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SEQUENTIAL ESTIMATION OF THE TRANSITION INTENSITIES IN MARKOV PROCESSES WITH MIGRATION

1. Statement of the problem. Assume that there is a flow of homogeneous objects arriving in a certain system A and each of the objects in the system may immigrate into one of n directions $B_1, \ldots, B_n$. We also assume that the arriving objects form a Poisson flow with intensity a. Further, if an object is in the system A at time $t > 0$, then it can immigrate during the time $(t, t+\Delta t)$, independently of its arrival time, in the direction B_j, $j = 1, \ldots, n$, with probability $\beta_j \Delta t + o(\Delta t)$.

Denote by $V(t)$ the number of objects which came into the system in the time interval $[0, t)$. Let $W_j(t)$ be the number of objects which immigrated during this time in the direction B_j, $j = 1, \ldots, n$, and let k_0 be the number of objects present in A at time $t = 0$.

Let $I = \{0, 1, \ldots\}$ and $T = [0, \infty)$. Next, we put $W_0(t) = k_0 + V(t)$ and denote by ϑ the vector $(a, \beta_1, \ldots, \beta_n) \in \Theta \subset (0, \infty)^{n+1}$.

Let $(\Omega, \mathscr{F}, P_\vartheta)$ be a probability space. Let us consider a homogeneous Markov process $\xi(t) = \big(W_0(t), W_1(t), \ldots, W_n(t)\big)$, $t \in T$, defined on $(\Omega, \mathscr{F}, P_\vartheta)$ and with values $x = (w_0, w_1, \ldots, w_n) \in \mathscr{X} = I^{n+1}$, describing the behaviour of the above-introduced system and satisfying the following conditions for every $\vartheta \in \Theta$:

(a) $P_\vartheta\big(\xi(0) = (k_0, 0, \ldots, 0)\big) = 1$;

(b) the transition probabilities are of the form

$$P_\vartheta\big(\xi(t+\Delta t) = y \mid \xi(t) = x\big)$$
$$= \begin{cases} a\Delta t + o(\Delta t) \\ \text{if } x = (w_0, w_1, \ldots, w_n) \text{ and } y = (w_0+1, w_1, \ldots, w_n), x \in \mathscr{X},\ y \in \mathscr{X}, \\ k\beta_j \Delta t + o(\Delta t) \quad \text{if } x = (w_0, w_1, \ldots, w_n) \text{ and} \\ y = (w_0, w_1, \ldots, w_{j-1}, w_j+1, w_{j+1}, \ldots, w_n), j = 1, \ldots, n,\ x \in \mathscr{X},\ y \in \mathscr{X}, \\ 1 - \Big(a + \sum_{j=1}^{n} k\beta_j\Big)\Delta t + o(\Delta t) \quad \text{if } x = y \in \mathscr{X}, \\ o(\Delta t) \quad \text{otherwise}, \end{cases}$$

where $k = w_0 - \sum_{j=1}^{n} w_j$ denotes the value of the random variable

$$K(t) = W_0(t) - \sum_{j=1}^{n} W_j(t),$$

determining the number of objects present in the system at time t;

(c) $P_\vartheta\big(K(t) \geqslant 0\big) = 1$ for every $t > 0$.

The above-described model of a Markov process appears in problems of demography and reliability theory when the long-life inspection of objects arriving in the inspection stand and leaving the inspection stand at random times takes place.

Our problem is to estimate the intensities $\alpha, \beta_1, \ldots, \beta_n$ or their functions using the observation of the process $\xi(t)$, $t \in T$, and applying the sequential approach.

2. Sufficient statistics. Let $(D, \mathscr{D})$ be a measurable space of $(n+1)$-dimensional vectors $x(s) = \big(w_0(s), w_1(s), \ldots, w_n(s)\big)\colon T \to \mathscr{X}$ whose components represent right-continuous functions with integer nonnegative values and unit jumps, having left-hand limits. By μ_ϑ we denote the measure on $(D, \mathscr{D})$ corresponding to the process $\xi(t)$, $t \in T$:

$$\mu_\vartheta(B) = P_\vartheta\big(\xi(\cdot) \in B\big), \quad B \in \mathscr{D}.$$

Let $\mu_{\vartheta,t}$ be the truncation of the measure μ_ϑ on the set

$$\mathscr{D}_t = \sigma\{x(s)\colon s \leqslant t,\ s \in T,\ t \in T\}.$$

Let us consider the sequential statistical space $(D, \mathscr{D}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$, corresponding to the process $\xi(t)$, $t \in T$. Let R be the real line, $\mathscr{Y} \subset R^m$, and let $\mathscr{B}_{\mathscr{Y}}$ denote the σ-algebra of Borel subsets of $\mathscr{Y}$. A function $Z\big(t, x(\cdot)\big)\colon T \times D \to \mathscr{Y}$ such that for every $t \in T$ the transformation $Z(t, \cdot)$ is $(\mathscr{D}_t, \mathscr{B}_{\mathscr{Y}})$-measurable is called an (m-dimensional) *statistic* on the space $(D, \mathscr{D}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$.

Let $\vartheta_0 = (\alpha_0, \beta_{01}, \ldots, \beta_{0n})$ be any fixed value of the parameter ϑ. It follows from the Skorohod theorems ([4], Section 8, and [5], Chapter 7, Section 6) that:

1° the statistical space $(D, \mathscr{D}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$, is *dominated*, i.e., for every $t \in T$ all the measures $\mu_{\vartheta,t}$, $\vartheta \in \Theta$, are absolutely continuous with respect to the measure $\mu_{\vartheta,t}$;

2° the densities $d\mu_{\vartheta,t}/d\mu_{\vartheta_0,t}$ are defined by

$$(1)\quad \frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}\big(x(\cdot)\big) = \left(\frac{\alpha}{\alpha_0}\right)^{v(t)} \exp\Big[-(\alpha-\alpha_0)t - \\ - \sum_{j=1}^{n}(\beta_j - \beta_{0j})\Big(k(t)t + \sum_{j=1}^{n}\sum_{l=1}^{w_j(t)}\sigma_{jl} - \sum_{r=1}^{v(t)}\nu_r\Big)\Big]\prod_{j=1}^{n}\left(\frac{\beta_j}{\beta_{0j}}\right)^{w_j(t)}$$

where ν_r's denote the arrival times ($0 < \nu_1 < \ldots < \nu_{v(t)} < t$) and σ_{jl}'s are the exit times in directions B_j: $0 < \sigma_{j1} < \ldots < \sigma_{jw_j(t)} < t$, $j = 1, \ldots, n$.

Let us put

$$w(t) = \big(w_1(t), \ldots, w_n(t)\big), \tag{2}$$

$$S\big(t, x(\cdot)\big) = k(t)t + \sum_{j=1}^{n} \sum_{l=1}^{w_j(t)} \sigma_{jl} - \sum_{r=1}^{v(t)} \nu_r, \tag{3}$$

$$\beta = \sum_{j=1}^{n} \beta_j, \quad \beta_0 = \sum_{j=1}^{n} \beta_{0j}, \tag{4}$$

$$Z\big(t, x(\cdot)\big) = \big(v(t), w(t), S\big(t, x(\cdot)\big)\big). \tag{5}$$

The function S determines the total time spent in the system by the objects which arrived during the time $[0, t)$ or were present in the system at time $t = 0$. Using (2)-(5) we can rewrite density (1) in the form

$$\begin{aligned} &\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}\big(x(\cdot)\big) \\ &= \left(\frac{a}{a_0}\right)^{v(t)} \exp\big[-(a-a_0)t - (\beta-\beta_0)S\big(t, x(\cdot)\big)\big] \prod_{j=1}^{n} \left(\frac{\beta_j}{\beta_{0j}}\right)^{w_j(t)} \\ &= C\big(t, Z\big(t, x(\cdot)\big); \vartheta_0\big) a^{v(t)} \exp\big[-at - \beta S\big(t, x(\cdot)\big)\big] \prod_{j=1}^{n} \beta_j^{w_j(t)} \\ &= h\big(t, Z\big(t, x(\cdot)\big); \vartheta, \vartheta_0\big), \end{aligned} \tag{6}$$

where the function C does not depend on ϑ.

It follows from the Fisher-Neyman theorem on factorization (see, e.g., [2], p. 29-30) that $Z\big(t, x(\cdot)\big) = \big(v(t), w(t), S\big(t, x(\cdot)\big)\big)$ is an ($(n+2)$-dimensional) sufficient statistic on the space $(D, \mathscr{D}_t, \{\mu_{\vartheta,t}, \vartheta \in \Theta\})$, $t \in T$.

3. Absolute continuity of the measures generated by a Markov stopping time and a sufficient statistic. Let $\tau = \tau\big(x(\cdot)\big)$ be a finite Markov time with respect to the family $\mathscr{D}_t$, $t \in T$, i.e., $\tau: D \to [0, \infty]$ so that $\{x(\cdot) \cdot \tau\big(x(\cdot)\big) \leqslant t\} \in \mathscr{D}_t$ for every $t \in T$ and

$$\mu_\vartheta\big(\{x(\cdot): \tau\big(x(\cdot)\big) < \infty\}\big) = 1 \quad \text{for all } \vartheta \in \Theta.$$

The statistic $Z\big(t, x(\cdot)\big) = \big(v(t), w(t), S\big(t, x(\cdot)\big)\big)$ is a mapping $T \times D \to \mathscr{X} \times T = \mathscr{Y}$, right-continuous with respect to t μ_ϑ-a.e. for every $\vartheta \in \Theta$. Let $U = T \times \mathscr{Y}$, $U \ni u = \big(t(u), z(u)\big)$, $t(u) \in T$, $z(u) = \big(v(u), w(u), s(u)\big) \in \mathscr{Y}$, where $v(u) \in I$, $w(u) = \big(w_1(u), \ldots, w_n(u)\big) \in I^n$, and $s(u) \in T$. The pair

$$\mathscr{Z}\big(x(\cdot)\big) = \Big(\tau\big(x(\cdot)\big), Z\big(\tau\big(x(\cdot)\big), x(\cdot)\big)\Big)$$

of both $\mathscr{D}_\tau$-measurable functions generates, for every $\vartheta \in \Theta$, the measure

m_ϑ on $(U, \mathscr{B}_U)$ in the standard way: for every $A \in \mathscr{B}_U$,

$$m_\vartheta(A) = \mu_\vartheta(\mathscr{X}^{-1}(A)) = \mu_\vartheta\Big(\big(\tau(x(\cdot)), Z(\tau(x(\cdot)), x(\cdot))\big) \in A\Big).$$

From the modification of the Sudakov lemma obtained in [6] for right-continuous functionals it follows that the measures m_ϑ, $\vartheta \in \Theta$, are absolutely continuous with respect to the measure m_{ϑ_0} and

$$\frac{dm_\vartheta}{dm_{\vartheta_0}}(u) = h\big(t(u), z(u); \vartheta, \vartheta_0\big),$$

i.e. (see formula (6)),

$$\frac{dm_\vartheta}{dm_{\vartheta_0}}(u) = C(u; \vartheta_0)\, a^{v(u)} \exp[-at(u) - \beta s(u)] \prod_{j=1}^{n} \beta_j^{w_j(u)}.$$

Thus we have the following

LEMMA 1. *For every finite Markov time τ there exists a σ-finite measure m_τ on $(U, \mathscr{B}_U)$, independent of ϑ and such that for every $A \in \mathscr{B}_U$ and each $\vartheta \in \Theta$*

$$m_\vartheta(A) = \int_A a^{v(u)} \exp[-at(u) - \beta s(u)] \prod_{j=1}^{n} \beta_j^{w_j(u)} m_\tau(du). \tag{7}$$

4. Sequential plans. Let $g(\vartheta)$ be a real-valued function of the parameter $\vartheta \in \Theta$. Observing the process $\xi(t)$, $t \in T$, up to time τ we have to find an optimal, in some sense, estimate of the value of the function $g(\vartheta)$. A $(\mathscr{B}_U, \mathscr{B}_R)$-measurable function $f\colon U \to R$ is called an *estimator* for $g(\vartheta)$.

Definition. By a *sequential estimation plan* for $g(\vartheta)$ we mean any pair (τ, f) consisting of a Markov time τ satisfying the condition

$$P_\vartheta\big(0 < \tau(\xi) < \infty\big) = 1 \tag{8}$$

for all $\vartheta \in \Theta$ and an estimator f such that, for every $\vartheta \in \Theta$,

$$\mathrm{E}_\vartheta f^2\big(\mathscr{X}(\xi)\big) = \int_U f^2(u)\, a^{v(u)} \exp[-at(u) - \beta s(u)] \prod_{j=1}^{n} \beta_j^{w_j(u)} m_\tau(du) < \infty$$

and

$$\mathrm{E}_\vartheta f\big(\mathscr{X}(\xi)\big) = \int_U f(u)\, a^{v(u)} \exp[-at(u) - \beta s(u)] \prod_{j=1}^{n} \beta_j^{w_j(u)} m_\tau(du) = g(\vartheta). \tag{9}$$

It follows from (8) that the observation of the process $\xi(t)$, $t \in T$, terminates in a finite time. Condition (9) means that the estimator f is unbiased for $g(\vartheta)$.

From (8) and Lemma 1 we have

$$\int_U a^{v(u)} \exp[-at(u) - \beta s(u)] \prod_{j=1}^{n} \beta_j^{w_j(u)} m_\tau(du) = 1 \tag{10}$$

for each $\vartheta \in \Theta$.

In the sequel the functional

$$Z(\tau(\xi), \xi) = \big(V(\tau(\xi)),\ W(\tau(\xi)),\ S(\tau(\xi), \xi)\big)$$

of the process will be simply denoted by

$$Z(\tau) = \big(V(\tau),\ W(\tau), S(\tau)\big).$$

Write $g'_a(\vartheta) = \partial g(\vartheta)/\partial a$ and $g'_j(\vartheta) = \partial g(\vartheta)/\partial \beta_j$, $j = 1, \ldots, n$. The following regularity conditions will be considered:

(i) $g(\vartheta)$ is a differentiable function of the variables $a, \beta_1, \ldots, \beta_n$ such that for every point $\vartheta = (a, \beta_1, \ldots, \beta_n) \in (0, \infty)^{n+1}$ the derivatives $g'_a(\vartheta)$ and $g'_j(\vartheta)$ $(j = 1, \ldots, n)$ do not vanish simultaneously;

(ii) $0 < \mathrm{E}_\vartheta[V(\tau) - a\tau]^2 < \infty$ for all $\theta \in \Theta$ and $0 < \mathrm{E}_\vartheta[W_j(\tau) - \beta_j S(\tau)]^2 < \infty$ for every $j = 1, \ldots, n$ and all $\vartheta \in \Theta$;

(iii) differentiation and repeated differentiation of the integrands with respect to parameters $a, \beta_1, \ldots, \beta_n$ in identities (9) and (10), respectively, are allowed.

The following lemma can easily be established:

LEMMA 2. *If for a sequential plan (τ, f) the regularity conditions* (i)-(iii) *are satisfied, then the following identities hold*:

$$\mathrm{E}_\vartheta V(\tau) = a\mathrm{E}_\vartheta \tau, \tag{11}$$

$$\mathrm{E}_\vartheta[V(\tau) - a\tau]^2 = \mathrm{E}_\vartheta V(\tau), \tag{12}$$

$$\mathrm{E}_\vartheta W_j(\tau) = \beta_j \mathrm{E}_\vartheta S(\tau), \quad j = 1, \ldots, n, \tag{13}$$

$$\mathrm{E}_\vartheta[W_j(\tau) - \beta_j S(\tau)]^2 = \mathrm{E}_\vartheta W_j(\tau), \quad j = 1, \ldots, n, \tag{14}$$

$$\mathrm{E}_\vartheta\big[f\big(\tau, Z(\tau)\big)\big(V(\tau) - a\tau\big)\big] = a g'_a(\vartheta), \tag{15}$$

$$\mathrm{E}_\vartheta\big[f\big(\tau, Z(\tau)\big)\big(W_j(\tau) - \beta_j S(\tau)\big)\big] = \beta_j g'_j(\vartheta), \quad j = 1, \ldots, n, \tag{16}$$

$$\mathrm{E}_\vartheta\big[\big(V(\tau) - a\tau\big)\big(W_j(\tau) - \beta_j S(\tau)\big)\big] = 0, \quad j = 1, \ldots, n, \tag{17}$$

$$\mathrm{E}_\vartheta\big[\big(W_i(\tau) - \beta_i S(\tau)\big)\big(W_j(\tau) - \beta_j S(\tau)\big)\big] = 0, \quad i, j = 1, \ldots, n,\ i \neq j. \tag{18}$$

Let

$$\Lambda = \left(\frac{\partial \log h\big(\tau, Z(\tau); \vartheta, \vartheta_0\big)}{\partial a}, \frac{\partial \log h\big(\tau, Z(\tau); \vartheta, \vartheta_0\big)}{\partial \beta_1}, \ldots, \frac{\partial \log h\big(\tau, Z(\tau); \vartheta, \vartheta_0\big)}{\partial \beta_n}\right)$$

$$= \left(\frac{V(\tau) - a\tau}{a}, \frac{W_1(\tau) - \beta_1 S(\tau)}{\beta_1}, \ldots, \frac{W_n(\tau) - \beta_n S(\tau)}{\beta_n}\right),$$

let $J = \mathrm{E}_\vartheta(\Lambda^*\Lambda)$, where Λ^* denotes the transposed matrix to Λ, and put $G = \big(g'_a(\vartheta), g'_1(\vartheta), \ldots, g'_n(\vartheta)\big)$. Assume that for a sequential plan (τ, f) the regularity conditions (i)-(iii) are satisfied. Then applying methods used in [2], p. 52, or in [8] we obtain the inequality

$$\mathrm{D}_\vartheta f\big(\tau, Z(\tau)\big) = \mathrm{E}_\vartheta\big[f\big(\tau, Z(\tau)\big) - g(\vartheta)\big]^2 \geqslant GJ^{-1}G^*,$$

where equality holds for a particular value of ϑ if and only if $f\big(\tau, Z(\tau)\big) = GJ^{-1}\Lambda^* + g(\vartheta)$ with probability 1. Using Lemma 2 we have the following

THEOREM 1. *For every sequential plan* (τ, f) *satisfying conditions* (i)-(iii) *we have*

$$(19) \qquad \mathrm{D}_\vartheta f\big(\tau, Z(\tau)\big) \geqslant \frac{a}{\mathrm{E}_\vartheta \tau}[g'_a(\vartheta)]^2 + \frac{1}{\mathrm{E}_\vartheta S(\tau)} \sum_{j=1}^{n} \beta_j [g'_j(\vartheta)]^2$$

for all $\vartheta \in \Theta$. *The equality holds for a particular value of* ϑ *if and only if*

$$(20) \qquad f\big(\tau, Z(\tau)\big) = \frac{g'_a(\vartheta)}{\mathrm{E}_\vartheta \tau}[V(\tau) - a\tau] + \frac{1}{\mathrm{E}_\vartheta S(\tau)} \sum_{j=1}^{n} g'_j(\vartheta)[W_j(\tau) - \beta_j S(\tau)] + g(\vartheta)$$

with probability 1.

A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient at* (a fixed value) ϑ if (19) becomes equality for ϑ. The estimator f is then called *efficient at the value* ϑ, and the function $g(\vartheta)$ is *efficiently estimable at the point* ϑ.

A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient* if it is efficient at each $\vartheta \in \Theta$. The estimator f is then called *efficient*, and the function $g(\vartheta)$ is *efficiently estimable*.

Two distinct values $\vartheta^{(1)}$ and $\vartheta^{(2)}$ are said to be *equivalent with respect to* $g(\vartheta)$ if $g(\vartheta^{(1)}) = g(\vartheta^{(2)})$.

As an immediate consequence of the second part of Theorem 1 we have the following corollary:

A sequential estimation plan (τ, f) *for* $g(\vartheta)$ *is efficient at a point* ϑ *if and only if there exist constants* $c, d_1, \ldots, d_n$ *not all equal to zero such that*

$$(21) \qquad f(u) = c[v(u) - at(u)] + \sum_{j=1}^{n} d_j [w_j(u) - \beta_j s(u)] + g(\vartheta) \qquad m_\tau\text{-a.e.}$$

Using this fact, in an analogous way as in [1] we obtain the following result:

THEOREM 2. *If a sequential estimation plan (τ, f) for $g(\vartheta)$ is efficient at two values of ϑ which are not equivalent with respect to $g(\vartheta)$, then there exist constants $\gamma_1, \ldots, \gamma_n, \delta_1, \delta_2, \delta_3$ not all equal to zero and $\delta_4 \neq 0$ such that*

$$(22)\qquad \sum_{j=1}^{n} \gamma_j w_j(u) + \delta_1 s(u) + \delta_2 v(u) + \delta_3 t(u) + \delta_4 = 0 \ \ m_\tau\text{-a.e.}$$

It follows from Theorem 2 that one should seek the efficient sequential plans for $g(\vartheta)$ among the plans determined by the Markov stopping times for which (22) holds.

Theorem 1 implies that for a given Markov stopping time τ the only efficient sequential estimators at a point $\vartheta^0 = (a^0, \beta_1^0, \ldots, \beta_n^0)$ are those which take the form (see (21))

$$(23)\qquad f(\tau, Z(\tau)) = c^0[V(\tau) - a^0\tau] + \sum_{j=1}^{n} d_j^0 [W_j(\tau) - \beta_j^0 S(\tau)] + g(\vartheta^0)$$

with probability 1, where the constants $c^0, d_1^0, \ldots, d_n^0$ do not vanish simultaneously. Thus the function $g(\vartheta)$ is efficiently estimable at $\vartheta = \vartheta^0$ if and only if it is equal to the expected value of the estimator defined by (23). Therefore, we have

$$\begin{aligned} g(\vartheta) &= \mathrm{E}_\vartheta f(\tau, Z(\tau)) \\ &= c^0 \mathrm{E}_\vartheta[V(\tau) - a^0\tau] + \sum_{j=1}^{n} d_j^0 \mathrm{E}_\vartheta[W_j(\tau) - \beta_j^0 S(\tau)] + g(\vartheta^0). \end{aligned}$$

Hence, using (11) and (13) we obtain the following

THEOREM 3. *In a given sequential plan (τ, f) the function $g(\vartheta)$ is efficiently estimable at a point $\vartheta^0 = (a^0, \beta_1^0, \ldots, \beta_n^0)$ if and only if there exist constants $c^0, d_1^0, \ldots, d_n^0$ not all equal to zero such that*

$$(24)\qquad g(\vartheta) = c^0(a - a^0)\mathrm{E}_\vartheta \tau + \sum_{j=1}^{n} d_j^0 (\beta_j - \beta_j^0) \mathrm{E}_\vartheta S(\tau) + g(\vartheta^0).$$

The study of functions efficiently estimable at a point was initiated by DeGroot in [3] for the binomial process.

In connection with Theorem 2 let us consider the following Markov stopping times:

$$(25)\qquad \tau^{(1)}(x(\cdot)) = T_0,$$

where T_0 is a positive real number;

$$(26)\qquad \tau^{(2)}(x(\cdot)) = \inf\{t\colon v(t) = v_0\},$$

where v_0 is a positive integer;

$$\tau^{(3)}(x(\cdot)) = \inf\{t\colon\ S(t, x(\cdot)) = s_0\}, \tag{27}$$

where s_0 is a positive real number;

$$\tau^{(4)}(x(\cdot)) = \inf\Big\{t\colon\ \sum_{i=1}^{k} w_{\sigma(i)}(t) = m_0\Big\}, \tag{28}$$

where m_0 is a positive integer, $(\sigma(1), \ldots, \sigma(n))$ is a permutation of $(1, \ldots, n)$, and k is an integer, $2 \leqslant k \leqslant n$;

$$\tau^{(5)}(x(\cdot)) = \inf\{t\colon\ w_j(t) = l_0\}, \tag{29}$$

where l_0 is a positive integer.

Let us take into consideration, e.g., the sequential plan determined by the Markov stopping time $\tau^{(4)}$. For this plan we have

$$\sum_{i=1}^{k} W_{\sigma(i)}(\tau^{(4)}) = m_0 \tag{30}$$

with probability 1. It can be easily checked that if $\mathrm{E}_\vartheta S^2(\tau^{(4)}) < \infty$ for all $\vartheta \in \Theta$, then there exist appropriate integrable majorants for the derivatives of the integrands in (9) and (10) in the case of the plan determined by $\tau^{(4)}$, so that the regularity condition (iii) is satisfied for this plan. From (13) we then have

$$m_0 = \sum_{i=1}^{k} \mathrm{E}_\vartheta W_{\sigma(i)}(\tau^{(4)}) = \sum_{i=1}^{k} \beta_{\sigma(i)} \mathrm{E}_\vartheta S(\tau^{(4)}),$$

whence

$$\mathrm{E}_\vartheta S(\tau^{(4)}) = \frac{m_0}{\sum_{i=1}^{k} \beta_{\sigma(i)}}. \tag{31}$$

We estimate now the function

$$g(\vartheta) = \frac{\sum_{i=1}^{k} c_i \beta_{\sigma(i)}}{\sum_{i=1}^{k} \beta_{\sigma(i)}}, \tag{32}$$

where $c_1, \ldots, c_k$ are arbitrary constants not all equal to zero. It follows from Theorem 1 that the sequential estimation plan $(\tau^{(4)}, f)$ for $g(\vartheta)$ given by (32) is efficient if and only if the estimator f takes the form

$$f(\tau^{(4)}, Z(\tau^{(4)})) = \frac{1}{\mathrm{E}_\vartheta S(\tau^{(4)})} \sum_{i=1}^{k} g'_{\sigma(i)}(\vartheta)[W_{\sigma(i)}(\tau^{(4)}) - \beta_{\sigma(i)} S(\tau^{(4)})] + g(\vartheta) \tag{33}$$

with probability 1. Using (30)-(32) and

$$g'_{\sigma(i)}(\vartheta) = \frac{c_i - g(\vartheta)}{\sum_{i=1}^{k} \beta_{\sigma(i)}},$$

we infer from (33) that this estimator is of the form

$$f\big(\tau^{(4)}, Z(\tau^{(4)})\big) = \frac{1}{m_0} \sum_{i=1}^{k} c_i W_{\sigma(i)}(\tau^{(4)}).$$

Under the sequential plan regularity conditions for the respective Markov times defined by (25)-(29) we have the following class of efficient sequential plans:

(a) $(\tau^{(1)}, f^{(1)})$ with $f^{(1)} = c_1 T_0^{-1} V(T_0) + c_2$ is efficient for $g(\vartheta) = c_1 a + c_2$, where $c_1 \neq 0$ and c_2 are arbitrary constants;

(b) $(\tau^{(2)}, f^{(2)})$ with $f^{(2)} = c_1 v_0^{-1} \tau^{(2)} + c_2$ is efficient for $g(\vartheta) = c_1/a + c_2$, where $c_1 \neq 0$ and c_2 are arbitrary constants;

(c) $(\tau^{(3)}, f^{(3)})$ with $f^{(3)} = s_0^{-1} \sum_{j=1}^{n} c_j W_j(\tau^{(3)}) + d$ is efficient for

$$g(\vartheta) = \sum_{j=1}^{n} c_j \beta_j + d,$$

where $c_1, \ldots, c_n, d$ are constants such that $c_1, \ldots, c_n$ do not vanish simultaneously;

(d) $(\tau^{(4)}, f^{(4)})$ with $f^{(4)} = m_0^{-1} \sum_{i=1}^{k} c_i W_{\sigma(i)}(\tau^{(4)}) + d$ is efficient for

$$g(\vartheta) = \Big(\sum_{i=1}^{k} c_i \beta_{\sigma(i)}\Big)\Big(\sum_{i=1}^{k} \beta_{\sigma(i)}\Big)^{-1} + d,$$

where $c_1, \ldots, c_k, d$ are constants such that $c_1, \ldots, c_k$ do not vanish simultaneously;

(e) $(\tau^{(5)}, f^{(5)})$ with $f^{(5)} = c_1 l_0^{-1} S(\tau^{(5)}) + c_2$ is efficient for $g(\vartheta) = c_1/\beta_j + c_2$, where $c_1 \neq 0$ and c_2 are arbitrary constants.

In our study of efficient sequential estimation for Markov processes with migration, the plan $(\tau^{(3)}, f^{(3)})$ is essentially different from the plans already investigated (see [7] and [8]) for the Poisson and finite-state Markov processes.

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