x})} \right)$$

$$= \int_{t_1}^{t_2} \left( \frac{u(t)g(t)}{f(\hat{x})} - u(t)c(t) + \frac{u'(t)}{f(\hat{x})}\hat{x}' - \frac{u\hat{F}'(t)}{f^2(\hat{x})} \right) dt,$$

$$\int_{t_1}^{2} \psi'(t)\,dt = \psi(t_2)-\psi(t_1)$$

$$= \int_{t_1}^{t_2} \frac{1}{f(\hat{x})}\left[u(t)g(t)-f(\hat{x})u(t)c(t)+u'(t)\hat{x}'-\frac{1}{f(\hat{x})}u(t)\hat{F}'\right]dt.$$

Hence

$$\psi(t_2)-\psi(t_1) = \int_{t_1}^{t_2} \big(-u(t)c(t)+\xi(t)\big)dt,$$

where $\xi(t)$ is an infinitesimal, because $u(t)g(t)\in {}^*\boldsymbol{R}_{bd'}$, and

$$\frac{u'(t)\hat{x}'(t)}{f\big(\hat{x}(t)\big)} - \frac{u(t)f(\hat{x})\hat{x}'(t)}{f^2(\hat{x})} = \frac{u-u'}{f(x)}\hat{x}'(t) \approx 0$$

by assumption. Therefore, since $u(t)c(t)$ is essentially not infinitesimal, and is of constant sign, $\int_{t_1}^{t_2} \psi'(t)\,dt \not\approx 0$. But by construction

$$\frac{u(t_1)}{f\big(\hat{x}(t_1)\big)}\hat{x}'(t_1) - \frac{u(t_2)}{f\big(\hat{x}(t_2)\big)}\hat{x}'(t_2) \approx 0,$$

which is a contradiction.

THEOREM 2.6. *If* $\liminf_{t\to\infty}|c(t)| > 0$, *while* $\limsup|g(t)| < \infty$, $\lim_{x\to\infty}|f(x)| = \infty$, *then there is no solution* $\hat{x}(t)$ *of equation* (4) *with the property*

$$\limsup_{t\to\infty}|\hat{x}(t)| = +\infty, \quad 0 < \liminf_{t\to\infty}\big|\big(x(t)\big)\big| < \infty, \quad \limsup|\hat{x}'(t)| < \infty.$$

Proof. Choose $u(t)\equiv 1$. Then all hypothesis of Theorem 1 are satisfied, but the conclusion $\liminf\big|u(t)/f\big(x(t)\big)\big| > 0$ is impossible.

Note. Observe that the "spike property" of solutions defined later in this paper was one of the main sources of worry in the proof of the lemma and Theorem 2.6.

Similar arguments are easily applied to higher order equations. Consider for example the equation

$$x^{(n)}+c(t)f(x) = g(t), \tag{5}$$

where $n$ is an integer $n > 2$, with identical assumptions made about $f(x)$, $c(t)$, $g(t)$ as in (4). Then the following type of lemmas are not too hard to prove.

LEMMA 2.3 (STANDARD VERSION). *Suppose that there exists a function* $u(t)\in C[T,\infty)$ *for some* $T > t_0$, *such that*:

$$\limsup_{t\to\infty}\big|\big(u(t)g(t)\big)\big| < \infty \quad \textit{and} \quad \liminf_{t\to\infty}\big(-u(t)c(t)\big) > 0,$$

$$\limsup_{t\to\infty}|u'(t)| < \infty, \quad \limsup_{t\to\infty}|u(t)| < \infty.$$

*Then any solution* $x(t)$ *which has the property* $\limsup|\hat{x}(t)| = +\infty$, $\liminf|\hat{x}(t)| < \infty$, *must have the property*:

*Given* $\varepsilon > 0$, $K > 0$, $T > t_0$, *and* $M > 0$ *such that* $t > T$ *whenever* $f(x(t)) > M$ *then either*

$$\left|\frac{x^{(n-1)}(t)}{f(x(t))}\right| > K \quad or \quad \left|\frac{x'(t)}{f(x(t))}\right| > K,$$

*except possibly on an interval of length smaller than* $\varepsilon$.

*The non-standard (equivalent) version of this statement is that*

$$\frac{x^{(n-1)}}{f(x)} \not\approx 0 \quad or \quad \frac{x'}{f(x)} \not\approx 0$$

*on any interval* $[t_1, t_2]$, $t_2 - t_1 \not\approx 0$, *on which* $x(t) \in {}^*\boldsymbol{R}_\infty$.

Proof. Assume to the contrary that there exists an interval $[t_1, t_2]$, $t_2 - t_1 \not\approx 0$ such that $f(\hat{x}(t_1)) = M_1 \in {}^*\boldsymbol{R}_\infty$, $f(\hat{x}(t_2)) = M_2 \in {}^*\boldsymbol{R}_\infty$ and $f(x(t)) \in {}^*\boldsymbol{R}_\infty$, while

$$\frac{x}{f(x)} \approx \frac{x^{(n-1)}}{f(x)} \approx 0 \quad \text{for all } t \in [t_1, t_2].$$

It may be assumed without any loss of generality that $\hat{x}(t_1) = \hat{x}(t_2)$, $f(\hat{x}(t_1)) = f(\hat{x}(t_2)) = M$, where $M = \min[M_1, M_2]$. (It is easily shown that such points $t_1, t_2$ must exist.) We examine the behavior of the function

$$\psi(t) = \frac{u(t)\hat{x}^{n-1}(t)}{f(\hat{x}(t))}$$

on the interval $[t_1, t_2]$.

It is clear that $\psi(t)$ is an infinitesimal on $[t_1, t_2]$, and $\psi(t_1) \approx \psi(t_2) \approx 0$. However,

$$\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} \psi'(t)\,dt$$
$$= \int_{t_1}^{t_2} \left\{\frac{u(t)g(t)}{f(\hat{x}(t))} - u(t)c(t) + \frac{\hat{x}^{(n-1)}(t)}{f(\hat{x}(t))}\left[u' - u\frac{\hat{x}'(t)}{f(\hat{x}(t))}\right]\right\}dt.$$

All terms of the integrand are infinitesimal except $-u(t)c(t)$ which is bounded away from zero by some standard number. This implies that $\psi(t_2) - \psi(t_1) \approx 0$ only if $t_2 - t_1 \approx 0$, which contradicts our assumptions.

COROLLARY (NON-STANDARD). *Suppose* $\limsup|g(t)| < \infty$, $\liminf|g(t)| > 0$. *Then* $\liminf|\hat{x}(t)| < \infty$, $\limsup|\hat{x}(t)| = \infty$ *implies that along the trajectory* $\hat{x}(t)$ *given any* $\tilde{t} \in {}^*\boldsymbol{R}_\infty$ *such that* $f(x(\tilde{t})) \in {}^*\boldsymbol{R}_\infty$ *it must be true that either*

$$\frac{x'(t_1)}{f(x(t_1))} \not\approx 0 \quad or \quad \frac{x^{(n-1)}(t_1)}{f(x(t_1))} \not\approx 0$$

*for some $t_1$ sufficiently close to $t$, except if $f(x(t)) \in {}^*\boldsymbol{R}_\infty$ only in an infinitesimal neighborhood of $\tilde{t}$, and $f(x(t)) \in {}^*\boldsymbol{R}_{bd}$ for some $t$ in any right or left neighborhood of $\tilde{t}$ of standard length.*

Proof. Observe that $u(t) \equiv 1$ satisfies the conditions of Lemma 2.3.

DEFINITION. A solution of a differential equation $x(t)$, $t \in [t_0, \infty)$ is said to have *asymptotic upward spike behavior* (or to have *asymptotic spikes*) if given any $\varepsilon > 0$, any $M > 0$, there exists $T > t_0$, and a number $K > 0$ such that $|f(x(T))| > M$, but for some $t_1, t_2$ such that $t_1 < T < t_2$, $T - t_1 < \varepsilon$, $t_2 - T < \varepsilon$, $|f(x(t_1))| < K$ and $|f(x(t_2))| < K$. An equivalent non-standard definition is: There exists $T \in {}^*\boldsymbol{R}_\infty$ such that $\hat{x}(T) \in {}^*\boldsymbol{R}_\infty$ but $\hat{x}(T+\xi_1) \in {}^*\boldsymbol{R}_{bd}$, $\hat{x}(T-\xi_2) \in {}^*\boldsymbol{R}_{bd}$ for some infinitesimals $\xi_1, \xi_2$. Similarly we define a downward spike if there exists $T \in {}^*\boldsymbol{R}_\infty$ such that $\hat{x}(T) \in {}^*\boldsymbol{R}_{bd}$ but $\hat{x}(T+\xi_1) \in {}^*\boldsymbol{R}_\infty$, $\hat{x}(T-\xi_2) \in {}^*\boldsymbol{R}_\infty$ for some infinitesimals $\xi_1, \xi_2$.

This idea is roughly illustrated on Figs. 2 and 3.

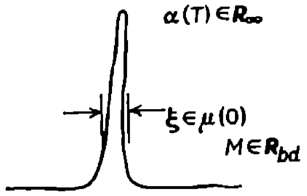


Fig. 2 (upward spike)

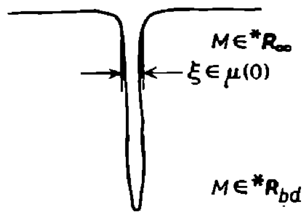


Fig. 3 (downward spike)

We say that a solution $\hat{x}(t)$ has asymptotic spikes if it has either upward or downward asymptotic spike behavior.

THEOREM 2. *We consider the equation*

(6) $$x'' + c(t)f(x) = g(t).$$

*Let there exist a function $u(t) \in C^1[t_1, \infty)$ for some $t_1 > t_0$, such that $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = 0$, and such that $\liminf u(t)c(t) > 0$, while $\limsup u(t)g(t) < \infty$. Then a solution $\hat{x}(t)$ of (6) such that $f(\hat{x}(t))$ is non-oscillatory will have the property* $\liminf |\hat{x}(t)| < \infty$, $\limsup |\hat{x}(t)| = \infty$, *only if $\hat{x}(t)$ has the asymptotic spike property.*

Proof. Consider the possibility that $\hat{x}(t)$ has the property $\liminf |\hat{x}(t)| < \infty$, $\limsup |\hat{x}(t)| = \infty$, but $\hat{x}(t)$ fails to have the spike property.

We examine the behavior of the function

$$\psi(\hat{x}(t), t) = \frac{u(t)\hat{x}'(t)}{f(\hat{x}(t))}$$

on the infinite part of ${}^*\boldsymbol{R}_{+\infty} \cdot f(\hat{x}(t)) \neq 0$ because of the assumption concerning non-oscillatory behavior of $f(x(t))$,

$$\psi'(t) = \frac{u(t)}{f(\hat{x}(t))}\big(g(t) - c(t)f(\hat{x})\big) + \frac{u'(t)\hat{x}'(t)}{f(x)} - \frac{u(t)f'(x)(x')^2}{f^2(\hat{x}(t))}.$$

It follows from our assumptions that there must exist points $t_1, t_2 \in {}^*\boldsymbol{R}_{+\infty}$ such that $|x(t_1)| = |x(t_2)| = m \not\approx 0$, $m \in {}^*\boldsymbol{R}_{bd}$, $|x(t)| > m$ for all $t \in (t_1, t_2)$, and $\hat{x}(\tilde{t}) = M \in {}^*\boldsymbol{R}_\infty$ for some $\tilde{t} \in (t_1, t_2)$. Without any loss of generality assume $\hat{x}(t) > 0$ on $[t_1, t_2]$, and $\hat{x}'(t_1) \geqslant 0$, $\hat{x}'(t_2) \leqslant 0$. Suppose that $t_2 - t_1$ is not an infinitesimal. We claim that in some infinitesimal neighborhood of $t_1$ (and of $t_2$) the function $\psi(t)$ will assume infinitesimal value. If $\hat{x}'(t_1) \in {}^*\boldsymbol{R}_{bd}$, then $u(t_1)\dfrac{\hat{x}'(t_1)}{f(\hat{x}(t_1))} \approx 0$ since $\dfrac{\hat{x}'(t_1)}{f(\hat{x}(t_1))} \in {}^*\boldsymbol{R}_{bd}$, while $u(t) \approx 0$, $\forall t \in {}^*\boldsymbol{R}_{+\infty}$. If $\hat{x}'(t_1) \in {}^*\boldsymbol{R}_{+\infty}$ in some neighborhood $N_\varepsilon(t_1) = [t_1 - \varepsilon, t_1 + \varepsilon]$ for some standard number $\varepsilon > 0$ while $\hat{x}(t)$ is finite in $N_\varepsilon(t_1)$, then we can easily get a contradictory statement

$$\min_{t \in [t_1 - \varepsilon, t_1]} \big(x'(t)\big) = -\eta \in \boldsymbol{R}_{-\infty} \Rightarrow \big(x(t-\varepsilon)\big) \in {}^*\boldsymbol{R}_{-\infty} \Rightarrow f\big(\hat{x}(t-\varepsilon)\big) \in {}^*\boldsymbol{R}_{-\infty},$$

since $\lim_{x\to\infty} f(x) = \infty$, $xf(x) > 0$ for sufficiently large $x$. Hence $\hat{x}'(t) \in {}^*\boldsymbol{R}_\infty$, $\forall t \in [\tilde{t}_1, \tilde{t}_1 - \varepsilon]$ is some interval $[\tilde{t}_1, \tilde{t}_1 - \varepsilon]$ by *continuity of $\hat{x}'(t)$, and $\hat{x}(t) \in {}^*\boldsymbol{R}_{-\infty}$ for all $t \in [t_1, t_1 - \varepsilon)$, for a suitable choice of $\tilde{t}_1$. A similar argument can be applied with regard to the behavior of $\hat{x}(t)$, and $\hat{x}'(t)$ on $[t_1, t_1 + \varepsilon)$ which shows that $\hat{x}(t)$ and therefore $f(\hat{x}(t))$ has the spike property, since at some point on such interval $f(\hat{x}(T)) \in {}^*\boldsymbol{R}_{bd}$ for some $T \in [t_1, t_2]$ and $x'(T) = 0$. (An identical argument applies to $t_2$.) Hence if $\hat{x}(t)$ fails to have the spike property in $[t_1, t_2]$ we have $T_1 \approx t_1$ and $T_2 \approx t_2$ such that $\psi(T_1) \approx \psi(T_2) \approx 0$, $T_2 - T_1 \not\approx 0$. But

$$\psi(T_2) - \psi(T_1) = \int_{T_1}^{T_2} \psi'(t)\,dt = \int_{T_1}^{T_2} \frac{u(t)g(t)}{f(\hat{x}(t))} - u(t)c(t) + \frac{\hat{x}'(t)}{f(\hat{x}(t))}\left[u' - u\frac{\hat{x}'(t)}{f(\hat{x}(t))}\right]dt.$$

By our previous arguments every term except $-u(t)c(t)$ is infinitesimal on $[t_1, t_2]$. Hence

$$\int_{T_1}^{T_2} -\big(u(t)c(t)\big)dt = \int_{T_1}^{T_2} [\psi'(t) + \xi]\,dt, \qquad \text{where } \xi \approx 0,$$

and $\psi(T_2) - \psi(T_1) \not\approx 0$ which is a contradiction.

With additional assumptions concerning behavior of $x^{(n-1)}$ it is easy to copy this proof for equation (5). It should be pointed out that non-linear differential equations do have the assymptotic spike property and examples of such behavior can be constructed even for linear first order inhomogeneous equations with $C^\infty$ forcing term.

For example the equation

$$x'' + 4tx' + 4x = 12t\sin(2t^2) + 4 + 16t^3\cos(4t^2), \qquad t \geqslant \pi/2,$$

turns out to have a solution $x = t\sin^2(t^2)+1$ which is positive and with the asymptotic spike property. This can be checked by substitution.

Other examples are easily constructed by picking a function $(C^\infty[t_0, \infty))$ with the asymptotic spike property and constructing the generally non-linear equation it satisfies.

It is not hard to construct even a first order equation whose solution has asymptotic spikes. Let

$$x(t) = \sum_{n=1}^{\infty}\left\{n\pi\left[U\left(n\pi-\frac{1}{n\pi}\right)-U\left(n\pi+\frac{1}{n\pi}\right)\right]\exp\left(\frac{-\frac{1}{n^2\pi^2}}{\frac{1}{n^2\pi^2}-(n\pi-t)^2}\right)\right\}t,$$

where $U$ stands for the unit step function. It is easy to see that $x(t)\in C^\infty$ for all $x > \pi$. Denoting by $\psi(n, t)$ the function:

$$\psi(n, t) = \exp\left(\frac{-\frac{1}{n^2\pi^2}}{\frac{1}{n^2\pi^2}-(n\pi-t)^2}\right), \qquad \psi' \equiv \frac{d\psi(n, t)}{dt},$$

etc. ,..., we see that $x(t)$ satisfies the linear first order, differential equation $x' - \frac{1}{t}x(t) = g(t)$, where

$$g(t) = \sum_{n=1}^{\infty}\left\{n\pi\left[U\left(n\pi-\frac{1}{n\pi}\right)-U\left(n\pi+\frac{1}{n\pi}\right)\right]t\psi'(t)\right\}.$$

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