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Characterization of regular tempered distributions

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Abstract. This paper gives an elementary proof of the following property: a locally Lebesgue integrable complex valued function u is a regular tempered distribution, i.e.,

$$u[\sigma] = \int u(x)\sigma(x)\,dx \quad \text{for } \sigma\in S \text{ (with } u\sigma\in L_1)$$

iff for a certain non-negative integer k the function

$$R^n \ni x \mapsto \frac{u(x)}{(1+|x|^2)^k} \quad \text{is integrable on } R^n.$$

I presented this result in the course held in the year 1976/77 at the Warsaw University.

Basic notations. The variable in the n-dimensional real Euclidean space R^n will be denoted by $x = (x_1, \ldots, x_n)$. A complex valued function φ defined in R^n is said to be a *C^∞-function* if it possesses continuous partial derivatives of all orders. By C_0^∞ we denote the set of all C^∞-functions with compact supports in R^n. S denotes, as usually, the space of C^∞-functions rapidly decreasing at infinity. A continuous linear functional u on S is called a *tempered distribution*. The set of all tempered distributions is denoted by S'. As usual, we indentify S' with a subspace of the vector space D' of Schwartz' distributions. A linear bijection between S' and a subspace of D' is the operation of restriction of functionals $T\in S'$ from S to C_0^∞.

We denote by L_1^{loc} the space of locally Lebesgue integrable functions, i.e., Lebesgue integrable on any compact subset of R^n. We indetify every complex valued function $u = L_1^{\mathrm{loc}}$ with the distribution defined by

$$(1)\qquad u[\varphi] = \int_{R^n} u(x)\varphi(x)\,dx \quad \text{for } \varphi\in C_0^\infty.$$

A tempered distribution u is called *regular* if there exists a function $u\in L_1^{\mathrm{loc}}$ such that

$$(2)\qquad u[\sigma] = \int_{R^n} u(x)\sigma(x)\,dx \quad \text{for } \sigma\in S.$$

The set of all regular tempered distributions will be denoted by $(S')_r$.

EXAMPLE. Let u be the function defined by $u(x) = \frac{d}{dx}(\sin e^x)$ for $x \in R^1$. Clearly $u \in S'(R^1) \cap L_1^{\text{loc}}(R^1)$. Let $\tilde{u}[\sigma] = -\int_{-\infty}^{+\infty} \sigma'(x) \sin e^x \, dx$ for $\sigma \in S(R^1)$. It is easy to see that $\tilde{u} \in S'(R^1)$, $\tilde{u}[\varphi] = u[\varphi] = \int_{-\infty}^{+\infty} \varphi(x) \frac{d}{dx}(\sin e^x) \, dx$ for $\varphi \in C_0^\infty$, but nevertheless there exists a non-negative function $\sigma \in S(R^1)$ (¹) such that $\sigma u \notin L_1(R^1)$. Clearly, $\tilde{u} = u \in L_1^{\text{loc}} \cap S'$, $u \notin (S')_r$.

Let N_0 be the set of non-negative integers. Denote by Λ the set of locally integrable functions u such that for some $k \in N_0$ the function g defined by

$$g(x) = \frac{u(x)}{(1+|x|^2)^k} \quad \text{for } x \in R^n \tag{3}$$

is integrable on R^n.

We denote by S_∞ the vector space of functions f defined a.e. (almost everywhere) on R^n and such that

$$q_k(f) = \operatorname{ess\,sup}(1+|x|^2)^k |f(x)| < +\infty, \quad k = 1, 2, \ldots$$

and we equip this space with the convergence defined by the sequence of semi-norms $\{q_k\}$.

In this paper we give a characterization of the space $(S')_r$ in terms of the spaces, S, S_∞, L_1 and we establish that $(S')_r = \Lambda$. For the proof of the inclusion $(S')_r \subset \Lambda$ we need the following two lemmas.

LEMMA 1. *For every sequence of positive numbers*

$$r_0 = 1, \quad r_k > r_{k-1} + 1, \quad k = 1, 2, \ldots$$

there exists a non-negative function $\sigma \in S$ such that

$$\sigma(x) = \frac{1}{(1+|x|^2)^k} \quad \text{for } r_{k-1} + 1 \leqslant |x| \leqslant r_k, \ k = 1, 2, \ldots \tag{4}$$

A proof of Lemma 1 is given in [3](²).

LEMMA 2. *If $u \in L_1^{\text{loc}}$, $u \notin \Lambda$, then there exists a non-negative function $\sigma \in S$ such that $u\sigma \notin L_1$.*

(¹) See [3], Exercise 9, p. 171.

(²) See [3], p. 154, Exercise 15* with $1+|x|^2$ instead of $1+|x|$. I have already applied the function σ from Lemma 1 in the proof of the theorem characterizing the class O_M of C^∞-functions slowly increasing at infinity (see [3], p. 152, Proposition 6). Observe that for an arbitrary sequence of points $\{a_k\}$, $|a_k| = r_k > r_{k-1} + 1$, $k = 1, 2, \ldots$, this function verifies the condition $\sigma(a_k) = 1/(1+r_k^2)^k$, $k = 1, 2, \ldots$ A simpler example of a function in $S(E^n)$ satisfying the above condition can be found in [1], p. 104. It was also constructed with the purpose of characterizing the class O_M.

Proof. We begin with two special cases (a) and (b) of Lemma 2.

(a) Proof of Lemma 2 when $u \geqslant 0$ a.e. Let

$$r_0 = 1, \quad r_k > r_{k-1}+1, \quad k = 1, 2, \dots$$

be a sequence of positive numbers such that

$$\int_{P_k} \frac{u(x)}{(1+|x|^2)^k} dx \geqslant 1, \quad k = 1, 2, \dots,$$

where $P_k = \{x: r_{k-1}+1 \leqslant |x| \leqslant r_k\}$, $k = 1, 2, \dots$

Let σ be the function with the properties stated in Lemma 1 for the sequence $\{r_k\}$. Then

$$\int u(x)\sigma(x)\,dx \geqslant \sum_{k=1}^{\infty} \int_{P_k} \frac{u(x)}{(1+|x|^2)^k} dx \geqslant \infty,$$

i.e., $u\sigma \notin L_1$.

(b) Proof of Lemma 2 for a real valued function u. Let

$$u = u^+ - u^-, \quad u^+ \geqslant 0, \quad u^- \geqslant 0 \quad \text{a.e.}$$

Since $u \notin \Lambda$, one of the functions u^+, u^- does not belong to Λ. Assume $u^+ \notin \Lambda$. Thus $u^+ \in L_1^{\text{loc}}$, $u^+ \notin \Lambda$, $u^+ \geqslant 0$ a.e., and so by (a) there exists a non-negative function $\sigma \in S$ such that $\sigma u^+ \notin L_1$. Observe that $(u\sigma)^+ = u^+\sigma$. Hence $(u\sigma)^+ \notin L_1$ and consequently $u\sigma \notin L_1$.

In the general case u is a complex valued function, $u \in L_1^{\text{loc}}$, $u \notin \Lambda$. Then its real and imaginary parts are locally integrable real valued functions and one of them does not belong to Λ. Thus (b) implies the assertion of Lemma 2.

THEOREM 1(³). *Let $u \in L_1^{\text{loc}}$. The following conditions are equivalent:*

(i) $u \in (S')_r$,

(ii) $u \in \Lambda$,

(iii) $S \ni \sigma \mapsto u\sigma \in L_1$,

(iv) $S_\infty \ni \sigma \mapsto u\sigma \in L_1$,

(v) *the mapping* (iv) *is continuous.*

Proof. Let $u \in \Lambda$ and let k be a non-negative integer such that the function (3) is integrable on R^n. Choose arbitrarily a sequence $\{\sigma_j\}$ converging to zero in S_∞. Then $\lim_{j\to+\infty} u\sigma_j = 0$ in L_1 because

$$\int_{R^n} |u(x)\sigma_j(x)|\,dx \leqslant \int_{R^n} |g(x)|(1+|x|^2)^k |\sigma_j(x)|\,dx \leqslant q_k(\sigma_j) \int_{R^n} |g(x)|\,dx.$$

(³) Added in proof. Another proof of the equivalence of conditions (ii) and (iii) is given in the Habilitationthesis of P. Dierolf (*Zwei Räume regulärer temperierter Distributionen*, preprint, Ludwig-Maximilians-Universität München 1978), suggested by my paper: *On regular temperate distributions*, Studia Math. 44 (1972).

Hence (ii) implies (v). Similarily (ii) implies (i). The implications (i) $\Rightarrow$ (iii), (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) are trivial. Thus we only have to prove that (iii) implies (ii). Assume (iii) and suppose, contrary to our assertion, that $u \notin \Lambda$. By Lemma 2 there exists a function $\sigma \in S$ such that $u\sigma \notin L_1$, and this contradicts condition (iii).

COROLLARY. *Let u be a complex-valued function, locally integrable on* R^n. *The following conditions are equivalent*:

(i) $u\sigma \in L_1$ *for each* $\sigma \in S(R^n)$,

(ii) *there exists a non-negative integer k such that the function*: $R^n \ni x \mapsto u(x)/(1+|x|^2)^k$ *is integrable on* R^n.

To emphasize the significance of Theorem 1 let us recall the following well-known(4)

THEOREM 2. *Let u be a locally integrable function non-negative a.e. Then* $u \in S'$ *iff* $u \in \Lambda$.

Note that the hypothesis that $u \geqslant 0$, essential(5) in Theorem 2, is irrelevant to Theorem 1 which deals with complex valued functions $u \in L_1^{loc}$.

References

[1] H. Hogbe-Nlend, *Distributions et bornologie*, Sao Paulo 1973.

[2] L. Schwartz, *Théorie des distributions*, Hermann, Paris 1966.

[3] Z. Szmydt, *Fourier transformation and linear differential equations*, PWN, Warszawa, Reidel Publ. Company, Dordrecht, Holland (1977).

[4] W. S. Vladimirov, *Generalized functions in mathematical physics* (in Russian) (1976).

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(4) See [2], Theorem VII, p. 242. Other proofs of this theorem are given in [3], p. 171, Exercise 6, and in [4], p. 92.

(5) The function $R^1 \ni x \mapsto \frac{d}{dx}\sin e^x$ considered in the Example above does not belong to Λ, but nevertheless it is a tempered distribution.