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# UNIFORM LIMITS OF DARBOUX FUNCTIONS

BY

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**1. Introduction.** In 1953 Sierpiński [11] proved that any real-valued function defined on an interval $I$ is the limit of a sequence of Darboux functions. This result was also established by Fast [3] and has been extended to very general spaces by Marcus [6]. Sierpiński mentioned in [11] that not every function is the *uniform* limit of a sequence of Darboux functions, although there are functions which are such limits without themselves satisfying the Darboux condition. The main purpose of this article is to provide an intrinsic characterization of those functions which are uniform limits of Darboux functions.

**2. Preliminaries.** Throughout this article we shall be concerned with real-valued functions defined on an interval $I$ (which can be taken to be the entire real line). Such a function $f$ is called a *Darboux function* provided the set $f(J)$ is connected for every interval $J \subset I$. We shall be concerned with the two generalizations of the notion of Darboux function which we now define.

Definition. A function $f$ is in the class $\mathscr{U}_0$ if for every interval $J \subset I$, the set $f(J)$ is dense in the interval $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$. The function $f$ is in the class $\mathscr{U}$ if for every interval $J \subset I$ and every set $A$ of cardinality less than $c$, the set $f(J-A)$ is dense in the interval $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$. (In these definitions the interval $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$ can be replaced by the interval $[f(a), f(b)]$ where $[a, b] = J$.) As we shall see below, the class $\mathscr{U}$ is the uniform closure of the class of Darboux functions. The class $\mathscr{U}_0$ has been considered by Ellis [4], Massera [7], and Radakovic [9], and will be of use in our study of the class $\mathscr{U}$.

If $A$ and $B$ are sets, then $A$ is said to be *c-dense* in $B$ provided every open interval which intersects $B$ contains $c$ points of $A$. We shall use

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freely those properties of Borel sets, and Baire functions which can be found in Kuratowski [5]. We shall use the notation card $A$ to mean the cardinality of the set $A$.

**3. Characterizations of the classes $\mathscr{U}_0$ and $\mathscr{U}$.** In this section we obtain various characterizations of the classes $\mathscr{U}_0$ and $\mathscr{U}$ which we shall find useful in the sequel. We begin with a definition.

Definition 3.1. For a function $f$ on $I$ and each point $x \in I$ define the sets $C_0(f, x)$ and $C(f, x)$ of extended real numbers by: $y \in C_0(f, x)$ provided for each neighborhood $N$ of $y$ and each neighborhood $M$ of $x$ the set $f^{-1}[N] \cap M$ is non-empty; similarly, $y \in C(f, x)$ provided the set $f^{-1}[N] \cap M$ has cardinality $c$. The one sided cluster sets $C_0^+(f, x)$, $C_0^-(f, x)$, $C^+(f, x)$, $C^-(f, x)$ are defined similarly with one sided neighborhoods $M$.

Below we make use of the easily verified fact that $y \in C_0(f, x)$ if and only if for some sequence $\{x_n\}$, $x_n \to x$, we have $f(x_n) \to y$. Note that $f(x) \in C_0^+(f, x) \cap C_0^-(f, x)$.

Lemma 3.1. *If $f \in \mathscr{U}$, then $C(f, x) = C_0(f, x)$ for each $x \in I$.*

Proof. Suppose $y_0 \in C_0(f, x)$, say, $y_0 = \lim f(x_n)$, $x_n \to x_0$. Let $N$ be a neighborhood of $y_0$, $M$ of $x_0$ and let $C = f^{-1}[N] \cap M$. Choose $n$ so large that the interval between $x_n$ and $x_{n+1}$ is contained in $M$ and that between $f(x_n)$ and $f(x_{n+1})$ in $N$. Since $f \in \mathscr{U}$ that subset of $C$ consisting of those points between $x_n$ and $x_{n+1}$ which map between $f(x_n)$ and $f(x_{n+1})$ already has cardinality $c$. Hence $y_0 \in C(f, x_0)$.

Theorem 3.1. *For a function $f$ defined on $I$ the following conditions are equivalent:*

a) $f \in \mathscr{U}_0$;

b) *each of the cluster sets $C_0^-(f, x)$, $C_0^+(f, x)$ is a closed interval for each $x \in I$;*

c) *for each $a, b \in I$ with $a < b$,*

$$\bigcup_{x \in [a,b]} C_0(f, x) = [\inf_{x \in [a,b]} f(x), \sup_{x \in [a,b]} f(x)],$$

*where $C_0(f, a)$ and $C_0(f, b)$ must be interpreted as $C_0^+(f, a)$ and $C_0^-(f, b)$ respectively.*

Proof. a) $\Rightarrow$ b): Let $f \in \mathscr{U}_0$, let $x_0 \in I$, and let $\alpha = \inf C_0^+(f, x_0)$, $\beta = \sup C_0^+(f, x_0)$. If $\alpha = \beta$, there is nothing to prove. Otherwise let $\gamma \in (\alpha, \beta)$. Choose $x_n \to x_0^+$, $y_n \to x_0^+$ with $f(x_n) \to \alpha$, $f(y_n) \to \beta$, and let $I_n$ be the closed interval determined by $x_n$ and $y_n$. Let $N$ be a neighborhood of $\gamma$, $M$ a one sided neighborhood of $x_0$. For $n$ large enough we have $I_n \subset M$ and $f(x_n) < \gamma < f(y_n)$. Since $f \in \mathscr{U}_0$, there is a point $z_n$ between $x_n$ and $y_n$ with $|f(z_n) - \gamma| < 1/n$. Clearly $\gamma = \lim f(z_n)$ and $z_n \to x_0^+$, so

$\gamma \in C_0^+(f, x_0)$. Consequently, $[\alpha, \beta] \subset C_0^+(f, x_0)$ and since the reverse inclusion is clear we have proved b).

b) $\Rightarrow$ c): Suppose each of $C_0^+(f, x)$, $C_0^-(f, x)$ is a closed interval for each $x \in I$. Let $a, b \in I$ with $a < b$, and let

$$K = \bigcup_{x \in [a,b]} C_0(f, x).$$

We first show that $K$ is dense in its closed convex hull $\hat{K}$. If this were not the case there would be an interval $(\alpha, \beta) \subset \hat{K} - K$. Since each $C_0(f, x)$ is an interval, it must lie either entirely above $\beta$ or entirely below $\alpha$. Let $C_0(f, x_0)$ lie below $\alpha$. If every interval $[x_0, x_0+\delta]$, $\delta > 0$, contained an $x_\delta \in [a, b]$ with $C_0(f, x_\delta)$ above $\beta$, it would follow that $C_0(f, x_0)$ had points above $\beta$ — an impossibility. Hence, the supremum $\delta_0$ of all $\delta$ such that $C_0(f, x)$ is below $\alpha$ for all $x$ in $[x_0, x_0+\delta] \cap [a, b]$ is positive. If $\delta_0 \neq \infty$, then $x_0+\delta_0 \in (a, b)$ and since every interval about $x_0+\delta_0$ contains points $x \in [a, b]$ with $C_0(f, x)$ below $\alpha$, we would have $C_0(f, x_0+\delta_0)$ below $\alpha$. But the argument above on the point $x_0$ may be repeated for $x_0+\delta_0$ contradicting the definition of $\delta_0$. Hence $\delta_0 = \infty$. A similar argument shows that $C_0(f, x)$ lies below $\alpha$ for all $x < x_0$. This contradiction establishes our assertion.

Let $\gamma \in \hat{K}$. Then, by what we have just proved, for each $n$ there is an $x_n \in [a, b]$ and a $y_n \in C_0(f, x_n) \cap (\gamma - 1/n, \gamma + 1/n)$. We may assume that $x_n \to x_0 \in [a, b]$. Hence, there are $z_n \in (x_n - 1/n,\ x_n + 1/n) \cap [a, b]$ with $f(z_n) \in (\gamma - 1/n, \gamma + 1/n)$. Thus $\gamma \in C_0(f, x_0) \subset K$. This proves that $K = \hat{K}$. Since $\hat{K}$ is easily seen to be $[\inf_{x \in [a,b]} f(x),\ \sup_{x \in [a,b]} f(x)]$, we have proved c).

c) $\Rightarrow$ a): Suppose c) holds for $f$. Let $a, b \in I$, $a < b$,

$$\alpha = \inf_{x \in [a,b]} f(x), \qquad \beta = \sup_{x \in [a,b]} f(x), \qquad K = \bigcup_{x \in [a,b]} C_0(f, x).$$

Then, if $\gamma \in [\alpha, \beta]$, $\gamma \in K$. Thus every interval about $\gamma$ contains the image of a point in $[a, b]$, and $f \in \mathscr{U}_0$.

We turn now to characterizing the class $\mathscr{U}$. The proof of Theorem 3.2 below makes use of the results obtained in Theorem 3.1.

THEOREM 3.2. *For a function $f$ defined on $I$ the following conditions are equivalent:*

a) $f \in \mathscr{U}$;

b) *for each $a, b \in I$ with $a < b$,*

$$\bigcup_{x \in [a,b]} C(f, x) = [\inf_{x \in [a,b]} f(x),\ \sup_{x \in [a,b]} f(x)],$$

*where $C(f, a)$ and $C(f, b)$ must be interpreted as $C^+(f, a)$ and $C^-(f, b)$ respectively;*

c) $f \in \mathscr{U}_0$ *and for each open interval* $N$, $f^{-1}[N]$ *is empty or c-dense in itself*;

d) $f \in \mathscr{U}_0$ *and* $f$ (*i. e. the graph of* $f$) *is c-dense in itself.*

Proof. a) $\Rightarrow$ b): If $f \in \mathscr{U}$, then $f \in \mathscr{U}_0$ and by Lemma 3.1 we see that b) follows from Theorem 3.1 part c).

b) $\Rightarrow$ c): Since $\bigcup_{x \in [a,b]} C_0(f, x)$ is always a subset of $[\inf_{x \in [a,b]} f, \sup_{x \in [a,b]} f]$ and $C(f, x) \subset C_0(f, x)$, b) implies condition c) in Theorem 3.1. Hence $f \in \mathscr{U}_0$. Let $N$ be an interval for which $f^{-1}[N] \neq \emptyset$. Let $x_0 \in f^{-1}[N]$, and let $M$ be an interval about $x_0$. Clearly there exists $x' \in M$ with $f(x_0) \in C(f, x')$, so $\operatorname{card}[f^{-1}[N] \cap M] = c$. Thus $f^{-1}[N]$ is $c$-dense in itself.

c) $\Rightarrow$ d): Fix a point $x_0$. Let $N$ be an interval about $f(x_0)$, $M$ an interval about $x_0$. Since $f^{-1}[N] \cap M \neq \emptyset$, it must contain $c$ points. Consequently, the rectangle $M \times N$ contains $c$ points of $f$.

d) $\Rightarrow$ a): Let $a, b \in I$, $a < b$, and suppose $f(a) \neq f(b)$. Since $f \in \mathscr{U}_0$, $f[a, b]$ is dense in $(f(a), f(b))$. Thus, if $\gamma \in (f(a), f(b))$ and $N$ is a neighborhood of $\gamma$ there are $x_1, x_2$ in $[a, b]$ with $f(x_1), f(x_2) \in N$ and $f(x_1) < \gamma < f(x_2)$. Since there are $c$-points of $f$ in the rectangle $(x_1, x_2) \times (f(x_1), f(x_2))$, there are $c$-points of $(x_1, x_2)$ (and hence of $[a, b]$) which map into $N$. Consequently $f \in \mathscr{U}$.

**4. Uniform limits of Darboux functions.** We are now ready to prove the main results of this article. Thus in Theorem 4.3 we prove that $\mathscr{U}$ is the uniform closure of the class of Darboux functions. Theorem 4.3 also asserts that a Baire or measurable function is in $\mathscr{U}$ if and only if it is the uniform limit of a sequence of Darboux Baire functions or Darboux measurable functions respectively. Theorems 4.4 and 4.5 provide information about certain classes of functions in $\mathscr{U}$.

We begin with a lemma.

LEMMA 4.1. *Any c-dense in itself subset* $A$ *of* $I$ *is the union of countably many disjoint, non-void subsets each of which is c-dense in* $A$. *Moreover, if* $A$ *is Borel* $\alpha$ *or Lebesgue measurable, then the subsets may be taken to be Borel of class* $\max(\alpha, 2)$ *or Lebesgue measurable respectively.*

Proof. I. The first statement is proved in Boboc and Marcus [2].

II. Now suppose $A$ is any Borel set which is $c$-dense in itself. First we show that any $c$-dense in itself Borel set $B$ has two disjoint Borel subsets each $c$-dense in $B$. Let $\{J_n\}_{n=1}^{\infty}$ be an enumeration of all rational open intervals which intersect $B$. It is well known (see Kuratowski [5], p. 387) that each uncountable Borel set contains a perfect subset, which in turn contains a nowhere dense (relative to the original set) perfect subset. So we pick $P_1$ to be a nowhere dense perfect subset of $J_1 \cap B$ and $Q_1$ to be a nowhere dense perfect subset of $J_1 \cap B - P_1$. By induction we pick $P_n$ to be a nowhere dense perfect subset of

$$J_n \cap B - \bigcup_{i<n} P_i - \bigcup_{i<n} Q_i$$

and $Q_n$ to be a nowhere dense perfect subset of

$$J_n \cap B - \bigcup_{i\leqslant n} P_i - \bigcup_{i<n} Q_i.$$

Then $\bigcup_{n=1}^{\infty} P_n$ and $\bigcup_{n=1}^{\infty} Q_n$ are two disjoint, non-void subsets of $B$ each $c$-dense in $B$.

Now by the above construction there are disjoint subsets $A_1$ and $B_1$ of $A$ each $c$-dense in $A$. In turn, $B_1$ has disjoint subsets $A_2$ and $B_2$ $c$-dense in $B_1$. By induction, $B_n$ has disjoint subsets $A_{n+1}$ and $B_{n+1}$ each $c$-dense in $B_n$ and hence in $A$. Then the sequence $\{A_i\}_{i=2}^{\infty}$ together with $A_1 \cup (A - \bigcup_{i=1}^{\infty} A_i)$ yields the desired sequence decomposing $A$ into disjoint, non-void Borel sets each $c$-dense in $A$. It is easily checked that $A_i$ where $i \geqslant 2$ is an $F_\sigma$ and $A_1 \cup (A - \bigcup_{i=1}^{\infty} A_i)$ is a Borel set of class $\max(\alpha, 2)$.

III. Let $A$ be any Lebesgue measurable subset of $I$ $c$-dense in itself. Let $\{J_n\}_{i=1}^{\infty}$ be an enumeration of all rational open intervals in $R$ which intersect $A$. If $J_n \cap A$ has zero measure, we put $B_n = J_n \cap A$. If $J_n \cap A$ has positive measure, we can find a perfect set $P_n$ of positive measure inside $J_n \cap A$. Then by the process of "removing middle thirds" we can find a nowhere dense perfect subset $B_n$ of $P_n$ which has zero measure. Next put $B = \bigcup_{n=1}^{\infty} B_n$ to obtain a measurable subset of $A$, which is $c$-dense in itself and in $A$ and has zero measure. By part I, $B$ can be decomposed into a sequence of subsets $\{C_n\}_{n=1}^{\infty}$ each $c$-dense in $B$. Then $\{C_n\}_{n=2}^{\infty}$ together with $C_1 \cup (A - B)$ is a disjoint sequence of non-void, measurable subsets of $A$ each of which is $c$-dense in $A$ and whose union is $A$.

THEOREM 4.1. *Let $f \in \mathscr{U}$ and $\varepsilon > 0$. Then there exists a $g \in \mathscr{U}$ such that $g$ is constant on no subinterval of $I$, the range of $g$ is countable and $\|f-g\| < \varepsilon$. Moreover, if $f$ is Baire $\alpha$ or measurable, then $g$ may be taken to be Baire of class* $\max(\alpha+1, 2)$ *or measurable respectively.*

Proof. If $f$ is a constant, then the construction of $g$ is clear. In case $f$ is not a constant, we can without loss of generality assume that the closure of the range of $f$ is $R$. Now decompose $R$ into disjoint half-open intervals $\{I_n\}_{n=1}^{\infty}$ each of length $\varepsilon$ and having irrational endpoints. Put $A_n = f^{-1}(I_n^0)$. Enumerate the rationals in $I_n$ as $\{r_{nk}\}_{k=1}^{\infty}$. Since each $A_n$ is $c$-dense in itself (this follows from $f \in \mathscr{U}$), we may decompose $A_n$ into sets $\{A_{nk}\}_{k=1}^{\infty}$ as given by Lemma 4.1. Now define $g$ by $g(x) = r_{nk}$

if $x \in A_{nk}$ and put $g(x) = f(x)$ otherwise. Since each $A_{nk}$ is $c$-dense in $A_n$, $g$ can not be constant on any subinterval. In addition, it is obvious that $\|f-g\| < \varepsilon$ and the range of $g$ is countable. And it is easily verified that if $f$ is Baire $\alpha$ or measurable, then $g$ is Baire $\max(\alpha+1, 2)$ or measurable respectively.

To show $g \in \mathscr{U}$ we suppose without loss of generality that $g(a) < g(b)$. Let $\{I_k\}_{k=1}^{m}$ be the set of intervals which intersect $[g(a), g(b)]$ so that $a \in f^{-1}(I_1)$ and $b \in f^{-1}(I_m)$. Let $C$ be any set with card $C < c$. Because $f \in \mathscr{U}$, the set $f\big((a, b)-C\big)$ must intersect each $I_n^0$ for $n < m$. But it is easily shown that whenever $f\big((a, b)-C\big)$ intersects some $I_n^0$, $1 \leqslant n \leqslant m$, we can infer that $g([a, b]-C)$ contains all rationals in $I_n$ as follows: Let $x \in (a, b)-C$ so that $f(x) \in I_n^0$. Then $x \in A_n$, so that by the $c$-density of the sets $A_{nk}$ we have $(A_{nk}-C) \cap (a, b) \neq \emptyset$ for each $k$. It follows then that $g$ assumes all rationals in $I_n$ over $(a, b)-C$. Thus, in case $f(b) \in I_m^0$, $g([a, b]-C)$ contains all rationals in $[g(a), g(b)]$. If $f(b) \notin I_m^0$, then $g([a, b]-C)$ contains all rationals in $\bigcup_{k=1}^{m-1} I_k$ and, hence, in $[g(a), g(b)]$ because $g(b) = f(b)$. So in any case $g([a, b]-C)$ is dense in $[g(a), g(b)]$, completing the proof.

THEOREM 4.2. *Let $f \in \mathscr{U}$ be such that $f$ is not constant on any subinterval and the range of $f$ is countable. Then $f$ is the uniform limit of a sequence of Darboux functions. Moreover, if $f$ is Baire $\alpha$ or measurable, then the approximating functions can be taken to be Baire of order* $\max(\alpha, 2)$ *or measurable respectively.*

Proof. I. First of all suppose $f$ is a Baire function of order $\alpha$. Let $\varepsilon > 0$. Then it suffices to find a Darboux, Baire function $g$ of order $\max(\alpha, 2)$ such that $\|f-g\| < \varepsilon$. Without loss of generality we can assume that $I = [0, 1]$. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the range of $f$. Put $A_n = f^{-1}(r_n)$ and $A_n^m = [(m-1)/2^n, m/2^n) \cap A_n$ for each $m$ with $0 < m \leqslant 2^n$. Clearly each $A_n^m$ is a Borel set of order $\alpha$. For each $A_n^m$ for which card $A_n^m = c$ we pick a nowhere dense perfect set $P_n^m \subset A_n^m$ (see proof of Lemma 4.1, part II) and a corresponding continuous function $g_n^m$ which maps $P_n^m$ onto the closed interval $[r_n-\varepsilon, r_n+\varepsilon]$. Now define $g$ as follows:

$$g(x) = \begin{cases} g_n^m(x), & \text{if} \quad x \in P_n^m, \\ f(x), & \text{if} \quad x \text{ belongs to no } P_n^m. \end{cases}$$

That $\|f-g\| < \varepsilon$ is obvious. To show that $g$ is a Baire $\alpha$ function, let $V$ be any closed set in $R$. Then

$$g^{-1}(V) = \Big(\bigcup_{m,n} (g_n^m)^{-1}(V)\Big) \cup \Big(f^{-1}(V) - \bigcup_{m,n} P_n^m\Big)$$

which is clearly a Borel set of order $\max(\alpha, 2)$.

To show $g$ is Darboux, let $a < b$ with $g(a) \neq g(b)$. Put

$$\alpha = \inf_{x\epsilon[a,b]} f(x) \quad \text{and} \quad \beta = \sup_{x\epsilon[a,b]} f(x).$$

Since $f$ is not constant on any subinterval, we have $\alpha < \beta$. Since $f \epsilon \mathscr{U}$, we can pick $r_n$ and $r_m$ so that $\alpha < r_n < r_m < \beta$. Choose $x \epsilon f^{-1}(r_n)$ and $y \epsilon f^{-1}(r_m)$. Supposing $x < y$ pick an open interval $U$ so that $[x, y] \subseteq U \subseteq \bar{U} \subseteq (a, b)$. Put $B = \{k: r_n < r_k < r_m\}$. If, for each $k \epsilon B$, card $(A_k \cap (x, y)) < c$, we would have $f \notin \mathscr{U}$. Therefore, it follows that there are infinitely many $k$'s in $B$ for which $\operatorname{card}(A_k \cap (x, y)) = c$. Now choose a $k$ in $B$ so large that the appropriate subintervals $[(m-1)/2^k, m/2^k)]$ which contain $x$ and $y$, lie entirely inside $U$. Hence, there exists an $m$ so that $P_k^m \subseteq U$ and thus $g(P_k^m) = [r_k - \varepsilon, r_k + \varepsilon]$. Then it is clear that by varying the choices of $r_n$ and $r_m$ in $(\alpha, \beta)$ we can prove that $(\alpha - \varepsilon, \beta + \varepsilon) \subseteq g([a, b])$. Since $g(a)$ and $g(b)$ belong to $[\alpha - \varepsilon, \beta + \varepsilon]$, any value $y$ between $g(a)$ and $g(b)$ must belong to $(\alpha - \varepsilon, \beta + \varepsilon)$ and hence to $g([a, b])$, which establishes the Darboux property for $g$.

II. Suppose $f$ is measurable now. Then the proof in I works with the following modifications: For any $A_n^m$ for which $\operatorname{card} A_n^m = c$ we pick a nowhere dense perfect subset $P_n^m$ of $A_n^m$ if $A_n^m$ has positive measure. In this case $g_n^m$ is defined as in I. If, however, $A_n^m$ has measure zero and $\operatorname{card} A_n^m = c$, we put $P_n^m = A_n^m$ and map $P_n^m$ onto $[r_n - \varepsilon, r_n + \varepsilon]$ by any function $g_n^m$. Since $A_n^m$ has measure zero, $g_m^n$ will be measurable. Thus the measurability of $g$ is immediate. The rest of the proof is the same as in I.

III. Lastly suppose $f$ is any function. Then the proof in I or II works with the following modification: For any $A_n^m$, for which $\operatorname{card} A_n^m = c$, put $P_n^m = A_n^m$ and map it by any function $g_n^m$ onto $[r_n - \varepsilon, r_n + \varepsilon]$.

We now put Theorems 4.1 and 4.2 together to obtain the results promised at the beginning of this section.

THEOREM 4.3. *A necessary and sufficient condition that $f \epsilon \mathscr{U}$ is that $f$ be the uniform limit of a sequence of Darboux functions. Moreover, if $f$ is Baire $\alpha$ where $\alpha \geqslant 1$ or measurable, then the approximating functions may be taken to be Baire $\alpha + 1$ or measurable respectively.*

Proof. The necessity is proved by applying Theorems 4.1 and 4.2. For the sufficiency, let $J = [a, b]$ be a closed subinterval of $I$ and $A$ be a set with card $A < c$. Let $U$ be any open interval whose closure is contained in $(f(a), f(b))$. For the proof we must show that $f(J - A) \cap U \neq \emptyset$. Express $U$ as $(y - \varepsilon, y + \varepsilon)$. There exists an $n$ such that $|f_n(x) - f(x)| < \varepsilon/3$ for all $x \epsilon J$ and such that $f_n(a) < y - \varepsilon$ and $y + \varepsilon < f_n(b)$. (Without loss of generality we are assuming $f(a) < f(b)$.) Since $f_n$ takes on each value in $(y - \varepsilon/2, y + \varepsilon/2)$ over $J$, it follows that there exists an $x_0 \epsilon J - A$ such that $y - \varepsilon/2 < f_n(x_0) < y + \varepsilon/2$. This in turn implies that $f(x_0) \epsilon U$, completing the proof.

THEOREM 4.4. *A Baire* 1 *function in* $\mathscr{U}$ *satisfies the Darboux condition.*

Proof. Massera [7] (p. 656) has shown that a Baire 1 function with the property that

$$f(x)\in[\varliminf_{z\to x^-} f(z),\ \varlimsup_{z\to x^-} f(z)]\cap[\varliminf_{z\to x^+} f(z),\ \varlimsup_{z\to x^+} f(z)]$$

for each $x\in I$, must be a Darboux function. A Baire 1 function in $\mathscr{U}$ obviously satisfies this condition, completing the proof.

COROLLARY. *A Baire* 1 *function* $f$ *belongs to* $\mathscr{U}$ *if and only if* $f$ *is the uniform limit of a sequence of Darboux, Baire* 1 *functions.*

We do not know (**P 546**) whether Theorem 4.4 is valid for Baire 2 functions. However, Theorem 4.4 is not valid for functions in Baire class 3. To see this, let

$$I=\bigcup_{n=1}^{\infty}A_n,$$

where the sets $A_n$ are pairwise disjoint, non-void, Borel 2 sets, each $c$-dense in $I$. That this can be done is a consequence of Lemma 4.1. Let $r_1, r_2, \ldots$ be an enumeration of all rational numbers. Define a function $f$ by $f(x)=r_n$ if $x\in A_n$. Then $f$ is a Baire 3 function in $\mathscr{U}$ but $f$ is not a Darboux function.

A function $f$ is said to be *convex in the sense of Jensen* if

$$f\left(\frac{x+y}{2}\right)\leqslant\frac{f(x)+f(y)}{2}\quad\text{ for all } x, y\in I.$$

A function is said to be *additive* if $f(x+y)=f(x)+f(y)$ for all $x, y\in I$ such that $x+y\in I$. Every function which is additive on the entire real line $R$ is also convex in the sense of Jensen on $R$.

THEOREM 4.5. *Let* $f$ *be a function convex in the sense of Jensen on an open interval* $I$ (*which may be the entire real line*). *Then* $f\in\mathscr{U}$.

Proof. Let $x, y\in I$. Let $D$ denote the set of rational numbers in $[0, 1]$. Then, according to [1] (p. 224), there is a *continuous* convex function $g$ on $[x, y]$ such that $g=f$ on the set $S=\{t\colon t=\lambda x+(1-\lambda)y$ for some $\lambda\in D\}$. It is clear that the projection of the graph of $f$ over $S$ onto the $y$-axis is dense in the interval $[g(x), g(y)]=[f(x), f(y)]$. It follows in particular that $f\in\mathscr{U}_0$. Thus, according to theorem 3.2 it suffices to show that the graph of $f$ is $c$-dense in itself.

Let $x_0\in I$. For $x\in I$, define

$$Z_x=\left\{\left(x_0+\frac{x-x_0}{2^n}, f\left(x_0+\frac{x-x_0}{2^n}\right)\right)\colon n=1, 2, \ldots\right\}.$$

Let $\delta$ be a subinterval of $I$ not containing $x_0$ and so small that $x$ and $(x+x_0)/2$ cannot be in $\delta$ simultaneously. Then $Z_x \cap Z_{x'} = \emptyset$ for $x, x' \in \delta$. Let $U$ be some neighborhood of the point $(x_0, f(x_0))$. Obviously, $Z_x \cap U \neq \emptyset$ for all $x \in I$. Therefore $Z_x \cap U$, $x \in \delta$, is a family of non-empty and disjoint sets and

$$\operatorname{card}(U \cap f) \geqslant \operatorname{card}\Big(\bigcup_{x \in \delta} x \cap U\Big) \geqslant \operatorname{card} \delta = c.$$

Remark 1. A function convex in the sense of Jensen need not be a Darboux function. To see this, let $H$ be a Hamel basis including the point $x = 1$. Define a function $f$ by the equation

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 1, \\ 0 & \text{if} \quad x \in H,\ x \neq 1, \\ \text{by additivity otherwise.} \end{cases}$$

The function $f$ is additive (and hence convex in the sense of Jensen) on $R$, yet has the rationals as range. Such a function cannot be a Darboux function.

Remark 2. The proof of theorem 4.5 is valid for any generalized convex function (see [1]).

**5. Algebraic structure of $\mathscr{U}_0$ and $\mathscr{U}$.** It is easy to provide examples to show that the classes $\mathscr{U}$ and $\mathscr{U}_0$ are not closed under the various algebraic operations. For example, Sierpiński [10] and Fast [8] have shown that *any* function on $I$ is the sum of two Darboux functions. However, as we show in this section, the classes $\mathscr{U}$ and $\mathscr{U}_0$ are closed under certain combinations of functions in $\mathscr{U}$ with *continuous* functions. Theorem 5.2 furnishes a general theorem of the desired kind and the usual algebraic combinations are considered in the corollary to Theorem 5.2. Finally, we consider the case in which the Darboux functions are also in the first class of Baire.

Definition 5.1. Let $F(x, y)$ be continuous in the entire plane. A point $y_0$ is a *singular point* of $F$ provided either

$$\lim_{\substack{y \to y_0 \\ x \to \infty}} F(x, y) \quad \text{or} \quad \lim_{\substack{y \to y_0 \\ x \to -\infty}} F(x, y)$$

fails to exist in the extended sense.

Examples. Each of the functions $x+y$, $\max(x, y)$, $\min(x, y)$ fails to have a singular point. The function $xy$ has 0 as its only singular point.

THEOREM 5.1. *A polynomial, $P(x, y)$, has at most a finite number of singular points.*

Proof. Write

$$P(x, y) = \sum_{k=0}^{n} p_k(y)x^k,$$

where each $p_k(y)$ is a polynomial in $y$ and $p_n(y)$ is not identically zero. We prove by induction on $n$ that if $p_n(y_0) \neq 0$, then $y_0$ is not a singular point of $P$ and

$$\lim_{\substack{y \to y_0 \\ x \to \pm\infty}} P(x, y) \neq 0.$$

This is clear for $n = 0$. Suppose the assertion holds for $n-1$, and $p_n(y_0) \neq 0$. Then

$$P(x, y) = p_0(y) + x \sum_{k=1}^{n} p_k(y)x^{k-1}.$$

By hypothesis

$$\lim_{\substack{y \to y_0 \\ x \to \pm\infty}} \sum_{k=1}^{n} p_k(y)x^{k-1}$$

exists and is different from zero. Therefore, $P(x, y)$ tends to an (infinite) limit as $y \to y_0$ and $x \to \pm\infty$. This verifies the induction step and the result is clear.

LEMMA 5.1. *Let $F(x, y)$ be continuous in the entire plane. Suppose for each $f \in \mathscr{U}_0$ and continuous $g$ that $F(f(x), g(x)) \in \mathscr{U}_0$. Then for each $f \in \mathscr{U}$ and continuous $g$, $F(f(x), g(x)) \in \mathscr{U}$.*

Proof. Let $f \in \mathscr{U}$, let $g$ be continuous. Since we are supposing that $h(x) = F(f(x), g(x))$ is in $\mathscr{U}_0$ it suffices, by Theorem 3.2, to show that $h$ is $c$-dense in itself. To this end fix a point $x_0$ and let $N$ and $M$ be neighborhoods respectively of $h(x_0)$ and $x_0$. Since $F$ is continuous, there are neighborhoods $M_1$ of $f(x_0)$ and $M_2$ of $g(x_0)$ for which $F[M_1 \times M_2] \subset N$. Since $g$ is continuous, there is a neighborhood $M_3$ of $x_0$, $M_3 \subset M$, so that $g[M_3] \subset M_2$. Since $f \in \mathscr{U}$, $f$ intersects $M_3 \times M_1$ in $c$-points. Thus there are $c$-points $x$ in $M_3$ with $(x, f(x)) \in M_3 \times M_1$. For each such point $h(x) = F[f(x), g(x)] \subset N$. Therefore, the set of points $(x, h(x))$ in $M \times N$ has cardinality $c$ as was to be proved.

THEOREM 5.2. *Let $F(x, y)$ be continuous in the entire plane and have at most a finite number of singular points. Then, if $f \in \mathscr{U}_0$ (resp. $\mathscr{U}$) and $g$ is continuous, the function $h$ defined by $h(x) = F(f(x), g(x))$ is in $\mathscr{U}_0$ (resp. $\mathscr{U}$).*

Proof. By Lemma 5.1 it suffices to prove the assertion for $\mathscr{U}_0$. We first show this in case $F$ has at most one singular point $y_0$.

Let $x_0 \in I$. Suppose $g(x_0) \neq y_0$. Then,

$$C_0(h, x_0) = \{y \mid y = \lim F(f(x_n), g(x_n)), x_n \to x_0\}$$
$$= \{y \mid y = \lim F(f(x_n), g(x_n)), x_n \to x_0, f(x_n) \to \gamma\}$$

where $-\infty \leqslant \gamma \leqslant \infty$. By the continuity of $F$ and the fact that, by Theorem 3.1, $C_0(f, x_0)$ is a closed interval we have $C_0(h, x_0) = F[C_0(f, x_0) \times \{g(x_0)\}]$, a closed interval.

Thus, in the special case that $F$ has no singular points or if the range of $g$ doesn't contain any singular points, $C_0(h, x)$ is a closed interval for each $x \in I$, proving that $h \in \mathscr{U}$.

Suppose $g(x_0) = y_0$. Then we may write

$$C_0^+(h, x_0) = A_0 \cup A_\infty \cup A_{-\infty},$$

where

$$A_0 = \{y \mid y = F(\gamma, y_0), \gamma \in C_0^+(f, x_0) \cap (-\infty, \infty)\},$$
$$A_\infty = \{y \mid y = \lim F(f(x_n), g(x_n)), x_n \to x_0^+, f(x_n) \to \infty\},$$
$$A_{-\infty} = \{y \mid y = \lim F(f(x_n), g(x_n)), x_n \to x_0^+, f(x_n) \to -\infty\}.$$

Since $F$ is continuous, and $C_0^+(f, x_0)$ is an interval, $A_0$ is also an interval.

Let $\alpha = \inf A_0$, $\beta = \sup A_0$. Let $\alpha' = \inf A_\infty$, $\beta' = \sup A_\infty$.

Suppose $\beta < \beta'$ and let $\gamma \in (\beta, \beta')$. Let $x_n \to x_0^+$ with $f(x_n) \to \infty$ and $F(f(x_n), g(x_n)) \to \beta'$. We claim that $x_n$ is eventually in the set where $g(x) \neq y_0$. Otherwise we would have $F(f(x_n), g(x_n)) = F(f(x_0), y_0)$ for a subsequence $J$ of indices. But for $n \in J$ sufficiently large, $f(x_n) \in C_0(f, x_0)$, implying the contradiction: $F(f(x_n), y_0) \leqslant \beta$ for infinitely many $n$. We may now assume $g(x_n) \neq y_0$ for all $n$. Let $(a_n, b_n)$ be the component of the set $\{x \mid g(x) \neq y_0\}$ in which $x_n$ lies. By choosing an appropriate subsequence (which we continue to denote by $I$) we may suppose $F(f(a_n), y_0) \leqslant (\gamma + \beta)/2$. Since $g(a_n) = y_0$ for each $n$, we may choose $y_n \in (a_n, b_n)$ so close to $a_n$ that $y_n < x_n$ and

$$|F(f(y_n), g(y_n)) - F(f(a_n), y_0)| < (\gamma - \beta)/2.$$

Then, for all large $n$

$$h(y_n) < \gamma < h(x_n).$$

On the interval $[y_n, x_n]$, the range of $g$ contains no singular points of $F$, so by a remark at the beginning of this proof $h$ is in $\mathscr{U}_0$ on this interval. Hence, for large $n$ there is a $z_n \in [y_n, x_n]$ with $|h(z_n) - \gamma| < 1/n$. Since $z_n \to x_0^+$, $\gamma \in C_0^+(h, x_0)$. We have, therefore, proved that the points of $A_\infty$ which lie above $A_0$ form an interval abutting on $A_0$. Similar statements hold for the points of $A_\infty$ lying below $A_0$, and, $A_\infty$ lying above or below $A_0$. This makes it clear that $C_0^+(h, x_0)$ is a closed interval containing $h(x_0)$.

Similarly, we may prove that $C_0^-(h, x_0)$ is a closed interval.

By Theorem 3.1 we have $h \in \mathcal{U}_0$.

Finally, we consider the general case when $F$ has an arbitrary but finite number of singular points. Let $x_0 \in I$. We restrict our attention to an interval about $x_0$ in which $g$ assumes at most one of the singular values of $F$. Then $h \in \mathcal{U}_0$ in this interval and so each of $C_0^+(h, x_0)$ and $C_0^-(h, x_0)$ is a closed interval. By Theorem 3.1, $h \in \mathcal{U}_0$.

COROLLARY. *If $f \in \mathcal{U}_0$ (resp. $\mathcal{U}$) and $g$ is continuous, then each of the following functions is in $\mathcal{U}_0$ (resp. $\mathcal{U}$): $g(f), f(g), P(f, g)$ for any polynomial $P$, (in particular $f+g$, $f \cdot g$),* $\max(f, g)$, $\min(f, g)$.

Proof. The assertions for $g(f)$ and $f(g)$ follow easily from the definitions of $\mathcal{U}_0$ and $\mathcal{U}$. The rest of the assertions are immediate from Theorems 5.1, 5.2 and Lemma 5.1.

The part of this corollary which deals with sums and $\mathcal{U}_0$ was known to Ellis [4], Massera [7], and Radakovic [9].

It follows from Theorem 4.4 and the corollary that the sum and product of a Darboux function in Baire class 1 and a continuous function are also Darboux functions in Baire class 1. This result also follows readily from a criterion of Maximoff ([8], p. 260).

On the other hand, if $f$ is any Darboux function in Baire class 1 which is discontinuous at a point $x_0$, then it is easy to define another Darboux function $g$ in Baire class 1 and also discontinuous at $x_0$ such that the function $f+g$ is not a Darboux function. However, there are *discontinuous* Darboux functions in Baire class 1 whose product with any Darboux function in Baire class 1 is again in that class. The function $f$, given by

$$f(x) = \sin\frac{1}{x}, \quad x \neq 0, \quad f(0) = 0,$$

is such a function. This function also has the property that its product with any function in $\mathcal{U}$ is again in $\mathcal{U}$. Again, the only functions $f$ with the property that if $g \in \mathcal{U}$, then $f+g \in \mathcal{U}$, are the continuous functions.

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