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Spaces with a covariant constant linear geometric object

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Abstract. In this note we prove the following theorem (Theorem A).

Let M be a connected paracompact manifold and A be a linear geometric object on $P(M, G)$. There is a connection in $P(M, G)$ such that A is covariant constant if and only if there exists a family of sections σ_a: $U_a \to P$, $a \in I$, such that $\{U_a\}_{a\in I}$ is an open covering of M and the coordinates of A with respect to σ_a are the same constants for all $a \in I$.

In the proof we use the arguments analogous to those of M. Kurita [3]. The above theorem generalizes Theorems 1 and 2 of M. Kurita [3].

1. Introduction. Let $P(M, G)$ be a principal fibre bundle, V a vector space of finite dimension and let

$$\varrho\colon G \to \mathrm{GL}(V)$$

be a homomorphism of Lie groups.

A ϱ-linear geometric object, or shortly a linear geometric object, on $P(M, G)$ is a mapping $A\colon P \to V$ such that

$$A(p\xi) = \varrho(\xi^{-1})A(p)$$

for all $p \in P$ and $\xi \in G$. If σ: $U \to P$ is a section of $P(M, G)$, then the vector-valued function $a = A\circ\sigma$: $U \to V$ is called a *system of coordinates of A with respect to* σ (see [1]).

The aim of this note is to prove the following theorem.

THEOREM A. *Let A be a ϱ-linear geometric object on $P(M, G)$, where the base M is connected and paracompact. The following conditions are equivalent.*

(A.1) *There is a connection Γ in $P(M, G)$ such that $\nabla_v A = 0$ for all vector fields v on M.*

(A.2) *There is a family of sections σ_a: $U_a \to P$, $a \in I$, such that $\{U_a\}_{a\in I}$ is an open covering of M and the coordinates of A with respect to σ_a are the same constants for all $a \in I$.*

(A.3) *There is a closed subgroup H of G and an H-structure $P_0(M, H)$ in $P(M, G)$ such that A is constant on P_0.*

2. Covariant differentiation. In this note we shall use the definition of Crittenden [1] of covariant derivative. We remind that if $A\colon P\to F$ is an object on $P(M, G)$ such that

$$A\circ R_\xi = \lambda_{\xi^{-1}}\circ A,$$

where $\lambda_\xi\colon F\to F$ is a left translation on the manifold F, $\xi\in G$, then

$$\nabla_v A = dA\circ H_v \text{ [1]},$$

where $v\colon M\to TM$ is a vector field on M and $H_v\colon P\to TP$ is the horizontal lift of v (with respect to a connection Γ in $P(M, G)$). Now, $\nabla_v A$ is a geometric object on $P(M, G)$ such that

$$\nabla_v A\circ R_\xi = d\lambda_{\xi^{-1}}\circ \nabla_v A$$

(see [1]). If $F = V$ is a vector space and $\lambda_\xi = \varrho(\xi)\colon V\to V$ is linear, then we have the natural identification $\varphi\colon TV\to V\times V$ and the diagram

$$\begin{array}{ccc} TV & \xrightarrow{d\lambda_\xi} & TV \\ \downarrow & & \downarrow\varrho \\ V\times V & \xrightarrow{\lambda_\xi\times\lambda_\xi} & V\times V \end{array}$$

is commutative. Thus

$$\tilde{\nabla}_v A = p_2\circ\varphi\circ\nabla_v A\colon P\to V, \tag{2.1}$$

where $p_2\colon V\times V\to V$ is the projection on the second factor, is a geometric object such that

$$(\tilde{\nabla}_v A)(p\cdot\xi) = \varrho(\xi^{-1})(\tilde{\nabla}_v A)(p).$$

Let $\sigma\colon U\to P$ be a section. We denote by $a = A\circ\sigma$ and $\nabla_v a = \tilde{\nabla}_v A\circ\sigma$ coordinates of A and $\tilde{\nabla}_v A$ with respect to σ. If we consider the diffeomorphism

$$\bar{\sigma}\colon U\times G\ni(x, \xi)\to\sigma(x)\cdot\xi\in P|U,$$

then

$$\begin{aligned}(\nabla_v A\circ\sigma)(x) &= d_{\sigma(x)}A(\tilde{v}_{\sigma(x)}) \\ &= d_{(x,e)}(A\circ\bar{\sigma})\circ(d_{(x,e)}\bar{\sigma})^{-1}(\tilde{v}_{\sigma(x)}),\end{aligned} \tag{2.2}$$

where $\tilde{v}_{\sigma(x)} = H_v(\sigma(x))$ is a horizontal vector. If

$$(d_{(x,e)}\bar{\sigma})^{-1}(\bar{v}_{\sigma(x)}) = (v_x, \bar{v})\in T_xM\oplus T_eG,$$

[1] If $A\colon P\to F$ and $p\in P$, then $d_pA\colon T_pP\to T_{A(p)}F$ denotes the linear homomorphism induced by A and $dA\colon TP\to TF$ is a mapping between tangent bundles such that $dA|T_pP = d_pA$.

then

$$\tilde{v}_{\sigma(x)} = d_{(x,e)}\bar{\sigma}(v_x, \bar{v}) = d_x\sigma(v_x) + d_e R_{\sigma(x)}(\bar{v}),$$

where $R_{\sigma(x)}\colon G \ni \xi \to \sigma(x)\cdot\xi \in P$. If ω denotes a connection form, then the last formula implies

$$0 = \omega(v_{\sigma(x)}) = (\omega\circ d_x\sigma)(v_x) + \bar{v},$$

because $d_e R_{\sigma(x)}(v)$ is vertical (we identify $\mathscr{L}(G)$ with T_eG). From the above formula and from (2.2) we have

$$(\nabla_v A\circ\sigma)(x) = d_{(x,e)}(A\circ\bar{\sigma})\big(v_x, -(\omega\circ d_x\sigma)(v_x)\big).$$

Since $(A\circ\bar{\sigma})(x, \xi) = A\big(\sigma(x)\cdot\xi\big) = \varrho(\xi^{-1})A\big(\sigma(x)\big) = \varrho(\xi^{-1})a(x)$ thus from the Leibniz formula ([2], p. 11) we obtain

$$(\nabla_v A\circ\sigma)(x) = d_x a(v_x) + (d_I\lambda_{a(x)}\circ d_e\varrho\circ\omega_{\sigma(x)}\circ d_x\sigma)(v_x),$$

where $\lambda_{a(x)}\colon \mathrm{GL}(V)\ni a\to a\cdot a(x)\in V$.

The action of $\mathrm{GL}(V)$ on V can be prolonged in the natural way to an action of $\mathrm{gl}(V)$ on V. If $X = \left(\frac{d}{dt}a_t\right)(0)\in\mathrm{gl}(V)$, where $a_t\in\mathrm{GL}(V)$ and $a_0 = I$, then $X\cdot v = \frac{d}{dt}(a_t\cdot v)(0)$ for $v\in V$. Now,

$$(p_2\circ\varphi\circ d_I\sigma_{a(x)}\circ d_e\varrho\circ\omega\circ d_x\sigma)(v_x) = (\bar{\varrho}\circ\omega\circ d_x\sigma)(v_x)\cdot a(x),$$

where $\bar{\varrho}\colon \mathscr{L}(G)\to\mathrm{gl}(V)$ is the Lie algebra homomorphism induced by ϱ. Thus we have

$$\nabla_v a = \tilde{\nabla}_v A\circ\sigma = (da)(v) + (\bar{\varrho}\circ\omega\circ d\sigma)(v)\cdot a, \tag{2.3}$$

where da denotes the exterior derivative of a.

Thereafter we shall write $\nabla_v A$ instead of $\tilde{\nabla}_v A$.

3. The proof of Theorem A. A proof of implication (A.1)$\Rightarrow$(A.2) will be based on the lemma.

LEMMA. *Let A be a linear geometric object such that $\nabla_v A = 0$ for all vector fields on M. Let $x_0\in M$, let U_0 be an open neighbourhood of x_0 and let $\varphi_0\colon U_0\to K(0,\varepsilon)$ be a diffeomorphism such that $\varphi_0(x_0) = 0$ ($K(0,\varepsilon)$ denotes a ball). If $x_1\in U_0$, $p_1\in P$ and $\pi(p_1) = x_1$, then there is a section $\sigma\colon U_0\to P$ such that $\sigma(x_1) = p_1$ and A has constant coordinates with respect to σ.*

Proof of the lemma. For $u\in K(0,\varepsilon)$ we denote $\gamma_u(t) = \varphi_0^{-1}(tu)$. γ_u is a curve such that $\gamma_u(0) = x_0$ and $\gamma_u(1) = \varphi_0^{-1}(u)$. Let $u_1 = \varphi_0(x_1)$ and let $\gamma^*_{u_1}$ be a horizontal lift of γ_{u_1} such that $\gamma^*_{u_1}(1) = p_1$. We write $p_0 = \gamma^*_{u_1}(0)$. For $u\in K(0,\varepsilon)$ we denote by γ^*_u a horizontal lift of γ_u such that $\gamma^*_u(0) = p_0$. Now,

$$\sigma(x) = \gamma^*_{\varphi_0(x)}(1),\qquad x\in U_0,$$

is a section of P. We calculate $\nabla_v a$ along a curve γ_u, $u \in K(0, \varepsilon)$, i.e., for $v = \dot{\gamma}_u$. Now, $d\sigma(\dot{\gamma}_u) = \dot{\gamma}_u^*$ and hence, using (2.3), we get

$$\nabla_v a = da(\dot{\gamma}_u) + (\bar{\varrho}\circ\omega)(\dot{\gamma}_u^*)a = \frac{d}{dt}(a\circ\gamma_u),$$

because $\dot{\gamma}_u^*$ is horizontal. This means that a is constant along γ_u for all $u \in K(0, \varepsilon)$, and thus a is constant in the whole U_0.

We now prove that (A.1) implies (A.2).

We fix $x_0 \in M$. For each $x \in M$ we find a neighbourhood U_x and a diffeomorphism $\varphi_x\colon U_x \to K(0,1)$, $\varphi_x(x) = 0$. For x_0 we choose a section $\sigma_0\colon U_0 \to P$ such that the coordinates of A with respect to σ_0 are constants denoted by a_0. For each point $x \in M$ we choose a curve γ from x_0 to x. We can find a finite sequence

$$U_{x_0}, U_{x_1}, \ldots, U_{x_n} = U_x$$

such that $U_{x_{i-1}} \cap U_{x_i} \neq \emptyset$ and $U_{x_0}, U_{x_1}, \ldots, U_{x_n}$ cover γ. We choose $y_i \in U_{x_{i-1}} \cap U_{x_i}$, $i = 1, \ldots, n$. By the lemma, for each i we can construct a section $\sigma_i\colon U_{x_i} \to P$ such that $\sigma_i(y_i) = \sigma_{i-1}(y_i)$ and coordinates of A with respect to σ_i are constant. Since $\sigma_{i-1}(y_i) = \sigma_i(y_i)$, these coordinates are a_0 in each U_{x_i}, $i = 0, 1, \ldots, n$.

Implication (A.2) $\Rightarrow$ (A.3) is trivial.

To prove that (A.3) implies (A.1) we consider any connection in $P_0(M, H)$ with connection from ω_0. Such a connection exists because M is paracompact (see [2], Theorem 2.1, p. 67). Next we extend ω_0 to a connection ω in $P(M, G)$ (see [2], Proposition 6.1, p. 79). If $w \in TP_0$, then $\omega(w) = \omega_0(w)$.

Since A is a constant a_0 on P_0, we have for each $X \in \mathscr{L}(H)$

$$\bar{\varrho}(X)\cdot a_0 = 0.$$

This means that

$$\nabla_v a = (da)(v) + (\bar{\varrho}\circ\omega\circ d\sigma)(v)\cdot a = 0$$

for each section σ of $P_0(M, G)$, because

$$(\omega\circ d\sigma)(v) = (\omega_0\circ d\sigma)(v) \in \mathscr{L}(H)$$

and $a = A\circ\sigma = a_0$ is constant.

The proof is now complete.

Remark. In the proof of implication (A.1) $\Rightarrow$ (A.2) we used the same arguments as Kurita [3] for the proof of his Theorem 1. Implication (A.3) $\Rightarrow$ (A.1) [or (A.2) $\Rightarrow$ (A.1)] means that the assumption that H is reductive (see Kurita [3], Theorem 2) is not necessary.

Theorem A implies immediately the following corollaries (we assume that M is a connected paracompact manifold).

COROLLARY 3.1. *Let A be a tensor field on M. There is a linear connection on M such that A is covariant constant if and only if there is a family of fields of frames such that coordinates of A are the same constants for each field of the family* (see Theorems 1 and 2 in [3]).

COROLLARY 3.2. *Let v be a vector field on M. There is a linear connection on M such that v is covariant constant if and only if either v vanishes identically or $v_x \neq 0$ for each point $x \in M$.*

COROLLARY 3.3. *If $v_1, \ldots, v_k$ are vector fields on M, linearly independent at each point of M, then there is a linear connection on M such that $v_1, \ldots, v_k$ are covariant constant.*

References

[1] R. Crittenden, *Covariant differentiation*, Quart. J. Math. Oxford (2) 13 (1972), p. 285–298.
[2] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, New York–London 1963.
[3] M. Kurita, *Tensor fields and their parallelism*, Nagoya Math. J. 18 (1961), p. 133–151.

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