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Some properties of a non-linear integral Volterra equation with deviated argument

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Abstract. The aim of this paper is to indicate some effect caused by the deviated, continuous argument ψ in the equation

$$u(x) = F\left[x, u(x), \int_a^x H\left(x, t, u\left(\psi(t)\right)\right)dt\right].$$

By the emission of deviation we understand a set consisting of all continuous solutions of this equation. By the region of emission of deviation we understand a set of points of the plane which belong to all curves generated by emission of deviation functions.

In Section 1 we establish the conditions for compactness and upper semi-continuity (with respect to inclusion) of emission. In Section 2 we give some theorems concerning the regions of emissions.

The solution of our equation can be considered as an operation (multivalued, in general) defined on the space of points (F, H, ψ). In Section 3 we give sufficient conditions for this operation to be continuous, and also sufficient conditions under which the solution depends continuously on a functional.

In this paper we consider a collection of non-linear integral equations of the Volterra type

$$\text{(I)} \qquad u(x) = F\left[x, u(x), \int_a^x H\left(x, t, u\left(\psi(t)\right)\right)dt\right] \qquad (\psi \in \Phi),$$

where u denotes an unknown function and F, H, ψ are given real functions. Our aim is to indicate some effects which are caused by deviated argument in equation (I).

As regards theorems on the existence and uniqueness of solutions for an equation of the type (I), see [1], [9], [10] and [3]. The problem of the existence and uniqueness of the solution for equation (I) when $F(x, y, z) = z$ has been treated in [2], [5], [7] and [8].

0. Assumptions (a), (b) and (c), given below, are valid throughout this paper and will not be repeated in formulations of particular theorems.

Suppose that:

(a) the function $F(x, s, z)$ is defined and continuous on $a \leqslant x \leqslant b$, $-\infty < s, z < \infty$,

(b) the function $H(x, t, v)$ is defined and continuous on $a \leqslant x \leqslant b$, $a \leqslant t \leqslant x$, $-\infty < v < \infty$,

(c) Φ denotes the set of all continuous functions from $[a, b]$ into $[a, b]$.

By $\|\cdot\|$ we denote the usual supremum norm on the set of bounded and continuous functions. If A, B are non-empty subsets of a plane with metric ϱ, then

$$\operatorname{dist}(A, B) = \max[\sup_{x \in A} \varrho(x, B), \sup_{y \in B} \varrho(y, A)],$$

where $\varrho(x, Z) = \inf\{\varrho(x, z): z \in Z\}$.

By a *solution* of (I) we mean any function $u \in C[a, b]$ satisfying equation (I).

Let $\psi \in \Phi$ and $\Delta \subseteq \Phi$. By the *emission of a deviation* ψ we understand the set $\mathscr{E}_\psi$ consisting of all solutions of (I). An *emission* $\mathscr{E}_\Delta$ *of a set* Δ is said to be the set

$$\mathscr{E}_\Delta = \bigcup \{\mathscr{E}_\psi: \psi \in \Delta\}.$$

By the *region of the emission* of a deviation ψ, or of the set of deviations Δ, we mean a set of all those points in $[a, b] \times (-\infty, \infty)$ through which pass the curves generated by the functions belonging to the corresponding emission. These regions will be denoted by e_ψ, e_Δ, respectively.

Let $a \leqslant x_0 \leqslant b$. Denote by $e_\Delta(x_0)$ the intersection of the region of the emission e_Δ of a set Δ with the hyperplane $x = x_0$.

1. In this section we assume that the investigated emissions are non-empty.

1.1. THEOREM. *Let $\Delta \subseteq \Phi$ and let a function F satisfy the Lipschitz condition with a constant $\lambda < 1$ with respect to the second variable. If $\mathscr{E}_\Delta$ is a bounded set, then the functions $\mathscr{E}_\Delta$, being its members, are equicontinuous.*

Proof. Let $x_1, x_2 \in [a, b]$ and $u \in \mathscr{E}_\Delta$. On account of the Lipschitz condition one can easily get the inequality

$$(1-\lambda)\cdot|u(x_1)-u(x_2)| \leqslant \Big|F\Big[x_2, u(x_2), \int_a^{x_2} H\big(x_2, t, u(\psi(t))\big)\Big]dt - \\ -F\Big[x_1, u(x_2), \int_a^{x_1} H\big(x_1, t, u(\psi(t))\big)dt\Big]\Big|,$$

where $\psi \in \Delta$ is such that $u \in \mathscr{E}_\psi$. Therefore, by uniform continuity of functions F and H on corresponding compact sets, our assertion follows.

1.2. THEOREM. *Let a set of deviations $\Delta \subset \Phi$ consist of equi-continuous functions. If the set $\mathscr{E}_\Delta$ is bounded, then the set $\mathscr{E}_{\bar{\Delta}}$ is closed* [1].

Proof. Let (u_i), $u_i \in \mathscr{E}_{\bar{\Delta}}$, be a sequence convergent to $u_0 \in C[a, b]$. For any $i = 1, 2, \ldots$ there exists $\psi_i \in \bar{\Delta}$ such that $u_i \in \mathscr{E}_{\psi_i}$. By the Arzelà theorem we can choose from it a convergent subsequence (ψ_n). Let $(\psi_n) \to \psi_0$. We consider the restrictions of H and F to the suitable compact set. Because

$$\lim u_n(\psi_n(x)) = u_0(\psi_0(x)) \quad \text{uniformly on } [a, b],$$

and

$$\Big| u_0(x) - F\Big[x, u_0(x), \int_a^x H\big(x, t, u_0(\psi_0(t))\big)dt\Big]\Big|$$

$$\leqslant |u_0(x) - u_n(x)| + \Big| F\Big[x, u_n(x), \int_a^x H\big(x, t, u_n(\psi_n(t))\big)dt\Big] -$$

$$- F\Big[x, u_0(x), \int_a^x H\big(x, t, u_0(\psi_0(t))\big)dt\Big]\Big|$$

we have

$$u_0(x) = F\Big[x, u_0(x), \int_a^x H\big(x, t, u_0(\psi_0(t))\big)dt\Big] \quad \text{for } x \in [a, b].$$

From 1.1 and 1.2 there follows:

1.3. THEOREM. *Let a function F satisfy the Lipschitz condition with a constant $\lambda < 1$ with respect to the second variable, and let a set of deviations $\Delta \subset \Phi$ consist of equi-continuous functions. If the set $\mathscr{E}_\Delta$ is bounded, then the set $\mathscr{E}_{\bar{\Delta}}$ is compact.*

Now we formulate conditions concerning upper semi-continuity with respect to inclusion of the emission of deviations.

1.4. THEOREM. *Let $\Delta \subseteq \Phi$ and let $\mathscr{E}_\Delta$ be a bounded set consisting of equi-continuous functions. Then for any deviation $\psi \in \Delta$ and for any $\varepsilon > 0$ there exists such $\delta > 0$ that if $\varphi \in \Delta$, $\|\psi - \varphi\| < \delta$, then*

$$\mathscr{E}_\varphi \subset \{v \in C[a, b]\colon \inf_{u \in \mathscr{E}_\psi} \|v - u\| < \varepsilon\}.$$

Proof. Fix $\psi \in \Delta$ and $\varepsilon > 0$. We have to prove the existence of such a number $\delta > 0$ that conditions $\varphi \in \Delta$, $\|\psi - \varphi\| < \delta$ imply $\mathscr{E}_\varphi \subset K(\mathscr{E}_\psi, \varepsilon)$, where $K(\mathscr{E}_\psi, \varepsilon)$ denotes the generalized open ball with its center in $\mathscr{E}_\psi$ and radius ε.

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[1] $\bar{\Delta}$ denotes a closure of Δ (in $C[a, b]$).

Suppose that the theorem is false. Let $\delta = i^{-1}$. Then for any $i = 1, 2, \ldots$ there exist $\eta_i \in \Delta$ and $u_i \in \mathscr{E}_{\eta_i}$ so that

$$\|\eta_i - \psi\| < i^{-1}, \quad u_i \notin K(\mathscr{E}_\psi, \varepsilon).$$

Because $\mathscr{E}_\Delta$ is a conditionally compact set, (u_i) has a convergent subsequence (u_n). Let $(u_n) \to u_0$. We prove that $u_0 \in \mathscr{E}_\psi$.

If suffices to prove that

$$\lim \int_a^x H[x, t, u_n(\eta_n(t))]\, dt = \int_a^x H[x, t, u_0(\psi(t))]\, dt.$$

We have $\|\eta_n - \psi\| \to 0$ and $\|u_n - u_0\| \to 0$. If we restrict H to a suitable compact set, then for any $\varepsilon > 0$ there exists a natural number N such that

$$\Big| \int_a^x H[x, t, u_n(\eta_n(t))]\, dt - \int_a^x H[x, t, u_0(\psi(t))]\, dt \Big|$$
$$\leqslant \int_a^x \big| H[x, t, u_n(\eta_n(t))] - H[x, t, u_0(\psi(t))] \big|\, dt \leqslant \varepsilon |b - a|$$

for $n > N$ and $x \in [a, b]$. Thus

$$u_0(x) = F\Big[x, u_0(x), \int_a^x H\big(x, t, u_0(\psi(t))\big)\, dt\Big]$$

for $x \in [a, b]$.

Because $\inf\{\|u_n - u\|: u \in \mathscr{E}_\psi\} \leqslant \|u_n - u_0\|$, for ε there exists $N(\varepsilon)$ such that $u_n \in K(\mathscr{E}_\psi, \varepsilon)$, where $n > N(\varepsilon)$.

1.5. Corollary. *Let the assumptions from* 1.4 *be satisfied. Assume that $\psi_0 \in \Delta$ and equation* (I) *has only one solution u_0 for $\psi = \psi_0$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi \in \Delta$ implies $\|\varphi - \psi_0\| < \delta$, then for $\psi = \varphi$ any solution u of* (I) *fulfils the condition*

$$\|u - u_0\| < \varepsilon.$$

2. In this section we give some theorems concerning the regions of emissions.

2.1. Theorem. *If the emission of a set of deviations $\Delta \subseteq \Phi$ is compact, then its region e_Δ is compact.*

Proof. Let (P_i) be a sequence of points beloging to e_Δ. For any $i = 1, 2, \ldots$ there exist the function $u_i \in \mathscr{E}_\Delta$ and the number $x_i \in [a, b]$ such that $P_i = (x_i, u_i(x_i))$.

From the sequence (u_i) we can choose a subsequence (u_n) convergent uniformly in $[a, b]$ to some function $u_0 \in \mathscr{E}_\Delta$. Without loss of generality we can assume that (x_n) converges to $x_0 \in [a, b]$. Let $P_n = (x_n, u_n(x_n))$, $P_0 = (x_0, u_0(x_0))$. Then $P_0 \in e_\Delta$ and $(P_n) \to P_0$.

2.2. THEOREM. *Let the assumptions from* 1.4 *be satisfied. Then for any deviation* $\psi \in \Delta$ *and for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\varphi \in \Delta$, $\|\psi - \varphi\| < \delta$, *then*

$$e_\varphi \subset \{(x, y) \in [a, b] \times (-\infty, \infty): \inf\{|x - \bar{x}| + |y - \bar{y}|: (\bar{x}, \bar{y}) \in e_\psi\} < \varepsilon\}.$$

Proof. Fix $\varepsilon > 0$ and $\psi \in \Delta$. There exists such a number $\delta > 0$ that from $\varphi \in \Delta$ and $\|\varphi - \psi\| < \delta$ it follows that

$$\mathscr{E}_\varphi \subset \{v \in C[a, b]: \inf_{u \in \mathscr{E}_\psi} \|v - u\| < \varepsilon\}.$$

Let $Q_0 = (x_0, y_0) \in e_\varphi$. Then there exists $u_0 \in \mathscr{E}_\varphi$ such that $u_0(x_0) = y_0$ and $\inf\{\|u_0 - u\|: u \in \mathscr{E}_\psi\} < \varepsilon$.

Since

$$\begin{aligned}\inf\{\varrho(Q, Q_0): Q \in e_\psi\} &= \inf_{u \in \mathscr{E}_\psi} \inf\{\varrho(Q, Q_0): Q = (x, y),\ u(x) = y\} \\ &= \inf_{u \in \mathscr{E}_\psi} \inf\{|x - x_0| + |u(x) - u_0(x_0)|: (x, u(x)) \in [a, b] \times (-\infty, \infty)\} \\ &\leqslant \inf\{|x - x_0| + |u(x_0) - u_0(x_0)|: u \in \mathscr{E}_\psi\} = \inf_{u \in \mathscr{E}_\psi} |u(x_0) - u_0(x_0)| \\ &\leqslant \inf_{u \in \mathscr{E}_\psi} \|u - u_0\| < \varepsilon,\end{aligned}$$

we have

$$Q_0 \in K(e_\psi, \varepsilon).$$

This completes the proof.

2.3. THEOREM. *Let* (Δ_m) *be a sequence of non-empty and closed subsets of the set* Φ *such that* $\Delta_m \supseteq \Delta_{m+1}$ $(m = 1, 2, \ldots)$ *and* $\lim \delta(\Delta_m) = 0$ (2). *Assume, moreover, that* F *is bounded and satisfies the Lipschitz condition with a constant* $\lambda < 1$ *with respect to the second variable, and that* Δ_1 *is a set of equicontinuous functions.*

Then there exists exactly one deviation $\psi_0 \in \bigcap_1^\infty \Delta_m$ *and the following holds: if* $\mathscr{E}_{\psi_0} \neq \emptyset$, *then for any* $\varepsilon > 0$ *there exists a natural number* M *such that*

$$\operatorname{dist}[e_{\Delta_m}(x), e_{\psi_0}(x)] < \varepsilon,$$

where $m > M$ *and* $x \in [a, b]$.

Proof. Assume the existence of $\varepsilon_0 > 0$, a sequence of numbers (x_i), $a \leqslant x_i \leqslant b$ and a subsequence (Δ_i) such that

$$\operatorname{dist}[e_{\Delta_i}(x_i), e_{\psi_0}(x_i)] \geqslant \varepsilon_0.$$

It suffices to consider the case

$$\sup\{\varrho(P, e_{\psi_0}(x_i)): P \in e_{\Delta_i}(x_i)\} \geqslant \varepsilon_0.$$

(2) $\delta(\Delta_m)$ denotes a diameter of Δ_m.

Fix an index i. Then there exists a sequence of points $(P_n^{(i)})$,

$$P_n^{(i)} = (x_i, u_n^{(i)}(x_i)), \quad \text{where } u_n^{(i)} \in \mathscr{E}_{\Delta_i}$$

such that

$$\varepsilon_0 - n^{-1} \leqslant \varrho(P_n^{(i)}, e_{\varphi_0}(x_i))$$

for $n = 1, 2, \ldots$ Because the set $\mathscr{E}_{\Delta_i}$ is compact, the sequence $(u_n^{(i)})$ contains a subsequence $(u_k^{(i)})$ such that $(u_k^{(i)}) \rightrightarrows \bar{u}_i$ on $[a, b]$, where $\bar{u}_i \in \mathscr{E}_{\Delta_i}$. We denote

$$\bar{P}_i = (x_i, \bar{u}_i(x_i)).$$

Because $\lim\limits_{k\to\infty} \varrho(P_k^{(i)}, \bar{P}_i) = 0$, where $P_k^{(i)} = (x_i, u_k^{(i)}(x_i))$, we have $\lim\limits_{k\to\infty} \varrho(P_k^{(i)}, e_{\varphi_0}(x_i)) = \varrho(\bar{P}_i, e_{\varphi_0}(x_i))$. Hence

$$\varrho(\bar{P}_i, e_{\varphi_0}(x_i)) \geqslant \varepsilon_0.$$

Since $\mathscr{E}_{\Delta_1} \supseteq \mathscr{E}_{\Delta_i}$, the sequence $(\bar{u}_i)$ contains a subsequence $(\bar{u}_k)$ such that $(\bar{u}_k) \rightrightarrows \bar{u}$ on $[a, b]$. Consider the sequence $(\bar{P}_k)$ and let $\lim x_k = \bar{x}$. Then

$$\varrho(\bar{P}_k, \bar{P}) \to 0, \quad \text{where } \bar{P} = (\bar{x}, u(\bar{x})).$$

It can be easily proved that

$$\varrho(\bar{P}, e_{\varphi_0}(\bar{x})) > \varepsilon_0.$$

It suffices to prove that $\bar{u} \in \mathscr{E}_{\varphi_0}$. Since $\bar{u}_k \in \mathscr{E}_{\Delta_k}$, there exists $\bar{\varphi}_k \in \Delta_k$ such that $\bar{u}_k \in \mathscr{E}_{\bar{\varphi}_k}$. Without loss of generality we can assume that $(\bar{\varphi}_k) \rightrightarrows \bar{\varphi}_0$ on $[a, b]$. We have

$$\lim \|\varphi_0 - \bar{\varphi}_k\| \leqslant \lim \delta(\Delta_k) = 0.$$

Hence

$$\bar{\varphi}_0(x) = \varphi_0(x) \quad \text{for } x \in [a, b].$$

Now consider the restriction of H to a suitable compact set. Then for $\varepsilon > 0$ there exists $K(\varepsilon)$ such that

$$\left| \int_a^x H[x, t, \bar{u}_k(\bar{\varphi}_k(t))]\,dt - \int_a^x H[x, t, \bar{u}(\varphi_0(t))]\,dt \right|$$
$$\leqslant \int_a^x \big|H[x, t, \bar{u}_k(\bar{\varphi}_k(t))] - H[x, t, \bar{u}(\varphi_0(t))]\big|\,dt < \varepsilon|b-a|$$

for $k > K(\varepsilon)$ and $x \in [a, b]$. From this it follows that

$$\bar{u}(x) = F\Big[x, \bar{u}(x), \int_a^x H\big(x, t, \bar{u}(\varphi_0(t))\big)\,dt\Big] \quad \text{for } x \in [a, b].$$

2.4. THEOREM. *Let a function F satisfy the Lipschitz condition with a constant $\lambda < 1$ with respect to the second variable. Let a set of deviations $\Delta \subset \Phi$ be closed and consist of equicontinuous functions, and let its emission $\mathcal{E}_\Delta$ be non-empty and uniformly bounded. Then the function*

$$x \mapsto e_\Delta(x)$$

maps continuously $[a, b]$ into the metric space of non-empty compact subsets of the plane.

Proof. Let $P_1 \in e_\Delta(x_1)$ and $\varepsilon > 0$. Then there exist a solution $u \in \mathcal{E}_\Delta$ and a number $\delta = \delta(\varepsilon) > 0$ such that

$$P_1 = (x_1, u(x_1)) \quad \text{and} \quad |u(x) - u(x_1)| < \varepsilon \quad \text{for } |x - x_1| < \delta.$$

It follows that the distance of any point $P_1 \in e_\Delta(x_1)$ from the point $P = (x, u(x)) \in e_\Delta(x)$ is less than ε whenever $|x - x_1| < \delta$. Analogously, the distance between each point from the set $e_\Delta(x)$ and the set $e_\Delta(x_1)$ is less than ε if $\delta > 0$ is small enough. So

$$\operatorname{dist}[e_\Delta(x_1), e_\Delta(x)] < \varepsilon \quad \text{for } |x - x_1| < \delta(\varepsilon).$$

This completes the proof of the theorem.

3. The solution of (I) is dependent on the functions F, H and ψ. This solution can be considered as an operation (multivalued, in general) defined on the space of points (F, H, ψ). In this section we give sufficient conditions for this operation to be continuous, and also sufficient conditions under which the solution depends continuously on a functional parameter.

Let:

$\mathcal{E}(\bar{F}, \bar{H}, \bar{\psi})$ be the set of solutions of (I) for $F = \bar{F}$, $H = \bar{H}$, $\psi = \bar{\psi}$.

$\mathcal{F}$ is the set of real functions $F(x, s, z)$ defined, continuous and bounded on $a \leqslant x \leqslant b$, $-\infty < s, z < \infty$, and in the second variable satisfying the Lipschitz condition, with a common constant $\lambda < 1$.

$\mathcal{H}$ is the set of real functions $H(x, t, v)$ defined, continuous and bounded for $a \leqslant x \leqslant b$, $a \leqslant t \leqslant x$, $-\infty < v < \infty$.

Let us assume, moreover, that $\mathcal{E}(F, H, \psi) \neq \emptyset$ for any $F \in \mathcal{F}$, $H \in \mathcal{H}$, $\psi \in \Phi$.

3.1. THEOREM. *For any functions $F_0 \in \mathcal{F}$, $H_0 \in \mathcal{H}$, $\psi_0 \in \Phi$ and $\varepsilon > 0$ there exists such a number $\delta > 0$ that if $F \in \mathcal{F}$, $H \in \mathcal{H}$, $\psi \in \Phi$*

$$\|F - F_0\| < \delta, \quad \|H - H_0\| < \delta, \quad \|\psi - \psi_0\| < \delta,$$

then

$$\mathscr{E}(F, H, \psi) \subset \{v \in C[a, b]:\ \inf[\|v-u\|:\ u \in \mathscr{E}(F_0, H_0, \psi_0)] < \varepsilon\}.$$

Proof. Fix $F_0 \in \mathscr{F}$, $H_0 \in \mathscr{H}$, $\psi_0 \in \Phi$ and $\varepsilon > 0$. Suppose that the theorem is false. Then for any $n = 1, 2, \ldots$ there exist $F_n \in \mathscr{F}$, $H_n \in \mathscr{H}$, $\psi_n \in \Phi$ and $u_n \in \mathscr{E}(F_n, H_n, \psi_n)$ such that

$$\|F_n - F_0\| < n^{-1}, \qquad \|H_n - H_0\| < n^{-1}, \qquad \|\psi_n - \psi_0\| < n^{-1}$$

and

$$u_n \notin K_\varepsilon,$$

where K_ε denotes the generalized open ball having its centre in $\mathscr{E}(F_0, H_0, \psi_0)$ and radius ε.

Let C_1, C_2 be numbers that bound the functions H_n $(n = 1, 2, \ldots)$, F_n $(n = 1, 2, \ldots)$, respectively. Let us write

$$Q_1 = \{(x, t, v):\ a \leqslant x \leqslant b,\ a \leqslant t \leqslant x,\ |v| \leqslant C_1\},$$
$$Q_2 = \{(x, s, z):\ a \leqslant x \leqslant b,\ |s| \leqslant C_2,\ |z| \leqslant C_1 |b-a|\}.$$

Because the sequence of the restrictions H_n, $n = 1, 2, \ldots$, to the set Q_1 consists of equi-continuous functions, for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\Big|\int_a^{x_1} H_n[x_1, t, u_n(\psi_n(t))]\,dt - \int_a^{x_2} H_n[x_2, t, u_n(\psi_n(t))]\,dt\Big|$$
$$\leqslant \int_a^{x_1} \big|H_n[x_1, t, u_n(\psi_n(t))] - H_n[x_2, t, u_n(\psi_n(t))]\big|\,dt + \int_{x_1}^{x_2} \big|H_n[x_2, t, u_n(\psi_n(t))]\big|\,dt$$
$$\leqslant C_1 |x_2 - x_1| + \int_a^{x_1} \big|H_n[x_1, t, u_n(\psi_n(t))] - H_n[x_2, t, u_n(\psi_n(t))]\big|\,dt$$
$$< C_1 \varepsilon + \varepsilon |b-a|$$

for $|x_1 - x_2| < \eta$ and $n = 1, 2, \ldots$ The sequence of the restrictions F_n, $n = 1, 2, \ldots$, to Q_2 consists of equi-continuous functions. We obtain that for $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\Big|F_n\Big[x_1, u_n(x_2), \int_a^{x_1} H_n\big(x_1, t, u_n(\psi_n(t))\big)\,dt\Big] - F_n\Big[x_2, u_n(x_2), \int_a^{x_2} H_n\big(x_2, t, u_n(\psi_n(t))\big)\,dt\Big]\Big| < \varepsilon$$

for $|x_1-x_2|<\delta$ and $n=1,2,\ldots$ Hence we have

$$|u_n(x_1)-u_n(x_2)| \leqslant \Big|F_n\Big[x_1, u_n(x_1), \int_a^{x_1} H_n\big(x_1, t, u_n(\psi_n(t))\big)dt\Big] - F_n\Big[x_1, u_n(x_2), \int_a^{x_1} H_n\big(x_1, t, u_n(\psi_n(t))\big)dt\Big]\Big| + \Big|F_n\Big[x_1, u_n(x_2), \int_a^{x_1} H_n\big(x_1, t, u_n(\psi_n(t))\big)dt\Big] - F_n\Big[x_2, u_n(x_2), \int_a^{x_2} H_n\big(x_2, t, u_n(\psi_n(t))\big)dt\Big]\Big|$$

$$\leqslant \lambda|u_n(x_1)-u_n(x_2)|+\varepsilon$$

if $|x_1-x_2|<\delta$, $n=1,2,\ldots$ Thus the functions u_n, $n=1,2,\ldots$, are equi-continuous on $[a,b]$. By the Arzelà Theorem each subsequence of (u_n) contains a convergent subsequence (u_k). Let $(u_k)\rightrightarrows u_0$ on $[a,b]$. We prove that $u_0\in\mathscr{E}(F_0, H_0, \psi_0)$.

Because $u_k\in\mathscr{E}(F_k, H_k, \psi_k)$ and $(u_k)\rightrightarrows u_0$ on $[a,b]$ and $(F_k)\rightrightarrows F_0$ on Q_2, it suffices to prove that

$$\lim \int_a^x H_k[x, t, u_k(\psi_k(t))]\,dt = \int_a^x H_0[x, t, u_0(\psi_0(t))]\,dt.$$

We have $(H_k)\rightrightarrows H_0$ on Q_1 and $(\psi_k)\rightrightarrows\psi_0$ on $[a,b]$. If we restrict H_0 to the set Q_1, then for any $\varepsilon>0$ there exists a natural number K such that

$$\Big|\int_a^x H_k[x, t, u_k(\psi_k(t))]\,dt - \int_a^x H_0[x, t, u_0(\psi_0(t))]\,dt\Big|$$

$$\leqslant \int_a^x \Big(\big|H_k[x, t, u_k(\psi_k(t))]-H_0[x, t, u_k(\psi_k(t))]\big| + \big|H_0[x, t, u_k(\psi_k(t))]-H_0[x, t, u_0(\psi_0(t))]\big|\Big)dt < 2\varepsilon|b-a|$$

for $k>K$ and $x\in[a,b]$. Thus $u_0\in\mathscr{E}(F_0, H_0, \psi_0)$.

Since

$$\inf[\|u_k-u\|:\ u\in\mathscr{E}(F_0, H_0, \psi_0)]\leqslant\|u_k-u_0\|,$$

for ε there exists $N(\varepsilon)$ such that $u_k\in K_\varepsilon$, where $k>N(\varepsilon)$.

3.2. Corollary. *Let*

$$\|F_n-F_0\|\to 0, \quad \|H_n-H_0\|\to 0, \quad \|\psi_n-\psi_0\|\to 0,$$

where $F_m \in \mathcal{F}$, $H_m \in \mathcal{H}$, $\psi_m \in \Phi$ $(m = 0, 1, \ldots)$. *Suppose that equation* (I) *has for* $F = F_0$, $H = H_0$ *and* $\psi = \psi_0$ *exactly one solution* u_0, *and* u_n *is a solution of* (I) *for* $F = F_n$, $H = H_n$ *and* $\psi = \psi_n$ $(n = 1, 2, \ldots)$. *Then*

$$\lim \|u_n - u_0\| = 0.$$

Let H and ψ fulfil our assumptions and $f(x, s, z, p)$ be a real function, continuous and bounded for $a \leqslant x \leqslant b$, $-\infty < s, z, p < \infty$, and satisfying the Lipschitz condition with a constant $\lambda < 1$ with respect to the second variable. We consider the equation

$$\text{(II)} \qquad u(x) = f\Big[x, u(x), \int_a^x H\big(x, t, u(\psi(t))\big)dt, \mu(x)\Big],$$

where $\mu \in C[a, b]$. Let $\mathcal{E}(\bar{\mu})$ denote the set of continuous solutions of equation (II) on $[a, b]$ for $\mu = \bar{\mu}$. We assume, moreover, that $\mathcal{E}(\bar{\mu}) \neq \emptyset$ for $\mu \in C[a, b]$.

3.3. THEOREM. *For any function* $\mu_0 \in C[a, b]$ *and any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for* $\mu \in C[a, b]$ *and* $\|\mu - \mu_0\| < \delta$

$$\mathcal{E}(\mu) \subset \{v \in C[a, b]: \inf_{u \in \mathcal{E}(\mu_0)} \|v - u\| < \varepsilon\}.$$

The proof is similar to that of Theorem 3.1 and therefore will be omitted.

3.4. COROLLARY. *Let equation* (II) *have only one solution* $u \in C[a, b]$ *tor any* $\mu \in C[a, b]$. *Then this solution, treated as a function of* μ, *is a continuous mapping from* $C[a, b]$ *to* $C[a, b]$.

4. Let R^n be an n-dimensional Euclidean space with the norm $|\cdot|$ and $C_n[a, b]$ the space of continuous functions from $[a, b]$ to R^n with the norm $\|y\| = \max\{|y(x)|: a \leqslant x \leqslant b\}$. Denote

$$H = (H^{(1)}, \ldots, H^{(n)}), \qquad \psi = (\psi^{(1)}, \ldots, \psi^{(n)}), \qquad y = (y^{(1)}, \ldots, y^{(n)}),$$

$$\int_a^x H[x, t, y(\psi(t))]\,dt = \Big(\int_a^x H^{(1)}[x, t, y^{(1)}(\psi^{(1)}(t)), \ldots, y^{(n)}(\psi^{(n)}(t))]\,dt, \ldots$$
$$\ldots, \int_a^x H^{(n)}[x, t, y^{(1)}(\psi^{(1)}(t)), \ldots, y^{(n)}(\psi^{(n)}(t))]\,dt\Big).$$

In this section we formulate a theorem of the Kneser type for an equation

$$\text{(III)} \qquad y(x) = \int_a^x H[x, t, y(\psi(t))]\,dt.$$

Suppose that $H^{(i)}$, $\psi^{(i)}$ $(i = 1, 2, \ldots, n)$ are given real functions such that

1° the function $H^{(i)}(x, t, v_1, \ldots, v_n)$ is defined, continuous and bounded for $a \leqslant x \leqslant b$, $a \leqslant t \leqslant x$, $-\infty < v_1, \ldots, v_n < \infty$,

2° the function $\psi^{(i)}(x)$ is defined, continuous and satisfies the inequality

$$a \leqslant \psi^{(i)}(x) \leqslant x$$

for $a \leqslant x \leqslant b$.

It is known ([4]) that under the above assumptions equation (III) has a solution in $C_n[a, b]$, and the set $\mathcal{E}$ of all its continuous solutions is compact. We shall now prove that $\mathcal{E}$ is a connected set.

Suppose that $\mathcal{E}$ is not connected in $C_n[a, b]$. Then there exist non-empty, closed and disjoint sets $\mathcal{E}_1, \mathcal{E}_2$ such that

$$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2.$$

For the restriction of H to a suitable compact set Q there exists ([4], [6]) a sequence (H_n) of equi-continuous functions satisfying the Lipschitz condition with respect to the last variable and $(H_n) \rightrightarrows H$ on the set Q.

Take $y_1 \in \mathcal{E}_1$, $y_2 \in \mathcal{E}_2$ and consider the integral equation

$$\text{(III}n\text{)} \qquad y(x) = \int_a^{\xi} H[x, t, y_i(\psi(t))]\,dt + \int_{\xi}^{x} H_n[x, t, y(\psi(t))]\,dt \qquad (i = 1, 2),$$

where $a \leqslant \xi \leqslant b$. This equation has exactly one solution $y_\xi^{(i,n)}$ ([4]). For $i = 1, 2$, $n = 1, 2, \ldots$, let

$$h_\xi^{(i,n)}(x) = \begin{cases} y_i(x) & \text{for } a \leqslant x \leqslant \xi, \\ y_\xi^{(i,n)}(x) & \text{for } \xi \leqslant x \leqslant b, \end{cases}$$

and

$$\gamma^{(i,n)}(\xi) = h_\xi^{(i,n)}.$$

Then $\gamma^{(i,n)}$ is a continuous function from $[a, b]$ to $C_n[a, b]$ and

$$\Gamma_i^{(n)} = \{h_\xi^{(i,n)}:\ \xi \in [a, b]\}$$

is the connected set. Because $h_a^{(1,n)} = h_a^{(2,n)}$ we obtain that

$$\Gamma_n = \Gamma_1^{(n)} \cup \Gamma_2^{(n)}$$

is the connected set.

By normality of $C_n[a, b]$ for the sets $\mathcal{E}_1, \mathcal{E}_2$ there exists a pair of open, disjoint sets A_1, A_2 such that

$$\mathcal{E}_i \subset A_i \quad (i = 1, 2).$$

Since Γ_m is non-empty $(y_1, y_2 \in \Gamma_m)$ and connected and A_1, A_2 are disjoint sets, $\Gamma_m \setminus \Gamma_m \cap (A_1 \cup A_2) \neq \emptyset$ for every m. Take a sequence (h_m),

$$h_m \in \Gamma_m, \quad h_m \notin A_1 \cup A_2 \quad (m = 1, 2, \ldots).$$

It is easy to verify that the functions forming Γ,

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_m \cup \ldots$$

are commonly bounded and equi-continuous, hence a sequence (h_m) contains a subsequence (h_n) uniformly convergent to some function h_0. Without loss of generality we can assume that $h_n = h_{\xi_n}^{(1,n)}$, where $(\xi_n) \to \xi_0$. Then $h_0(x) = y_1(x)$ for $a \leqslant x \leqslant \xi_0$ and

$$h_n(x) = \int_a^{\xi_n} H[x, t, y_1(\psi(t))]\,dt + \int_{\xi_n}^{x} H_n[x, t, h_n(\psi(t))]\,dt$$

for $\xi_0 < x \leqslant b$. Thus h_0 fulfils (IIIn) in $[a, \xi_0]$ and

$$h_0(x) = \lim \int_a^{\xi_n} H[x, t, y_1(\psi(t))]\,dt + \lim \int_{\xi_n}^{x} H_n[x, t, h_n(\psi(t))]\,dt$$

for $x \in (\xi_0, b]$. It can be easily proved that

$$\lim \int_a^{\xi_n} H[x, t, y_1(\psi(t))]\,dt = \int_a^{\xi_0} H[x, t, y_1(\psi(t))]\,dt$$

and

$$\lim \int_{\xi_n}^{x} H_n[x, t, h_n(\psi(t))]\,dt = \int_{\xi_0}^{x} H[x, t, h_0(\psi(t))]\,dt.$$

Because $a \leqslant \psi^{(i)}(t) \leqslant t \leqslant \xi_0$, then $y_1[\psi(t)] = h_0[\psi(t)]$. Hence

$$h_0(x) = \int_a^{x} H[x, t, h_0(\psi(t))]\,dt$$

for $\xi_0 < x \leqslant b$. Thus we have proved that $h_0 \in \mathscr{E}$.

On the other hand, if $h_0 \in \mathscr{E}$, then $h_0 \in A_1 \cup A_2$. Since $(h_n) \to h_0$ and $A_1 \cup A_2$ is open, there exists an index N such that $h_n \in A_1 \cup A_2$ for $n > N$. This contradiction proves that $h_0 \notin \mathscr{E}$.

So we have proved the following:

4.1. THEOREM. *Let assumptions* 1° *and* 2° *be fulfilled. Then the set* $\mathscr{E}$ *of all solutions of equation* (III) *is non-empty, compact and connected in* $C_n[a, b]$.

4.2. COROLLARY. *Under the assumptions of Theorem* 4.1, *for any* $x \in [a, b]$, *the set* $\{y(x) \colon y \in \mathscr{E}\}$ *is compact and connected in* R^n.

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