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On the extension of holomorphic maps on locally convex spaces with values in Fréchet spaces

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Abstract. It is shown that for every exact sequence

$$0 \to E \to F \to G \to 0$$

of Fréchet spaces and for every Fréchet space P the restriction map $O(F', P) \to O(G', P)$ is surjective if either G is nuclear or F is a Fréchet–Schwartz space having a basis of pre-Hilbertian neighbourhoods of zero.

The aim of this paper is to study the extension of holomorphic maps defined on locally convex spaces with values in Fréchet spaces. This problem has been investigated by several authors [1], [2].

In Section 1 we deal with the extension of holomorphic maps on DF-spaces with values in Fréchet spaces. It is proved that every locally bounded holomorphic map on a DFN-subspace F of a locally convex space L with values in a Fréchet space can be extended to a holomorphic map on some neighbourhood of F in L. We also prove that for every exact sequence of Fréchet spaces

$$0 \to E \to F \to G \to 0$$

and for every Fréchet space P the restriction map R: $O(F', P) \to O(G', P)$ is surjective if one of the two following conditions holds:

(FN) G is nuclear,

(FSH) F is an FSH-space i.e. a Fréchet–Schwartz space having a basis of pre-Hilbertian neighbourhoods of zero.

Note that when F is nuclear and $P = C$ this result has been established by Boland [2].

Section 2 is devoted to a proof of the fact that if X is a Stein space containing a subvariety W such that $\theta(W) \cong O(C^n)$ for some n, then there exist an s-nuclear Fréchet space F containing $O(X)$ as a subspace and a holomorphic function on $O(X)$ which cannot be extended to a holomorphic function on a neighbourhood of $O(X)$ in F.

1. The extension of holomorphic maps on DF**-spaces.** In this section the following two theorems are proved:

1.1. THEOREM. *Let F be a nuclear Fréchet space and L a locally convex space containing F' as a subspace. Then every locally bounded holomorphic map of F' into a Fréchet space P can be extended to a holomorphic map on a neighbourhood of F' in L.*

1.2. THEOREM. *Let*

$$0 \to E \xrightarrow{e} F \xrightarrow{\eta} G \to 0$$

be an exact sequence of Fréchet spaces and P a Fréchet space. Then the restriction map $O(F', P) \to O(G', P)$ is surjective if one of the two following conditions holds:

(FN) G is nuclear.

(FSH) F is an FSH-space.

The proof of Theorem 1.1 is based on the following

1.3. PROPOSITION. *Let E be a nuclear subspace of a locally convex space L. Then there exist an SH-space $\tilde{E}$ and continuous linear maps $h\colon L \to \tilde{E}$ and $\tilde{e}\colon E \to \tilde{E}$ such that*

$$h|E = \tilde{e} \quad \text{and } \tilde{e} \text{ is an embedding.}$$

Proof. By $\mathscr{U}(E)$ we denote the set of all balanced convex neighbourhoods of zero in E. For $U, V \in \mathscr{U}(E)$ we write $U < V$ if $U \supset V = \tilde{V} \cap E$, $\tilde{V} \in \mathscr{U}(L)$ and the canonical map $\omega(V, U)\colon E(V) \to E(U)$ belongs to $l^{1/2}$ [12], where $E(U)$ denotes the completion of $E/\mathscr{G}_U \stackrel{\text{def}}{=} E/\mathscr{G}_U^{-1}(0)$ equipped with the norm $\mathscr{G}_U$ generated by U.

Let $V, U \in \mathscr{U}(E)$, $U < V$. Since $\omega(V, U) \in l^{1/2}$, it can be represented in the form

$$\omega(V, U)u = \sum_{j=1}^{\infty} \lambda_j u_j'(u) v_j \quad \text{for all } u \in E(V),$$

where $\lambda_j > 0$, $\sum_{j=1}^{\infty} \sqrt{\lambda_j} < \infty$ and $\sup\{\|u_j'\| + \|v_j\|\} < \infty$, [12], Proposition 8.4.2. By the Hahn–Banach theorem we find $\tilde{u}_j \in L(\tilde{V})'$ such that $\tilde{u}_j|E(V) = u_j'$ and $\|\tilde{u}_j\| = \|u_j'\|$ for all j. Define continuous linear maps $\tilde{\omega}(V, U)\colon L(\tilde{V}) \to E(U)$, $P_1(V, U)\colon L(\tilde{V}) \to l^2$ and $Q_1(V, U)\colon l^2 \to E(U)$ by the formulas

$$\tilde{\omega}(V, U)u = \sum_{j=1}^{\infty} \lambda_j \tilde{u}_j(u) v_j \qquad \text{for } u \in L(\tilde{V}),$$

$$P_l(V, U)u = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \tilde{u}_j(u) e_j \quad \text{for } u \in L(\tilde{V}),$$

$$Q_l(V, U)\xi = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \xi_j v_j \qquad \text{for } \xi = (\xi_j) \in l^2,$$

where $\{e_j\}$ is the canonical basis of l^2. Note that $\tilde{\omega}(V, U) = Q_l(V, U) P_l(V, U)$ and $P_l(V, U)$, $Q_l(V, U)$ are nuclear.

We write $\sqrt{\lambda_j} = \beta_j \alpha_j$, with $\sum_{j=1}^{\infty} |\beta_j| < \infty$, $\alpha_j \to 0$, and define continuous linear maps $P_2(V, U)$: $L(\tilde{V}) \to l^2$, $Q_2(V, U) = l^2 \to l^2$ by the formulas

$$P_2(V, U)u = \sum_{j=1}^{\infty} \beta_j \tilde{u}_j(u) e_j, \qquad Q_2(V, U)\xi = \sum_{j=1}^{\infty} \alpha_j \xi_j e_j.$$

Then $P_2(V, U)$ is nuclear, $Q_2(V, U)$ is compact and $P_1(V, U) = Q_2(V, U) P_2(V, U)$. Continuing this process we get two sequences of continuous linear maps $\{P_n(V, U)\}$ and $\{Q_n(V, U)\}$ such that $P_n(V, U)$ are nuclear and $Q_n(V, U)$ are compact and

$$\begin{aligned} &\tilde{\omega}(V, U) = Q_1(V, U) P_1(V, U) \\ (1.1) \qquad &\dots\dots\dots\dots\dots\dots \\ &P_j(V, U) = Q_{j+1}(V, U) P_{j+1}(V, U) \\ &\dots\dots\dots\dots\dots\dots \end{aligned}$$

Fix $\tilde{W} \in \mathscr{U}(L)$. Let $\mathscr{F}(\tilde{W})$ denote the set of all finite sequences $\mathscr{A} = \{(\tilde{V}_1, \tilde{U}_1, \dots, (\tilde{V}_n, \tilde{U}_n), k_1, \dots, k_n\}$, where $\tilde{V}_j, \tilde{U}_j \in \mathscr{U}(L)$, $U_j = \tilde{U}_j \cap E < V_j = \tilde{V}_j \cap E$, $k_j \in N$.

For each $\mathscr{A} \in \mathscr{F}(\tilde{W})$ we denote by $E(\tilde{W}, \mathscr{A})$ the completion of $L(\tilde{W})/\mathscr{G}_{\mathscr{A}}$ in the norm

$$(1.2) \qquad \mathscr{G}_{\mathscr{A}}(u) = \Big(\sum_{j=1}^{n} \|P_{k_j}(V_j, U_j)\omega(\tilde{W}, \tilde{V}_j)u\|^2\Big)^{1/2}.$$

Then $\tilde{E}(\tilde{W}, \mathscr{A})$ is a Hilbert space, since $P_n(V, U)$ have values in l^2. For $a_j = \{(\tilde{V}_1^j, \tilde{U}_1^j), \dots, (\tilde{V}_{n_j}^j, \tilde{U}_{n_j}^j); k_1^j, \dots, k_n^j\} \in \mathscr{F}(\tilde{W})$ we write $a_1 < a_2$ if

$$\{(\tilde{V}_1^1, \tilde{U}_1^1), \dots, (\tilde{V}_{n_1}^1, \tilde{U}_{n_1}^1)\} \subset \{(\tilde{V}_1^2, \tilde{U}_1^2), \dots, (\tilde{V}_{n_2}^2, \tilde{U}_{n_2}^2)\}$$

and $\max k_j^1 < \min k_j^2$. Note that if $a_1 < a_2$, then the identity map $L(\tilde{W}) \to L(\tilde{W})$ induces naturally a continuous linear map $\tilde{\omega}(a_2, a_1)$: $\tilde{E}(\tilde{W}, a_2) \to \tilde{E}(\tilde{W}, a_1)$. Put $\tilde{E}(\tilde{W}) = \varprojlim \{\tilde{E}(\tilde{W}, \mathscr{A}), \tilde{\omega}(\tilde{W}, a, b)\}$. From the construction of $\tilde{E}(\tilde{W})$ it follows that for $\tilde{W}_1, \tilde{W}_2 \in \mathscr{F}(L)$, $\tilde{W}_2 \subset \tilde{W}_1$, the map $\omega(\tilde{W}_2, \tilde{W}_1)$: $L(\tilde{W}_2) \to L(\tilde{W}_1)$ induces naturally a continuous linear map $\tilde{\omega}(\tilde{W}_2, \tilde{W}_1)$: $\tilde{E}(\tilde{W}_2) \to \tilde{E}(\tilde{W}_1)$ satisfying the relation $h(\tilde{W}_1)\omega(\tilde{W}_2, \tilde{W}_1) = \tilde{\omega}(\tilde{W}_2, \tilde{W}_1) h(\tilde{W}_2)$, where $h(\tilde{W})$: $L(\tilde{W}) \to \tilde{E}(\tilde{W})$ is the canonical map. Put $\tilde{E} - \varprojlim \{\tilde{E}(\tilde{W}), \tilde{\omega}(\tilde{W}, \tilde{W}')\}$, $h = \varprojlim h(\tilde{W})$ and $\tilde{e} = h|E$. Then, by (1.1) and (1.2) and since $Q_n(V, U)$ are compact, it is easy to check that $\tilde{e}$ is an embedding and $\tilde{E}$ is an SH-space. The lemma is proved.

1.4. LEMMA. *Every locally bounded holomorphic map from a normed space*

B into a complete locally convex space L can be extended to a holomorphic map on some neighbourhood of B in the completion $\hat{B}$.

Proof. Let f: $B \to L$ be a locally bounded holomorphic map. To prove the lemma it suffices to show that f can be extended to a holomorphic map on some neighbourhood of every point $x \in B$. We can assume that $x = 0$. Take $\varepsilon > 0$ such that f is bounded on $S(\varepsilon) = \{x \in B: \|x\| < \varepsilon\}$. Consider the Taylor expansion of f at zero

$$f(x) = \sum_{n=0}^{\infty} P_n f(x).$$

Since

$$P_n f(x) = \tfrac{1}{2}\pi i \int_{|\lambda|=1} \frac{f(\lambda x)\, d\lambda}{\lambda^{n+1}} \quad \text{for } x \in S(\varepsilon),$$

it follows that

$$q(P_n f) \leqslant \tfrac{1}{2} M_q \pi \varepsilon^n, \tag{1.3}$$

where q is a continuous seminorm on L and $M_q = \sup\{q(fx): x \in S(\varepsilon)\}$.

Let $\widehat{P_n f}$ denote the continuous extension of $P_n f$ onto $\hat{B}$. Then by (1.3) we have

$$q(\widehat{P_n f}) \leqslant \tfrac{1}{2} M_q n^n \pi n! \varepsilon^n. \tag{1.4}$$

By (1.4) it follows that the series $\sum_{n=0}^{\infty} P_n f$ converges uniformly on $S(\delta)$ to a holomorphic extension of f onto $S(\delta)$, where $\delta = \varepsilon/e$.

Proof of Theorem 1.1. First observe that every strong dual of a Fréchet–Montel space is Lindelöf and satisfies the following condition [6]:

(G) Given a sequence of neighbourhoods of zero $\{W_j\}$, there exists a sequence $\{\lambda_j\}$, $\lambda_j > 0$, such that $\bigcap_{j=1}^{\infty} \lambda_j W_j$ is again a neighbourhood of zero.

Take a countable open covering $\{u_j + U_j\}$, $U_j \in \mathscr{U}(F')$, of F' such that $f|u_j + U_j$ is bounded. By (G) there exists a sequence $\{\lambda_j\}$ such that $U = \bigcap \lambda_j U_j \in \mathscr{U}(F')$. Let K be a compact set in F such that span K is dense in F and let q be a continuous norm on F' given by

$$q(u) = \mathscr{G}_U(u) + \sup\{|u(v)|: v \in K\}.$$

Then f can be considered as a G-holomorphic function on F'/q. Since $U/\lambda_j \subset U_j$, it follows that U_j is open in F'/q. Hence by the boundedness of f on $u_j + U_j$ we infer that f is holomorphic and locally bounded on F'/q. Let $\tilde{F}'$, $\tilde{e}$ and h be as in Lemma 1.3. Since $\tilde{F}'$ has a basis of pre-Hilbertian neighbourhoods of zero, there exists a continuous seminorm $\tilde{q}$ on $\tilde{F}'$ such that $q \leqslant \tilde{q}|F'$ and $\hat{\tilde{F}}'/\tilde{q}$ is Hilbert. Let d: $\hat{\tilde{F}}'/\tilde{q} \to \hat{F}'/\tilde{q}$ be the orthogonal projection. Since f is holomorphic and locally bounded on $F'/\tilde{q}$, f can be extended to a holomorphic map $\tilde{f}$ on a neighbourhood Ω of $F'/\tilde{q}$ in $\widehat{F'/\tilde{q}}$. Setting $f = \tilde{f}dh|(dh)^{-1}(\Omega)$ we get a required extension of f. The theorem is proved.

The proof of Theorem 1.2 is based on the following

1.5. Lemma [5]. *Every separable Fréchet space is the image of a Fréchet–Montel space under a continuous linear map.*

1.6. Lemma. *Let g be a continuous linear map from a Fréchet space F onto a separable Fréchet space E. Then there exists a separable closed subspace G of F such that $g(G) = E$.*

Proof. By a theorem of Michael [8] there exists a continuous map q: $E \to F$ such that $gq = \mathrm{id}$. Then setting $G = \overline{\operatorname{span} q(E)}$ we get a separable closed subspace G of F such that $g(G) = E$.

1.7. Lemma. *Let F be a reflexive Fréchet space and Ω a subset of F' such that $\Omega \cap U^0$ is $\sigma(F', F)$-open for all $U \in \mathscr{U}(F)$. Then Ω is open.*

Proof. Let $\{U_n\}$ be a decreasing basis of balanced convex neighbourhoods of zero in F and $F_n = F(U_n)$. Since F is reflexive Fréchet, F' is bornological [13] and hence $F' = \varinjlim F'_n$. Thus it suffices to construct for each $u \in \Omega$ a sequence of balanced convex neighbourhoods V_n of zero in F'_n such that $u + \sum_{n=1}^{\infty} V_n \subset \Omega$.

Let $u \in \Omega$. Take n_0 such that $\|u\| U_{n_0} < 1$. Since $\Omega \cap U^0_{n_0}$ is $\sigma(F'_{n_0}, F_{n_0})$-open in $U^0_{n_0}$, there exists n_1 such that

$$u + V_1 \subset \Omega \cap U^0_{n_0} \quad \text{and} \quad \|u+v\| U_{n_0} < 1 \quad \text{for } v \in V_1,$$

where $V_1 = U^0_{n_0}/n_1$. Since $u + V_1$ is $\sigma(F'_{n_0}, F_{n_0})$-compact in $U^0_{n_0}$, and hence $\sigma(F'_{n_0+1}, F'_{n_0+1})$-compact in $U^0_{n_0+1}$, by the $\sigma(F'_{n_0+1}, F'_{n_0+1})$-openness of $\Omega \cap U^0_{n_0+1}$ in $U^0_{n_0+1}$ it follows that there exists n_2 such that

$$u + V_1 + V_2 \subset \Omega \cap U^0_{n_0+1} \quad \text{and} \quad \|u + v_1 + v_2\| U_{n_0+1} < 1$$

for $v_1 \in V_1$ and $v_2 \in V_2$, where $V_2 = U^0_{n_0+1}/n_2$.

Continuing this process we get a sequence of balanced convex neighbourhoods V_j of zero in F'_{n_0+j-1} such that

$$u + \sum_{j=1}^{\infty} V_j \subset \Omega.$$

This completes the proof.

Since every Fréchet–Montel is reflexive and since its dual is also Montel, the following corollary is an immediate consequence of Lemma 1.7.

1.8. COROLLARY. *The dual space of a Fréchet–Montel space is a k-space. Let L be a locally convex space and let $P^n(L)$ denote the set of continuous homogeneous polynomials of degree n on L. A homogeneous polynomial $f \in P^n(L)$ is called nuclear if it can be represented in the form*

$$(1.5) \qquad f(u) = \sum_{k=1}^{\infty} a_1^k(u) \dots a_n^k(u), \quad \text{where } a_j^k \in L(U)'$$

for some $U \in \mathscr{U}(L)$ and

$$(1.6) \qquad \sum_k \|a_1^k\| U \dots \|a_n^k\| U < \infty.$$

By $P_N^n(L)$ we denote the set of nuclear homogeneous polynomials of degree n on L. By (1.6) it follows that

$$(1.7) \qquad \Pi_B(f) = \inf\{\sum_k \|a_1^k\| B \dots \|a_n^k\| B : f = \sum_k a_1^k \dots a_n^k\} < \infty$$

for every bounded subset B of L and for every $f \in P_N^n(L)$.

It is known [2], Lemma 3.1, that if L is dual nuclear Fréchet, then $P_N^n(L) = P^n(L)$ for every $n \geq 0$ and the topology of $O(L)$ can be defined by the seminorms

$$\Pi_B(f) = \sum_n \Pi_B(d^n f(0))/n!, \quad f \in O(L).$$

Proof of Theorem 1.2 in the case where G is nuclear.

(a) First we assume that F is Montel. By Corollary 1.8 it follows that the spaces $O(F')$ and $O(G')$ are Fréchet. Hence, to prove the surjectivity of the restriction map $R: O(F') \to O(G')$ it suffices to show that R is almost open. Let K be a compact set in F'. Since the restriction map $R|F'' = F: F \to G = G''$ is open, there exist a compact set $\tilde{K}$ in G' such that

$$(1.8) \qquad R(\{a \in F: \|a\| K < \alpha < \gamma < 1\} \supset \{b \in G: \|b\| \tilde{K} < 1\}).$$

Let

$$U = \{f \in O(F'): \|f\| K < \alpha < 1\} \quad \text{and} \quad V = \{f \in O(G'): \Pi_{\tilde{K}}(f) < \beta\},$$

where $\beta = \alpha(1-\alpha)/2$.

We will show that $\overline{R(U)} = V$. Let $f \in V$. Consider the Taylor expansion of f at 0,

$$f = \sum_{n=0}^{\infty} d^n f(0)/n!.$$

Since the sequence $\{P_r = \sum_{n=0}^{r} d^n f(0)/n!\}$ converges to f in $O(G')$, it suffices to check that $P_r \in \overline{R(U)}$ for all $r \geq 0$.

Select a representative

$$P_r = \sum_{n=0}^{r} \sum_{k=1}^{\infty} a_{1n}^k \ldots a_{nn}^k$$

of P_r such that

$$\sum_{n=0}^{r} \sum_{k=1}^{\infty} \|a_{1n}^k\| \tilde{K} \ldots \|a_{nn}^k\| \tilde{K} < \beta.$$

By (1.8), for every (k, j, n) there exists $\tilde{a}_{jn}^k \in F$ such that

$$R(\tilde{a}_{jn}^k) = a_{jn}^k \quad \text{and} \quad \|\tilde{a}_{jn}^k\| K < \gamma \|a_{jn}^k\| \tilde{K}.$$

Put

$$P_n^q = \sum_{k \leqslant q} a_{1n}^k \ldots a_{nn}^k, \qquad \tilde{P}_n^q = \sum_{k \leqslant q} \tilde{a}_{1n}^k \ldots \tilde{a}_{nn}^k.$$

Then $R(\tilde{P}_n^q) = P_n^q$ for every $n \geqslant 0$ and $q \geqslant 1$ and

$$\Big\|\sum_{n \leqslant r} \tilde{P}_n^q\Big\| K \leqslant \sum_{n \leqslant r} \|\tilde{P}_n^q\| K \leqslant \sum_{n \leqslant r} \sum_{k \leqslant q} \gamma^n \|a_{1n}^k\| \tilde{K} \ldots \|a_{nn}^k\| \tilde{K} < \sum \gamma^n \beta < \alpha$$

for every $q \geqslant 1$. Hence $P_r \in \overline{R(U)}$ for every $r \geqslant 0$.

(b) In general, by Lemma 1.6 there exists a separable closed subspace F_0 of F such that $\eta(F_0) = G$. Lemma 1.5 implies that there exists a continuous linear map g from a Fréchet–Montel space $\tilde{E}$ onto F_0. Consider the commutative diagram

$$\begin{tikzcd} G' \arrow[r, hook] \arrow[d, hook] & F' \arrow[d, "r"] \\ \tilde{E}' & F_e''' \arrow[l, "g'"] \end{tikzcd}$$

where F_e''' denotes the vector space F''' equipped with the topology of uniform convergence on countable sets in F'' and r the canonical map. Since $\tilde{E}$ is separable, g' is continuous. On the other hand, since F is Fréchet, every countable bounded set in F'' is equicontinuous [13] and the map r is continuous. Hence by (a) we infer that the restriction map $O(F') \to O(G')$ is surjective.

(c) Since $O(G')$ is nuclear [3], by (b) and by the relations $O(F', P) \cong O(F')\varepsilon P$, $O(G', P) \cong O(G')\varepsilon P \cong O(G') \hat{\otimes}_\Pi P$, we infer that the restriction map $O(F', P) \to O(G', P)$ is surjective.

Now assume that F is an FSH-space. To prove the theorem in this case we need the following

1.9. Lemma [11] (Mittag-Leffer). *Let*

$$0 \to \{E_n, \beta_n^m\} \xrightarrow{\{f_n\}} \{F_n, \omega_n^m\} \xrightarrow{\{g_n\}} \{G_n, \alpha_n^m\} \to 0$$

be a complex of projective systems of Fréchet spaces. Let $k \geqslant 0$. Assume that

$$\ker f_n = 0, \quad \operatorname{Im} f_n = \operatorname{Ker} g_n \quad \text{for every } n \geqslant 1$$

and

$$(ML_1) \qquad \operatorname{Im} g_n \geqslant \operatorname{Im} \alpha^n_{n+k} \quad \text{for every } n \geqslant 1,$$

$$(ML_2) \qquad \operatorname{Im} \beta^n_{n+k+1} \quad \text{is dense in} \quad \operatorname{Im} \beta^n_{n+k} \quad \text{for every } n \geqslant 1.$$

Then the map $\varprojlim g_n$ is surjective.

Proof of Theorem 1.2 in the case where F is an FSH-space. Let $\{U_n\}$ be a decreasing basis of balanced convex neighbourhoods of zero in F such that $F_n = F(U_n)$ are Hilbert spaces and the maps $\omega^n_{n+1} = \omega(U_{n+1}, U_n)$ are compact. Let $G_n = G(\eta U_n)$ and $\alpha^m_n = \omega(\eta U_n, \eta U_m)$. By η_n we denote the map induced by η from F_n onto G_n. Consider the complex of projective systems of Fréchet spaces

$$(1.9) \qquad 0 \to \{\operatorname{Ker} \hat{\eta}'_n\} \to \{O_b(F'_n, P)\} \xrightarrow{\{\hat{\eta}'_n\}} \{O_b(G'_n, P)\} \to 0,$$

where, for every n, by $O_b(F'_n, P)$ we denote the space of holomorphic maps from F'_n into P which are bounded on every bounded set in F'_n. This space is equipped with the topology of uniform convergence on bounded sets in F'_n. Since $\omega^{n'}_{n+1}$ and $\alpha^{n'}_{n+1}$ are compact, we have

$$O(F', P) = \varprojlim O_b(F'_n, P) \quad \text{and} \quad O(G', P) = \varprojlim O_b(G'_n, P).$$

Thus by Lemma 1.9 it remains to check that (1.9) satisfies (ML_1) and (ML_2).

(ML_1) is trivial since G'_n is complemented in F'_n.

(ML_2) Without loss of generality we may assume that P is Banach.

Take $\sigma \in O_b(F'_n, P)$, $\sigma | G'_n = 0$, suppose $K \subset F'_{n-1}$ is bounded and $\varepsilon > 0$. Let $d\colon F'_n \to G'_n$ be the orthogonal projection. Define the isomorphism $\theta\colon F'_n \to G'_n \times E'_n$, where $E_n = E(U_n \cap E)$, by $\theta u = (du, e'_n u)$. Then $\sigma\theta^{-1} \in O(G'_n, O_0(E'_n, P)) \cong O(G'_n)\varepsilon O_0(E'_n)\varepsilon P$, where $O_0(E'_n) = \{\beta \in O(E'_n)\colon \beta(0) = 0\}$. Hence we find $\beta = \sum_{j=1}^{m} \beta_j \sigma_j b_j$, $\beta_j \in P[G'_n]$, $\sigma_j \in P_0[E'_n]$, $b_j \in P$, such that

$$\|\sigma\theta^{-1} - \beta\| \tilde{K} < \varepsilon, \quad \text{where } \tilde{K} = \theta\omega^{n-1'}_n(K).$$

Since G'_n and E'_n are Hilbert spaces, we may assume that β_j and σ_j are polynomials in $u_1, \ldots, u_k \in G_n$ and $v_1, \ldots, v_k \in E_n$, respectively. Then by the relations

$$\beta\theta(u) = \sum_{j=1}^{m} \beta_j\big(u_1(du), \ldots, u_k(du)\big)\sigma_j\big(v_1(e'_n(u)), \ldots, v_k(e'_n(u))b_j\big)$$

holding for all $u \in F'_n$, $F''_n = F_n$, $E''_n = E_n$ and

$$\overline{\operatorname{Im}[F_{n+1} \to F_n]} = F_n, \quad \overline{\operatorname{Im}[E_{n+1} \to E_n]} = E_n$$

we find $\tilde{u}_1, \ldots, \tilde{u}_k \in F_{n+1}$ and $\tilde{v}_1, \ldots, \tilde{v}_k \in E_{n+1}$ such that

$$\sup_{u \in K} \Big| \sum_{j=1}^{m} \beta_j(\tilde{u}_1(u), \ldots, \tilde{u}_k(u)) \sigma_j(\tilde{v}_1(e'_{n+1} u), \ldots, \tilde{v}_k(e'_{n+1} u)) b_j - \beta\theta u \Big| < \varepsilon.$$

This implies that (ML_2) holds with $k = 1$. The theorem is proved.

2. The extension of holomorphic functions on Fréchet spaces. Let L be a locally convex space. Then L is called *s-nuclear* [12] if for each $U \in \mathscr{U}(L)$ there exists $V \in \mathscr{U}(L)$ such that $V \subset U$ and the map $\omega(V, U)$ is s-nuclear, i.e., can represented in the form

$$\omega(V, U) u = \sum_{j=1}^{\infty} \lambda_j u'_j(u) v_j$$

with

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots > 0,$$

$$\sum_{j=1}^{\infty} \lambda_j p < \infty \qquad \text{for every } p > 0$$

and

$$\sup \{\|u'_j\| + \|v_j\|\} < \infty.$$

In this section we prove the following

2.1. THEOREM. *Let X be a Stein space containing a subvariety W such that $O(W) \cong O(C^n)$ for some n. Then there exist an s-nuclear Fréchet space F containing $O(X)$ as a subspace and a holomorphic function on $O(X)$ which cannot be extended to a holomorphic function on a neighbourhood of $O(X)$ in F.*

The proof of the theorem is based on the following

2.2 LEMMA [9]. *Let X be a paracompact analytic space and $\mathscr{S}$ a coherent analytic sheaf on X. Then the space $H^0(X, \mathscr{S})$ is s-nuclear.*

2.3. LEMMA [9]. *Given an s-nuclear closed subspace E of a Fréchet space F, there exist an s-nuclear Fréchet space $\tilde{E}$ and continuous linear maps $h\colon F \to \tilde{E}$ and $\tilde{e}\colon E \to \tilde{E}$ such that*

$$h|E = \tilde{e} \text{ and } \tilde{e} \text{ is an embedding.}$$

2.4. LEMMA. *Let E be a nuclear Fréchet space and $f \in O(E)$. Then f can be extended to a holomorphic function on some neighbourhood of E in every Fréchet space F containing E as a subspace if and only if f can be written in* *the form* $f = g\Pi(U)$ *for some* $U \in \mathscr{U}(E)$, *where* $\Pi(U)$ *is the canonical map from* E *into* $E(U)$ *and* $g \in O(E/\mathscr{G}_U)$.

Proof. To prove the necessity we consider the canonical embedding $\theta\colon E \to F = \prod_{j=1}^{\infty} E(U_j)$, where $\{U_j\}$ is a decreasing basis of balanced convex neighbourhoods of zero in E. By hypothesis f is extended to a holomorphic function $\tilde{f}$ on some connected neighbourhood Ω on E in F. For each $u \in \Omega$, put

$$n(u) = \min\{n\colon \text{there exists a neighbourhood } \Omega_u = \tilde{U} \times \prod_{j>n} E(U_j) \text{ of } u \text{ in } \Omega$$
$$\text{such that } \tilde{f}(u', v'') = \tilde{f}(u', v'') \text{ for all } (u', u''),\ (u', v'') \in \Omega_u\}.$$

Then $n(u)$ is locally constant on Ω. Hence by the connectedness of Ω we get $n(u) \equiv n_0$.

Take $n_1 \geqslant n_0$ such that f is bounded on U_{n_1}. Considering the Taylor expansion of f at zero we infer that f can be written in the form $f = g\Pi(U_{n_1})$, where g is G-holomorphic on $E/\mathscr{G}_{U_{n_1}}$. It remains to show that g is holomorphic on $E/\mathscr{G}_{U_{n_1}}$.

Let $u \in E$. Take a neighbourhood $\Omega_u = \prod_{j \leqslant n_1} D_j \times \prod_{j > n_1} E(U_j)$ of u in Ω such that $\tilde{f}(u', u'') = \tilde{f}(u', v'')$ for all (u', u''), $(u', v'') \in \Omega_u$, where D_j are neighbourhoods of $\Pi(U_j)u$ in $E(U_j)$ such that $\omega(U_{j+1}, U_j)D_{j+1} \leqslant D_j$. Then $g(u') = \tilde{f}(\omega(U_{n_1}, U_1)u', \ldots, u', 0)$ for all $u' \in D_{n_1} \cap E/\mathscr{G}_{U_{n_1}}$. We infer that g is holomorphic on $D_{n_1} \cap E/\mathscr{G}_{U_{n_1}}$ and hence g is holomorphic on $E/\mathscr{G}_{U_{n_1}}$.

Conversely, take $\tilde{V} \in \mathscr{U}(F)$ such that $V = \tilde{V} \cap E \subset U$ and $\omega(V, U)$ is nuclear. Let $\tilde{\omega}\colon F(\tilde{V}) \to E(U \cap E)$ be a continuous linear extension of $\omega(V, U)$ and $\tilde{\Omega} = [\tilde{\omega}\Pi(\tilde{V})]^{-1}(\Omega)$, where Ω is a neighbourhood of $E/\mathscr{G}_U$ such that f is extended to a holomorphic function on Ω. Then $\tilde{\Omega}$ is a neighbourhood of E in F and $g\tilde{\omega}|\tilde{\Omega}$ is a holomorphic extension of f. The lemma is proved.

2.5. LEMMA. *Let X be a Stein space satisfying the following condition*:

(*LE*) *every holomorphic function on $O(X)$ can be extended to a holomorphic function on some neighbourhood of $O(X)$ in every Fréchet space F containing $O(X)$ as a subspace.*

Then every irreducible subvariety W of X satisfies also condition (*LE*).

Proof. Let $f \in O(O(W))$ and $g = fR$, where $R\colon O(X) \to O(W)$ is the restriction map. By Lemma 2.4 there exist a relatively compact holomorphically convex domain Ω in X and $g_1 \in O(U)$, where U is a neighbourhood of $\operatorname{Im} R(X, \Omega)$ in $O(\Omega)$, such that

$$g(\sigma) = g_1(\sigma|\Omega) \quad \text{for all } \sigma \in O(X)$$

and $\Omega \cap W \neq \emptyset$.

Let $\sigma_1, \sigma_2 \in O(X)$ and $\sigma_1|_{\Omega\cap W} = \sigma_2|_{\Omega\cap W}$. Then $\sigma_1|_W = \sigma_2|_W$, since $\Omega \cap W \neq \emptyset$ and W is irreducible. Hence

$$g_1(\sigma_1|\Omega) = g(\sigma_1) = f(\sigma_1|_W) = f(\sigma_2|_W) = g(\sigma_2) = g_1(\sigma_2|\Omega).$$

Now assume that $\sigma_1, \sigma_2 \in U$, $\sigma_1|_{\Omega\cap W} = \sigma_2|_{\Omega\cap W}$. Consider the coherent analytic sheaf $\mathscr{S}$ on X defined by

$$\mathscr{S}_z = \{(\beta_{1z}, \beta_{2z}) \in O_z \oplus O_z \colon \beta_{1z} - \beta_{2z} \in J_z\},$$

where J denotes the ideal subsheaf of O associated with W. Note that $(\sigma_1, \sigma_2) \in H^0(\Omega, \mathscr{S})$. Since Ω is holomorphically convex, there exists a sequence $\{(\sigma_1^n, \sigma_2^n)\} \in H^0(X, \mathscr{S}) \oplus H^0(X, \mathscr{S})$ converging to (σ_1, σ_2) on Ω [7]. Hence, by the relation $\sigma_1^n|_{\Omega\cap W} = \sigma_2^n|_{\Omega\cap W}$, we have

$$(2.1) \qquad g_1(\sigma_1) = \lim g_1(\sigma_1^n|\Omega) = \lim g_1(\sigma_2^n|\Omega) = g_1(\sigma_2).$$

By (2.1) and by the openness of the restriction map $R\colon O(\Omega) \to O(\Omega \cap W)$ it follows that f is written in the form $f = hR(W, \Omega \cap W)$, where h is a holomorphic function on some neighbourhood of $\operatorname{Im} R(W, \Omega \cap W)$ in $O(\Omega \cap W)$. The lemma is proved.

Proof of Theorem 2.1. Contradicting the assertion we infer by Lemmas 2.2, 2.3 that $O(X)$ satisfies (LE). Since $O(W) \cong O(C^n)$, W is irreducible. Hence by Lemma 2.5 $O(C^n)$ satisfies (LE). We infer that $O(C)$ satisfies (LE). Since $C \times O(C) \cong O(C)$, it follows that $C \times O(C)$ satisfies (LE). Consider the holomorphic function Ev on $C \times O(C)$ given by

$$Ev(z, \sigma) = \sigma(z) \quad \text{for all } (z, \sigma) \in C \times O(C).$$

By Lemma 2.4 there exist $U \in \mathscr{U}(C)$, $V \in \mathscr{U}(O(C))$ and a holomorphic function g on $C \times O(C)/\mathscr{G}_{U\times V} = C \times O(C)/\mathscr{G}_V$ such that $g(z, \sigma) = \sigma(z)$ for all $(z, \sigma) \in C \times O(C)/\mathscr{G}_V$. Let $r > 0$ be such that $\sigma_n \to 0$ in $O(C)/\mathscr{G}_V$, where $\sigma_n = (rz)^n$. Then for $z_0 \in C$, $|rz_0| > 1$, we get

$$0 = \lim_n |g(z_0, \sigma_n)| = \lim_n |\sigma_n(z_0)| = \lim_n |rz_0|^n \to \infty.$$

This contradiction completes the proof.

In [4] Djakov and Mitjagin have proved that for every irreducible algebraic subset V of C^n the space $O(V)$ is isomorphic to $O(C^r)$, where $r = \dim V$. Hence the following is an immediate consequence of Theorem 2.1.

2.6. COROLLARY. *If X is a Stein space containing a subvariety, which is holomorphically isomorphic to an algebraic subset of C^n, then there exist an s-nuclear Fréchet space F containing $O(X)$ as a subspace and a holomorphic function on $O(X)$ which cannot be extended to a holomorphic function on a neighbourhood of $O(X)$ in F.*

Let X be an analytic space and let $R(X)$ denote the regular part of X. It is known that $O(X)$ is contained in $C^\infty(R(X))$ as a subspace. From the proof of Theorem 2.1 we get the following

2.7. COROLLARY. *Let X be a Stein space as in Theorem 2.1. Then the restriction map $O(C^\infty(R(X))) \to O(O(X))$ is not surjective.*

Proof. By the proof of Theorem 2.1, it suffices to show that for every holomorphic function f on $C^\infty(R(X))$ there exists a compact subset K of $R(X)$ and holomorphic function g on $O(X)/\mathscr{G}_K$ such that $f = g\Pi$, where $\mathscr{G}_K$ denotes the seminorm generated by K and Π: $O(X) \to O(X)/\mathscr{G}_K$ is the canonical map. Let $f \in O(C^\infty(R(X)))$. From the Taylor expansion of f at zero we see that f can be written in the form $f = gR(R(X), \Omega)$, where Ω is a relatively compact open subset of $R(X)$ and g is a C-holomorphic function on $\operatorname{Im} R(R(X), \Omega)$. Let Ω_1 and Ω_2 be relatively compact open subsets of $R(X)$ such that $\bar{\Omega} \subset \Omega_1 \subset \Omega_1 \subset \Omega_2$ and $\varphi \in C^\infty(R(X))$, $\varphi|\Omega = 1$, $\operatorname{supp} \varphi \subset \Omega_1$. By $\hat{\varphi}$ we denote the continuous linear map from $C^\infty(\Omega_1) \to C^\infty(R(X))$ defined by multiplication by φ. Then we have

$$f(\hat{\varphi}\sigma) = g(\hat{\varphi}\sigma|\Omega) \quad \text{for all } \sigma \in C^\infty(\Omega_1).$$

Combining this with the relation $\operatorname{Im} R(\Omega_1, \Omega) = \operatorname{Im} R(R(X), \Omega)$ we conclude that the map $gR(\Omega_1, \Omega)$: $C^\infty(\Omega_1) \to C$ is holomorphic. Let e: $O(X)/\mathscr{G}\bar{\Omega}_2 \to C^\infty(\Omega_1)$ denote the canonical map. Then $gR(\Omega_1, \Omega)e$: $O(X)/\mathscr{G}\bar{\Omega}_2 \to C$ is holomorphic and

$$f(\sigma) = gR(\Omega_1, \Omega)e(\sigma|\Omega_2) \quad \text{for all } \sigma \in O(X).$$

The corollary is proved.

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