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KRYSTYNA ZIĘTAK (Wrocław)

ON THE OPTIMUM RATIONAL FUNCTION CONNECTED WITH THE ADI-METHOD

1. Introduction. The following problem has been posed (see e.g. Todd [1]) in connection with the alternating-direction implicit iterative method (ADI-method). For given real k $(0 < k < 1)$ and natural m find parameters $r_1, r_2, \ldots, r_m$ such that

$$L = \max_{k \leqslant x \leqslant 1} \left| \prod_{j=1}^{m} \frac{x-r_j}{x+r_j} \right| \tag{1}$$

is minimum.

Todd [1] has given formulae which permit a determination of these parameters. However, those formulae contain elliptic functions, and their application is cumbersome in practice. We shall give in section 3 of this paper a system of differential equations allowing the numerical determination of the parameters $r_1, r_2, \ldots, r_m$ and of some auxiliary parameters $u_1, u_2, \ldots, u_{m-1}$. In section 2 some properties of these parameters and of the function

$$\prod_{j=1}^{m} \frac{x-r_j}{x+r_j}$$

will be proved. The limits of the derivatives $r'_1, r'_2, \ldots, r'_m$ and $u'_1, u'_2, \ldots, u'_{m-1}$ for $k \to 1$ will be dealt with in section 4. Since a direct numerical calculation of the parameters $r_1, r_2, \ldots, r_m$ and $u_1, u_2, \ldots, u_{m-1}$ from the equations of section 3 would induce great errors caused by the form of these equations (products of squares of nearly equal quantities), we shall present in section 5 differential equations of the unknowns $R_1, R_2, \ldots, R_m$ and $U_1, U_2, \ldots, U_{m-1}$ by means of which the parameters $r_1, r_2, \ldots, r_m$ and $u_1, u_2, \ldots, u_{m-1}$ may easily be expressed. In the final section 6 we shall determine the limits of $R'_1, R'_2, \ldots, R'_m$ and $U'_1, U'_2, \ldots, U'_{m-1}$ for $k \to 1$.

2. Properties of r_p, u_p and of the optimal function. The optimal parameters $r_1, r_2, \ldots, r_m$ may be treated as a function of L. It is also convenient to treat the parameter k as a function of L. Thus,

$$k = k(L), \quad r_p = r_p(L) \quad (p = 1, 2, \ldots, m).$$

Let

$$f(x; L) = \prod_{j=1}^{m} \frac{x-r_j}{x+r_j}$$

be the optimal function satisfying the conditions of the problem mentioned in section 1. It is known that the function $f(x; L)$ assumes in the interval $\langle k, 1\rangle$ alternatively the extremum values $\pm L$ in $m+1$ points

$$u_0 = k, \quad u_p = u_p(L) \quad (p = 1, 2, \ldots, m-1), \quad u_m = 1 \tag{2}$$

(where $u_0 < u_1 < \ldots < u_m$):

$$f(u_p; L) = (-1)^{m-p} L \quad (p = 0, 1, \ldots, m) \tag{3}$$

(see e.g. Todd [1]).

THEOREM 1. *The parameters $r_p (p = 1, 2, \ldots, m)$, u_p $(p = 1, 2, \ldots, m-1)$ satisfy the following identities*:

$$\sum_{j=1}^{m} \frac{r_j}{u_p^2 - r_j^2} = 0 \quad (p = 1, 2, \ldots, m-1). \tag{4}$$

Proof. The points $u_1, u_2, \ldots, u_{m-1}$ are zeros of the partial derivative of $f(x; L)$ with respect to x. Thus,

$$\frac{\partial}{\partial x} f(x; L)|_{x=u_p} = 0 \quad (p = 1, 2, \ldots, m-1). \tag{5}$$

Calculate now this partial derivative

$$\frac{\partial}{\partial x} f(x; L) = f(x; L) \sum_{j=1}^{m} \left(\frac{1}{x-r_j} - \frac{1}{x+r_j}\right) = f(x; L) \sum_{j=1}^{m} \frac{2r_j}{x^2 - r_j^2}. \tag{6}$$

Formula (4) follows from (5) and (6).

3. The system of differential equations with unknowns r_p and u_p.

THEOREM 2. *The functions $r_p = r_p(L)$ $(p = 1, 2, \ldots, m)$, $u_p = u_p(L)$ $(p = 1, 2, \ldots, m-1)$, $k = k(L)$ are the solutions of the following system* *of differential equations*:

(7) $$\frac{dr_p}{dL} = \frac{A_p}{2L} \quad (p = 1, 2, \ldots, m),$$

(8) $$\frac{du_p}{dL} = \frac{C_p}{2L} \quad (p = 1, 2, \ldots, m-1),$$

(9) $$\frac{dk}{dL} = \frac{1}{2LB},$$

where

$$A_p = (-1)^m \frac{\prod_{j=1}^{m}(u_j^2 - r_p^2)}{\prod_{j=1, j\neq p}^{m}(r_j^2 - r_p^2)} \sum_{l=1}^{m} \frac{1}{u_l} \cdot \frac{\prod_{j=1, j\neq p}^{m}(u_l^2 - r_j^2)}{\prod_{j=1, j\neq l}^{m}(u_j^2 - u_l^2)} \quad (p = 1, 2, \ldots, m),$$

$$C_p = \frac{1}{2u_p} \cdot \frac{\sum_{j=1}^{m} \frac{u_p^2 + r_j^2}{(u_p^2 - r_j^2)^2} A_j}{\sum_{j=1}^{m} \frac{r_j}{(u_p^2 - r_j^2)^2}} \quad (p = 1, 2, \ldots, m-1),$$

$$B = \frac{\sum_{j=1}^{m} \frac{r_j}{k^2 - r_j^2}}{1 + k \sum_{j=1}^{m} \frac{A_j}{k^2 - r_j^2}}.$$

Proof. Let us differentiate (4) with respect to L:

$$\sum_{j=1}^{m} \frac{(u_p^2 - r_j^2)\frac{dr_j}{dL} - r_j\left(2u_p \frac{du_p}{dL} - 2r_j \frac{dr_j}{dL}\right)}{(u_p^2 - r_j^2)^2} = 0 \quad (p = 1, 2, \ldots, m-1).$$

From this the derivatives du_p/dL may be determined as

(10) $$\frac{du_p}{dL} = \frac{1}{2u_p} \cdot \frac{\sum_{j=1}^{m} \frac{u_p^2 + r_j^2}{(u_p^2 - r_j^2)^2} \cdot \frac{dr_j}{dL}}{\sum_{j=1}^{m} \frac{r_j}{(u_p^2 - r_j^2)^2}} \quad (p = 1, 2, \ldots, m-1).$$

Now, differentiate identities (3) with respect to L:

(11) $$\frac{du_p}{dL} \cdot \frac{\partial}{\partial x} f(x; L)\bigg|_{x=u_p} + \frac{\partial}{\partial L} f(x; L)\bigg|_{x=u_p} = (-1)^{m-p} \quad (p = 0, 1, \ldots, m).$$

The first component of the left-hand side of (11) is equal to zero for $p = 1, 2, \ldots, m$. Hence

$$\frac{\partial}{\partial L} f(x; L)\Big|_{x=u_p} = (-1)^{m-p} \quad (p = 1, 2, \ldots, m). \tag{12}$$

For $p = 0$, equation (11) gives

$$\frac{dk}{dL} = \frac{(-1)^m - \frac{\partial}{\partial L} f(x; L)\Big|_{x=k}}{\frac{\partial}{\partial x} f(x; L)\Big|_{x=k}}. \tag{13}$$

Let us calculate the partial derivative

$$\frac{\partial}{\partial L} f(x; L)\Big|_{x=u_p} = \frac{\partial}{\partial L} \prod_{j=1}^{m} \frac{x - r_j}{x + r_j}\Big|_{x=u_p} \tag{14}$$

$$= f(u_p; L) \sum_{j=1}^{m} \left(-\frac{1}{u_p - r_j} \cdot \frac{dr_j}{dL} - \frac{1}{u_p + r_j} \cdot \frac{dr_j}{dL} \right)$$

$$= -f(u_p; L) \sum_{j=1}^{m} \frac{2u_p}{u_p^2 - r_j^2} \cdot \frac{dr_j}{dL} = -2(-1)^{m-p} u_p L \sum_{j=1}^{m} \frac{1}{u_p^2 - r_j^2} \cdot \frac{dr_j}{dL}.$$

From (6), (14), and (3), we may write (13) as follows:

$$\frac{dk}{dL} = \frac{(-1)^m + 2(-1)^m kL \sum_{j=1}^{m} \frac{1}{k^2 - r_j^2} \cdot \frac{dr_j}{dL}}{(-1)^m L \sum_{j=1}^{m} \frac{2r_j}{k^2 - r_j^2}} \tag{15}$$

$$= \frac{1 + 2kL \sum_{j=1}^{m} \frac{1}{k^2 - r_j^2} \cdot \frac{dr_j}{dL}}{2L \sum_{j=1}^{m} \frac{r_j}{k^2 - r_j^2}}.$$

Now, substitute (14) into (12). We have then

$$\sum_{j=1}^{m} \frac{1}{u_p^2 - r_j^2} \cdot \frac{dr_j}{dL} = -\frac{1}{2u_p L} \quad (p = 1, 2, \ldots, m). \tag{16}$$

This is a system of m non-homogeneous linear equations with unknowns dr_j/dL $(j = 1, 2, \ldots, m)$. They may be calculated from (16) by an application of Cramer's formulae. Denote Cauchy's determinant by the symbol

$$\operatorname{cau}(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) = \begin{vmatrix} \frac{1}{a_1-b_1} & \frac{1}{a_1-b_2} & \cdots & \frac{1}{a_1-b_n} \\ \frac{1}{a_2-b_1} & \frac{1}{a_2-b_2} & \cdots & \frac{1}{a_2-b_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a_n-b_1} & \frac{1}{a_n-b_2} & \cdots & \frac{1}{a_n-b_n} \end{vmatrix}$$

and Vandermonde's determinant by the symbol

$$\operatorname{van}(c_1, c_2, \ldots, c_n) = \begin{vmatrix} 1 & c_1 & c_1^2 & \ldots & c_1^{n-1} \\ 1 & c_2 & c_2^2 & \ldots & c_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & c_n & c_n^2 & \ldots & c_n^{n-1} \end{vmatrix}.$$

It is known that

$$\text{(17)} \quad \operatorname{cau}(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n) = (-1)^{n(n-1)/2} \frac{\operatorname{van}(a_1, a_2, \ldots, a_n)\operatorname{van}(b_1, b_2, \ldots, b_n)}{\prod_{j=1}^{n}\prod_{i=1}^{n}(a_j-b_i)}.$$

Denote by W the determinant of the system (16),

$$W = \begin{vmatrix} \frac{1}{u_1^2-r_1^2} & \frac{1}{u_1^2-r_2^2} & \cdots & \frac{1}{u_1^2-r_m^2} \\ \frac{1}{u_2^2-r_1^2} & \frac{1}{u_2^2-r_2^2} & \cdots & \frac{1}{u_2^2-r_m^2} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{u_m^2-r_1^2} & \frac{1}{u_m^2-r_2^2} & \cdots & \frac{1}{u_m^2-r_m^2} \end{vmatrix},$$

and by W_p the determinant which is obtained from W by replacing its p-th column by the column of the free terms of (16):

$$W_p = -\frac{1}{2L} \begin{vmatrix} \frac{1}{u_1^2-r_1^2} & \cdots & \frac{1}{u_1^2-r_{p-1}^2} & \frac{1}{u_1} & \frac{1}{u_1^2-r_{p+1}^2} & \cdots & \frac{1}{u_1^2-r_m^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{u_m^2-r_1^2} & \cdots & \frac{1}{u_m^2-r_{p-1}^2} & \frac{1}{u_m} & \frac{1}{u_m^2-r_{p+1}^2} & \cdots & \frac{1}{u_m^2-r_m^2} \end{vmatrix}.$$

The determinant W is a determinant of Cauchy, and from (17) we have

$$W = (-1)^{m(m-1)/2} \frac{\operatorname{van}(u_1^2, u_2^2, \ldots, u_m^2)\operatorname{van}(r_1^2, r_2^2, \ldots, r_m^2)}{\prod_{j=1}^{m}\prod_{i=1}^{m}(u_j^2-r_i^2)}.$$

The determinant W_p will be calculated by developing it with respect to the p-th column and by using (17):

$$\begin{aligned} W_p &= -\frac{1}{2L}\sum_{l=1}^{m}(-1)^{p+l}\frac{1}{u_l}\times \\ &\quad\times \operatorname{cau}(u_1^2, \ldots, u_{l-1}^2, u_{l+1}^2, \ldots, u_m^2; r_1^2, \ldots, r_{p-1}^2, r_{p+1}^2, \ldots, r_m^2) \\ &= -\frac{1}{2L}\sum_{l=1}^{m}(-1)^{p+l}(-1)^{(m-1)(m-2)/2}\frac{1}{u_l}\times \\ &\quad\times \frac{\operatorname{van}(u_1^2, \ldots, u_{l-1}^2, u_{l+1}^2, \ldots, u_m^2)\operatorname{van}(r_1^2, \ldots, r_{p-1}^2, r_{p+1}^2, \ldots, r_m^2)}{\prod_{\substack{j=1\\ j\neq l}}^{m}\prod_{\substack{i=1\\ i\neq p}}^{m}(u_j^2-r_i^2)}. \end{aligned}$$

Since

$$\operatorname{van}(c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) = \frac{\operatorname{van}(c_1, c_2, \ldots, c_n)}{(-1)^{i-1}\prod_{\substack{j=1\\ j\neq i}}^{m}(c_j-c_i)},$$

we have

$$\begin{aligned} W_p &= -\frac{1}{2L}\sum_{l=1}^{m}(-1)^{p+l+(m-1)/(m-2)/2}\frac{1}{u_l}\times \\ &\quad\times \frac{\operatorname{van}(u_1^2, \ldots, u_m^2)\operatorname{van}(r_1^2, \ldots, r_m^2)}{(-1)^{l-1}\prod_{\substack{j=1\\ j\neq l}}^{m}(u_j^2-u_l^2)(-1)^{p-1}\prod_{\substack{j=1\\ j\neq p}}^{m}(r_j^2-r_p^2)\prod_{\substack{j=1\\ j\neq l}}^{m}\prod_{\substack{i=1\\ i\neq p}}^{m}(u_j^2-r_i^2)} \end{aligned}$$

and, therefore,

$$\begin{aligned} \frac{dr_p}{dL} = \frac{W_p}{W} &= \frac{(-1)^m}{2L}\sum_{l=1}^{m}\frac{1}{u_l}\cdot\frac{\prod_{j=1}^{m}\prod_{i=1}^{m}(u_j^2-r_i^2)}{\prod_{\substack{j=1\\ j\neq l}}^{m}(u_j^2-u_l^2)\prod_{\substack{j=1\\ j\neq p}}^{m}(r_j^2-r_p^2)\prod_{\substack{j=1\\ j\neq l}}^{m}\prod_{\substack{i=1\\ i\neq p}}^{m}(u_j^2-r_i^2)} \\ &= \frac{(-1)^m}{2L}\cdot\frac{\prod_{j=1}^{m}(u_j^2-r_p^2)}{\prod_{\substack{j=1\\ j\neq p}}^{m}(r_j^2-r_p^2)}\sum_{l=1}^{m}\frac{1}{u_l}\cdot\frac{\prod_{j=1, j\neq p}^{m}(u_l^2-r_j^2)}{\prod_{\substack{j=1\\ j\neq l}}^{m}(u_j^2-u_l^2)} \qquad (p = 1, 2, \ldots, m). \end{aligned}$$

From this equations (7) follow. Substituting them into (10) and (15) gives as result (8) and (9), which completes the proof.

The expressions $A_p(p = 1, 2, \ldots, m)$, $C_p(p = 1, 2, \ldots, m-1)$, B do not contain explicite the variable L. Due to this, we may treat now r_p, u_p $(p = 1, 2, \ldots, m)$ and L as functions of k (the function symbols will not be changed), and obtain easily the following

THEOREM 3. *The parameters* r_p $(p = 1, 2, \ldots, m)$, u_p $(p = 1, 2, \ldots \ldots, m-1)$ *treated as functions of* k *satisfy the system of differential equations*

$$\frac{dr_p}{dk} = A_p B \quad (p = 1, 2, \ldots, m), \tag{18}$$

$$\frac{du_p}{dk} = C_p B \quad (p = 1, 2, \ldots, m-1) \tag{19}$$

with initial conditions

$$\begin{aligned} r_p(1) &= 1 \quad (p = 1, 2, \ldots, m), \\ u_p(1) &= 1 \quad (p = 1, 2, \ldots, m-1). \end{aligned} \tag{20}$$

Proof. From (9) we have $dL/dk = 2LB$. Thus, from (7), we obtain

$$\frac{dr_p}{dk} = \frac{dr_p}{dL} \cdot \frac{dL}{dk} = \frac{A_p}{2L} 2LB = A_p B \quad (p = 1, 2, \ldots, m).$$

Analogously, from (8) follows (19). Since r_p, u_p are in the interval $\langle k, 1\rangle$, we have

$$\begin{aligned} \lim_{k\to 1} r_p(k) &= 1 \quad (p = 1, 2, \ldots, m), \\ \lim_{k\to 1} u_p(k) &= 1 \quad (p = 1, 2, \ldots, m-1). \end{aligned}$$

4. Asymptotic properties of the optimal function. Limits of the derivations of r_p and u_p. Now, we shall find the limits

$$\lim_{k\to 1} \frac{dr_p}{dk} (p = 1, 2, \ldots, m) \quad \text{and} \quad \lim_{k\to 1} \frac{du_p}{dk} (p = 1, 2, \ldots, m-1).$$

The numerators and denominators of A_p $(p = 1, 2, \ldots, m)$ tend to zero as $k \to 1$. It is thus impossible to calculate directly the values of the right-hand sides of (18) and (19) for $k = 1$. In order to investigate the behaviour of the right-hand sides of (18) and (19) as $k \to 1$, introduce a variable t such that

$$t = \frac{x-k}{1-k} \quad \text{for } k \leqslant x \leqslant 1. \tag{21}$$

Hence $0 \leqslant t \leqslant 1$. After (21) the points r_p are replaced by

$$R_p = \frac{r_p - k}{1-k} \quad (p = 1, 2, \ldots, m), \tag{22}$$

and the points u_p by

$$U_p = \frac{u_p - k}{1-k} \quad (p = 1, 2, \ldots, m-1). \tag{23}$$

Let

$$g(t; k) = f(x; L) = \prod_{j=1}^{m} \frac{(1-k)t + k - r_j}{(1-k)t + k + r_j} = \prod_{j=1}^{m} \frac{t - R_j}{t + R_j + \dfrac{2k}{1-k}}.$$

With $k \to 1$ the poles of $g(t; k)$ tend to $-\infty$, and the value of the denominator

$$\prod_{j=1}^{m} \left(t + R_j + \frac{2k}{1-k}\right)$$

tends to $+\infty$ for $0 \leqslant t \leqslant 1$. Thus, the behaviour of the function $g(t; k)$ in the interval $\langle 0, 1\rangle$ is, for k sufficiently near to 1, determined by the numerator, i.e. by the polynomial

$$\prod_{j=1}^{m} (t - R_j),$$

because the denominator may be treated as constant in the limit. Asymptotically, $g(t; k)$ behaves therefore in the interval $\langle 0, 1\rangle$ for $k \to 1$ like a polynomial of degree m. As it attains the extremum values $\pm L$ in $m+1$ points of the interval $\langle 0, 1\rangle$, and since $g(0; k) = (-1)^m L$ and $g(1; k) = L$, the function $g(t; k)$ behaves asymptotically like the m-th Chebyshev polynomial $T_m(2t-1)$. Thus

$$\lim_{k\to 1} R_p = \frac{1}{2}\left(1 - \cos\frac{(2p-1)\pi}{2m}\right) \quad (p = 1, 2, \ldots, m), \tag{24}$$

$$\lim_{k\to 1} U_p = \frac{1}{2}\left(1 - \cos\frac{p\pi}{m}\right) \quad (p = 1, 2, \ldots, m-1). \tag{25}$$

Introduce now the auxiliary notation

$$\lim_{k\to 1} R_p = R_{1p}, \quad \lim_{k\to 1} U_p = U_{1p}, \quad \lim_{k\to 1} \frac{dR_p}{dk} = R'_{1p}, \quad \lim_{k\to 1} \frac{dU_p}{dk} = U'_{1p}.$$

THEOREM 4. *The following formulae are satisfied:*

$$\lim_{k\to 1}\frac{dr_p}{dk} = 1-R_{1p} \quad (p = 1, 2, \ldots, m), \tag{26}$$

$$\lim_{k\to 1}\frac{du_p}{dk} = 1-U_{1p} \quad (p = 1, 2, \ldots, m-1). \tag{27}$$

Proof. Formula (26) may be obtained after an application of de l'Hospital's rule to

$$R_{1p} = \lim_{k\to 1}\frac{r_p-k}{1-k} \quad (p = 1, 2, \ldots, m),$$

which follow from (22). Analogously, (27) follows from (23).

5. The system of differential equations for the functions R_p and U_p. Now we shall find, on the basis of (18), (19) and (22), (23), the differential equations which will allow a determination of R_p $(p = 1, 2, \ldots, m)$ and U_p $(p = 1, 2, \ldots, m-1)$.

THEOREM 5. *The functions R_p $(p = 1, 2, \ldots, m)$ and U_p $(p = 1, 2, \ldots, m-1)$ of the variable k form the solutions of the system of differential equations*

$$\frac{dR_p}{dk} = \frac{D_pF+1-R_p}{k-1} \quad (p = 1, 2, \ldots, m), \tag{28}$$

$$\frac{dU_p}{dk} = \frac{E_pF+1-U_p}{k-1} \quad (p = 1, 2, \ldots, m-1) \tag{29}$$

with the initial conditions

$$\begin{aligned} R_p(1) &= \frac{1}{2}\left(1-\cos\frac{(2p-1)\pi}{2m}\right) \quad (p = 1, 2, \ldots, m),\\ U_p(1) &= \frac{1}{2}\left(1-\cos\frac{p\pi}{m}\right) \quad (p = 1, 2, \ldots, m-1), \end{aligned} \tag{30}$$

where

$$\begin{aligned} D_p &= A_p/(1-k) \quad (p = 1, 2, \ldots, m),\\ E_p &= C_p/(1-k) \quad (p = 1, 2, \ldots, m-1),\\ F &= (k-1)B. \end{aligned}$$

Proof. From (22) we have

$$\frac{dR_p}{dk} = \frac{(1-k)\left(\frac{dr_p}{dk}-1\right)+(r_p-k)}{(1-k)^2} = \frac{(1-k)\left(\frac{dr_p}{dk}-1\right)+R_p(1-k)}{(1-k)^2},$$

or

$$\frac{dR_p}{dk} = \frac{A_pB+R_p-1}{1-k} = \frac{D_pF+1-R_p}{k-1} \quad (p = 1, 2, \ldots, m).$$

In an analogous manner

$$\frac{dU_p}{dk} = \frac{C_pB+U_p-1}{1-k} = \frac{E_pF+1-U_p}{k-1} \qquad (p = 1, 2, \ldots, m-1).$$

The initial conditions were obtained already earlier (formulae (24) and (25)).

Now, we shall express A_p, C_p, B by the functions R_j, U_j $(j = 1, 2, \ldots, m)$ and by the variable k, as follows:

$$A_p = (-1)^m \frac{\prod_{j=1}^{m}\left[\big((1-k)U_j+k\big)^2-\big((1-k)R_p+k\big)^2\right]}{\prod_{j=1,j\neq p}^{m}\left[\big((1-k)R_j+k\big)^2-\big((1-k)R_p+k\big)^2\right]} \times$$

$$\times \sum_{l=1}^{m} \frac{1}{(1-k)U_l+k} \cdot \frac{\prod_{j=1,j\neq p}^{m}\left[\big((1-k)U_l+k\big)^2-\big((k-1)R_j+k\big)^2\right]}{\prod_{j=1,j\neq l}^{m}\left[\big((1-k)U_j+k\big)^2-\big((1-k)U_l+k\big)^2\right]}$$

$$= (-1)^m(1-k) \frac{\prod_{j=1}^{m}(U_j-R_p)\big((1-k)(U_j+R_p)+2k\big)}{\prod_{j=1,j\neq p}^{m}(R_j-R_p)\big((1-k)(R_j+R_p)+2k\big)} \times$$

$$\times \sum_{l=1}^{m} \frac{1}{(1-k)U_l+k} \cdot \frac{\prod_{j=1,j\neq p}^{m}(U_l-R_j)\big((1-k)(U_l+R_j)+2k\big)}{\prod_{j=1,j\neq l}^{m}(U_j-U_l)\big((1-k)(U_l+U_j)+2k\big)},$$

$$C_p = \frac{1}{2\big((1-k)U_p+k\big)} \cdot \frac{\sum_{j=1}^{m} \frac{\big((1-k)U_p+k\big)^2+\big((1-k)R_j+k\big)^2}{\left[\big((1-k)U_p+k\big)^2-\big((1-k)R_j+k\big)^2\right]^2} A_j}{\sum_{j=1}^{m} \frac{(1-k)R_j+k}{\left[\big((1-k)U_p+k\big)^2-\big((1-k)R_j+k\big)^2\right]^2}}$$

$$= \frac{1-k}{2\big((1-k)U_p+k\big)} \cdot \frac{\sum_{j=1}^{m} \frac{\big((1-k)U_p+k\big)^2+\big((1-k)R_j+k\big)^2}{(U_p-R_j)^2\big((1-k)(U_p+R_j)+2k\big)^2} D_j}{\sum_{j=1}^{m} \frac{(1-k)R_j+k}{(U_p-R_j)^2\big((1-k)(U_p+R_j)+2k\big)^2}},$$

$$B = \frac{\sum_{j=1}^{m} \frac{(1-k)R_j+k}{k^2-\big((1-k)R_j+k\big)^2}}{1+k\sum_{j=1}^{m} \frac{A_j}{k^2-\big((1-k)R_j+k\big)^2}} = \frac{1}{k-1} \cdot \frac{\sum_{j=1}^{m} \frac{(1-k)R_j+k}{R_j\big((1-k)R_j+2k\big)}}{1-k\sum_{j=1}^{m} \frac{D_j}{R_j\big((1-k)R_j+2k\big)}}.$$

For $k = 1$ the right-hand sides of (28) and (29) are expressions of the type $0/0$ because

$$\lim_{k\to 1}(D_pF+1-R_p) = \lim_{k\to 1}(-A_pB+1-R_p) = \lim_{k\to 1}\left(-\frac{dr_p}{dk}+1-R_p\right).$$

Thus, from Theorem 4 we obtain

$$\lim_{k\to 1}(D_pF+1-R_p) = 0 \quad (p = 1, 2, \ldots, m), \tag{31}$$

Analogously,

$$\lim_{k\to 1}(E_pF+1-U_p) = 0 \quad (p = 1, 2, \ldots, m-1). \tag{32}$$

That is why it is impossible to determine the derivatives $R_p (p = 1, 2, \ldots, m)$ and U_p $(p = 1, 2, \ldots, m-1)$ for $k = 1$ directly from (28) and (29). We shall calculate the limits of R'_{1p}, U'_{1p}.

6. The limit values R'_{1p}, U'_{1p}.

THEOREM 6. *The limit values of U_{1p}, R_{1p} satisfy*

$$\sum_{j=1}^{m}\frac{1}{U_{1p}-R_{1j}} = 0 \quad (p = 1, 2, \ldots, m-1). \tag{33}$$

Proof. R_{1p} $(p = 1, 2, \ldots, m)$ form the zeros, and U_{1p} $(p = 1, 2, \ldots, m-1)$ form the extremum points of the m-th Chebyshev polynomial $T_m(2t-1)$. We have proved this while considering (24) and (25). Thus,

$$\frac{\partial}{\partial x}\prod_{j=1}^{m}(x-R_{1j})\Big|_{x=U_{1p}} = 0 \quad (p = 1, 2, \ldots, m-1).$$

The thesis of Theorem 6 follows immediately from the above.

THEOREM 7. *The quantities R'_{1p} $(p = 1, 2, \ldots, m)$ and U'_{1p} $(p = 1, 2, \ldots, m-1)$ form the solutions of the following system of non-homogeneous linear equations*:

$$\sum_{j=1}^{m}\frac{U'_{1p}-R'_{1j}}{(U_{1p}-R_{1j})^2} = \frac{m}{2} \quad (p = 1, 2, \ldots, m-1), \tag{34}$$

$$\sum_{j=1}^{m}\frac{R'_{1j}}{(1-R_{1j})(U_{1p}-R_{1j})} = -\frac{m}{2} \quad (p = 1, 2, \ldots, m-1), \tag{35}$$

$$\sum_{j=1}^{m}\frac{R'_{1j}}{R_{1j}(1-R_{1j})} = \frac{m}{2}. \tag{36}$$

Proof. The left-hand sides of (4) may be transformed into

$$\sum_{j=1}^{m} \frac{(1-k)R_j+k}{\big((1-k)U_p+k\big)^2-\big((1-k)R_j+k\big)^2}.$$

Hence

$$\sum_{j=1}^{m} \frac{(1-k)R_j+k}{(U_p-R_j)\big((1-k)(U_p+R_j)+2k\big)} = 0 \quad (p = 1, 2, \ldots, m-1). \tag{37}$$

Let us differentiate (37) with respect to k:

$$\sum_{j=1}^{m} \Big\{(U_p-R_j)\big((1-k)(U_p+R_j)+2k\big)\big((1-k)R_j'+1-R_j\big)-$$
$$-\big((1-k)R_j+k\big)\big[(U_p'-R_j')\big((1-k)(U_p+R_j)+2k\big)+(U_p-R_j)\times$$
$$\times\big((1-k)(U_p'+R_j')+2-U_p-R_j\big)\big]\Big\}/[(U_p-R_j)^2\times$$
$$\times\big((1-k)(U_p+R_j)+2k\big)^2] = 0.$$

For $k \to 1$ the limit is

$$\sum_{j=1}^{m} \frac{2(U_{1p}-R_{1j})(1-R_{1j})-\big(2(U_{1p}'-R_{1j}')+(U_{1p}-R_{1j})(2-U_{1p}-R_{1j})\big)}{(U_{1p}-R_{1j})^2}$$
$$= 0 \quad (p = 1, 2, \ldots, m-1). \tag{38}$$

Among others, we have used here the fact that

$$\begin{aligned} &\lim_{k\to 1}(1-k)R_p' = 0 \quad (p = 1, 2, \ldots, m), \\ &\lim_{k\to 1}(1-k)U_p' = 0 \quad (p = 1, 2, \ldots, m-1). \end{aligned} \tag{39}$$

This is a consequence of (31) and (32). Simplifying (38), we obtain

$$\sum_{j=1}^{m} \frac{-2(U_{1p}'-R_{1j}')+(U_{1p}-R_{1j})^2}{(U_{1p}-R_{1j})^2}$$
$$= m-2\sum_{j=1}^{m} \frac{U_{1p}'-R_{1j}'}{(U_{1p}-R_{1j})^2} \quad (p = 1, 2, \ldots, m-1),$$

and from this (34) follows.

To obtain (35) and (36) a transformation of (3) is necessary:

$$f(u_p; L) = \prod_{j=1}^{m} \frac{u_p-r_j}{u_p+r_j} = \prod_{j=1}^{m} \frac{(1-k)U_p+k-(1-k)R_j-k}{(1-k)U_p+k+(1-k)R_j+k}$$
$$= (1-k)^m \prod_{j=1}^{m} \frac{U_p-R_j}{(1-k)(U_p+R_j)+2k},$$

and, in particular,

$$L = f(u_m; L) = (1-k)^m \prod_{j=1}^{m} \frac{1-R_j}{(1-k)R_j+1+k}.$$

Thus

$$\prod_{j=1}^{m} \frac{U_p-R_j}{(1-k)(U_p+R_j)+2k} = (-1)^{m-p} \prod_{j=1}^{m} \frac{1-R_j}{(1-k)R_j+1+k} \quad (p = 0, 1, \ldots, m-1).$$

We calculate now the logarithmic derivative of this equality:

$$\sum_{j=1}^{m} \frac{U'_p-R'_j}{U_p-R_j} - \sum_{j=1}^{m} \frac{(1-k)(U'_p+R'_j)+2-U_p-R_j}{(1-k)(U_p+R_j)+2k} = -\sum_{j=1}^{m} \frac{R'_j}{1-R_j} - \sum_{j=1}^{m} \frac{(1-k)R'_j+1-R_j}{(1-k)R_j+1+k}.$$

Using (39), we obtain in the limit for $k \to 1$

$$\sum_{j=1}^{m} \frac{U'_{1p}-R'_{1j}}{U_{1p}-R_{1j}} - \sum_{j=1}^{m} \frac{2-U_{1p}-R_{1j}}{2} = -\sum_{j=1}^{m} \frac{R'_{1j}}{1-R_{1j}} - \sum_{j=1}^{m} \frac{1-R_{1j}}{2} \quad (p = 0, 1, \ldots, m-1).$$

Hence

$$(40) \qquad U'_{1p} \sum_{j=1}^{m} \frac{1}{U_{1p}-R_{1j}} + \sum_{j=1}^{m} \left(\frac{1}{1-R_{1j}} - \frac{1}{U_{1p}-R_{1j}} \right) R'_{1j} = \frac{m}{2}(1-U_{1p}) \quad (p = 0, 1, \ldots, m-1).$$

Theorem 6 asserts that for $p = 1, 2, \ldots, m-1$ the first sum of the left-hand side of (40) is equal to zero and that

$$(41) \qquad \sum_{j=1}^{m} \frac{(U_{1p}-1)R'_{1j}}{(1-R_{1j})(U_{1p}-R_{1j})} = \frac{m}{2}(1-U_{1p}).$$

In this way we have obtained (35). For $p = 0$ holds $U_0 = 0$, $U'_0 = 0$, thus (36) follows from (41).

THEOREM 8. *The solutions of the system* (34), (35), (36) *are expressed by the following formulae*:

$$(42)\qquad R'_{1p} = \frac{1}{2}R_{1p}(1-R_{1p}) = \frac{1}{8}\sin^2\frac{(2p-1)\pi}{2m} \qquad (p = 1, 2, \ldots, m),$$

$$(43)\qquad U'_{1p} = \frac{1}{2}U_{1p}(1-U_{1p}) = \frac{1}{8}\sin^2\frac{p\pi}{m} \qquad (p = 1, 2, \ldots, m-1).$$

Proof. Let us sum equations (36) and (35) side by side:

$$\sum_{j=1}^{m} \frac{R_{1j}+U_{1p}-R_{1j}}{R_{1j}(1-R_{1j})(U_{1p}-R_{1j})} R'_{1j} = 0.$$

We have thus

$$(44)\qquad \sum_{j=1}^{m} \frac{R'_{1j}}{R_{1j}(1-R_{1j})(U_{1p}-R_{1j})} = 0 \qquad (p = 1, 2, \ldots, m-1).$$

Let be

$$(45)\qquad \varrho_j = \frac{R'_{1j}}{R_{1j}(1-R_{1j})}.$$

The system (36), (44) has the form

$$\sum_{j=1}^{m} \varrho_j = \frac{m}{2},$$

$$\sum_{j=1}^{m} \frac{\varrho_j}{U_{1p}-R_{1j}} = 0 \qquad (p = 1, 2, \ldots, m-1).$$

A comparison of this and of (33) shows that the system will be satisfied provided

$$(46)\qquad \varrho_j = \tfrac{1}{2} \qquad (j = 1, 2, \ldots, m).$$

This is a unique solution. Really, the system has the determinant

$$W = \begin{vmatrix} 1 & 1 & & 1 \\ \dfrac{1}{U_{11}-R_{11}} & \dfrac{1}{U_{11}-R_{12}} & & \dfrac{1}{U_{11}-R_{1m}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{U_{1,m-1}-R_{11}} & \dfrac{1}{U_{1,m-1}-R_{12}} & \cdots & \dfrac{1}{U_{1,m-1}-R_{1m}} \end{vmatrix}.$$

Addition of all other columns to the first one and use of Theorem 6 results in

$$W = m\,\mathrm{cau}(U_{11}, U_{12}, \ldots, U_{1,m-1}; R_{12}, R_{13}, \ldots, R_{1m}) \neq 0.$$

From (45) and (46) we obtain immediately (42), and equation (42) may be used to transform (34) as follows:

$$(47)\qquad U'_{1p}\sum_{j=1}^{m}\frac{1}{(U_{1p}-R_{1j})^2} = \frac{m}{2}+\frac{1}{2}\sum_{j=1}^{m}\frac{R_{1j}(1-R_{1j})}{(U_{1p}-R_{1j})^2}$$

$$(p = 1, 2, \ldots, m-1).$$

The right-hand sides of (47) may be transformed by the use of Theorem 6, as follows:

$$\frac{1}{2}\sum_{j=1}^{m}\frac{(U_{1p}-R_{1j})^2+R_{1j}(1-R_{1j})}{(U_{1p}-R_{1j})^2} = \frac{1}{2}\sum_{j=1}^{m}\frac{U_{1p}^2-2U_{1p}R_{1j}+R_{1j}}{(U_{1p}-R_{1j})^2}$$

$$= \frac{1}{2}\sum_{j=1}^{m}\frac{(2U_{1p}-1)(U_{1p}-R_{1j})+U_{1p}(1-U_{1p})}{(U_{1p}-R_{1j})^2}$$

$$= \frac{1}{2}U_{1p}(1-U_{1p})\sum_{j=1}^{m}\frac{1}{(U_{1p}-R_{1j})^2}.$$

Thus, from (47) formulae (43) immediately follow.

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KRYSTYNA ZIĘTAK (Wrocław)

O OPTYMALNEJ FUNKCJI WYMIERNEJ ZWIĄZANEJ Z METODĄ ADI

STRESZCZENIE

W pracy zajmujemy się następującym zadaniem, sformułowanym m. in. w [1]:

Dla danych liczb k $(0 < k < 1)$ i naturalnego m znaleźć takie parametry rzeczywiste $r_1, r_2, \ldots, r_m$, aby liczba (1) była najmniejsza.

Dowodzi się, że parametry $r_1, r_2, \ldots, r_m$ i pomocnicze wielkości $u_1, u_2, \ldots, u_{m-1}$ (zob. (2)), które spełniają związki (3), są rozwiązaniami układu równań różniczkowych (18) i (19) z warunkami początkowymi (20) dla $k = 1$. Aby móc wyznaczyć granice pochodnych tych wielkości przy $k \to 1$, wprowadzono nową zmienną niezależną t związaną ze zmienną x wzorem (21). Po tym przekształceniu wielkościom $r_1, r_2, \ldots, r_m$ i $u_1, u_2, \ldots, u_{m-1}$ odpowiadają pomocnicze wielkości $R_1, R_2, \ldots, R_m$ i $U_1, U_2, \ldots, U_{m-1}$ określone wzorami (22) i (23). Granice pochodnych wielkości $r_1, r_2, \ldots, r_m$, $u_1, u_2, \ldots, u_{m-1}$ przy $k \to 1$ są określone wzorami (26) i (27) (twierdzenie 4). Ze względu na postać prawych stron równań (18) i (19) (iloczyny różnic kwadratów wielkości bardzo bliskich siebie), numeryczne rozwiązanie tego układu równań obarczone byłoby dużymi błędami. Dlatego podano układ równań różniczkowych (28) i (29) z warunkami początkowymi (30), którego rozwiązaniem są wielkości $R_1, R_2, \ldots, R_m$, $U_1, U_2, \ldots, U_{m-1}$. Granice pochodnych tych wielkości są określone w twierdzeniu 8 wzorami (42) i (43).

КРЫСТЫНА ЗЕНТАК (Вроцлав)

ОБ ОПТИМАЛЬНОЙ РАЦИОНАЛЬНОЙ ФУНКЦИИ СВЯЗАННОЙ С МЕТОДОМ АДИ

РЕЗЮМЕ

В работе рассмотрена следующая задача, сформулированная например в [1]:

Для данного $k (0 < k < 1)$ и натурального числа m найти вещественные параметры $r_1, r_2, \ldots, r_m$ такие, чтобы число (1) было минимальным.

Доказано, что параметры $r_1, r_2, \ldots, r_m$ и воспомогательные величины u_1, $u_2, \ldots, u_{m-1}$ (см. (2)), которые удовлетворяют условиям (3), выполняют систему дифференциальных уравнений (18) и (19) с начальными условиями (20) при $k = 1$. Для определения пределов производных этих величин при $k \to 1$, вводится новая независимая переменная t, связанна с x формулой (21). После этого преобразования величинам $r_1, r_2, \ldots, \ldots, r_m$ и $u_1, u_2, \ldots, u_{m-1}$ соответствуют воспомогательные величины $R_1, R_2, \ldots, R_m$ и $U_1, U_2, \ldots, U_{m-1}$, определенные формулами (22) и (23). Пределы производных величин $r_1, r_2, \ldots, r_m, u_1, u_2, \ldots, u_{m-1}$ при $k \to 1$ определяются формулами (26) и (27) (теорема 4). Учитывая вид правых частей уравнений (18) и (19) (произведения разностей квадратов очень близких величин), можно предполагать, что точность численного решения этой системы уравнений будет малой. Поэтому приведена система дифференциальных уравнений (28) и (29) с начальными условиями (30), решением которой являются величины $R_1, R_2, \ldots, R_m$, $U_1, U_2, \ldots, U_{m-1}$. Пределы производных этих величин определяются формулами (42) и (43) из теоремы 8.
