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AN INTEGRAL TRANSFORMATION FOR CHARACTERISTIC FUNCTIONS

1. Introduction. Let Y be a continuous random variable with distribution function $F(y)$ and characteristic function $\varphi(u)$. The integral

$$(1.1) \quad \gamma(u) = \frac{\alpha}{u^\alpha} \int_0^u \varphi(\omega) \omega^{\alpha-1} d\omega, \quad \alpha > 0,$$

defines the characteristic function of the distribution function $G(x)$ corresponding to the random variable $X = U^{1/\alpha} Y$, where U is a random variable uniformly distributed on $[0, 1]$ and independent of Y . According to Olsen and Savage [6] the distribution function $G(x)$ is α -unimodal.

The above integral represents a transformation for characteristic functions, which converts the characteristic function $\varphi(u)$ of an arbitrary distribution function $F(y)$ into the characteristic function $\gamma(u)$ of an α -unimodal distribution function $G(x)$. The purpose of the present paper is to establish certain properties of this integral transformation and to give applications of the resulting α -unimodal distributions.

2. The results. The first theorem investigates the mixtures of α -unimodal distributions.

THEOREM 1. *Let $\gamma(u)$ be the characteristic function of an α -unimodal distribution function $G(x)$. Then*

(i) *$\gamma(u)$ can be represented in the form*

$$(2.1) \quad \gamma(u) = p\theta(u) + q\psi(u),$$

where $\theta(u)$ and $\psi(u)$ are characteristic functions of β -unimodal distributions with $\beta > \alpha > 0$ and $p > 0$, $q > 0$, $p+q=1$;

(ii) *the function $\kappa(u)$ defined by*

$$(2.2) \quad \kappa(u) = p\gamma(u) + q \frac{\beta}{u^\beta} \int_0^u \gamma(\omega) \omega^{\beta-1} d\omega$$

is the characteristic function of a β -unimodal distribution function, where $\alpha > \beta > 0$ and $p > 0$, $q > 0$, $p+q=1$.

Proof. (i) The α -unimodality of the distribution function $G(x)$ implies that

$$(2.3) \quad \gamma(u) = \frac{\alpha}{u^\alpha} \int_0^u \varphi(\omega) \omega^{\alpha-1} d\omega,$$

where $\varphi(u)$ is a characteristic function. For $\beta > \alpha > 0$ consider the characteristic functions

$$(2.4) \quad \theta(u) = \frac{\beta}{u^\beta} \int_0^u \varphi(\omega) \omega^{\beta-1} d\omega$$

and

$$(2.5) \quad \psi(u) = \frac{\beta}{u^\beta} \int_0^u \gamma(\omega) \omega^{\beta-1} d\omega$$

which belong to the class of β -unimodal distribution functions. Using (2.3) in (2.5) and integrating by parts, it is readily shown that

$$\gamma(u) = p\theta(u) + q\psi(u),$$

where $p = \alpha/\beta$ and $q = (\beta - \alpha)/\beta$. Hence $\gamma(u)$, the characteristic function of an α -unimodal distribution, is the mixture of two characteristic functions of β -unimodal distribution functions.

(ii) Using (2.3) in (2.2) with $\alpha > \beta > 0$, $p = \beta/\alpha$, $q = (\alpha - \beta)/\alpha$ and integrating by parts, we get

$$\kappa(u) = \frac{\beta}{u^\beta} \int_0^u \varphi(\omega) \omega^{\beta-1} d\omega.$$

Hence $\kappa(u)$ is the characteristic function of a β -unimodal distribution function.

The second theorem establishes a relationship between the above integral transformation and a sequence of transformations introduced by Harkness and Shantaram [3]. This sequence of transformations is frequently encountered in renewal theory. The limiting behaviour of a particular case of the Harkness and Shantaram sequence is investigated in the third theorem.

THEOREM 2. *Let $\varphi(u)$ be the characteristic function of a distribution function $F(x)$ on $[0, \infty)$, for which the n -th non-central moment μ_n is finite, $n = 1, 2, \dots$. Then the function*

$$(2.6) \quad \gamma_n(u) = n! \left[\varphi(u) - \sum_{k=0}^n \frac{\varphi^{(k)}(0)}{k!} u^k \right] / \varphi^{(n)}(0) u^n$$

is the characteristic function of an α -unimodal distribution function with $\alpha = n$.

Proof. From the fact that $\mu_n < \infty$ it follows that the function $F_n(x)$ defined by

$$(2.7) \quad F_n(x) = \int_0^x \omega^n dF(\omega)/\mu_n$$

is a distribution function and $\varphi^{(n)}(u)/\varphi^{(n)}(0)$ is its characteristic function. Since

$$\begin{aligned} \gamma_n(u) &= n! \left[\varphi(u) - \sum_{k=0}^n \frac{\varphi^{(k)}(0)}{k!} u^k \right] / \varphi^{(n)}(0) u^n \\ &= \frac{n!}{u^n} \int_0^u \int_0^{u_{n-1}} \cdots \int_0^{u_1} \frac{\varphi^{(n)}(x)}{\varphi^{(n)}(0)} dx du_1 \cdots du_{n-1} \\ &= \frac{n}{u^n} \int_0^u \left[\frac{n-1}{u_{n-1}^{n-1}} \int_0^{u_{n-1}} \cdots \left[\frac{1}{u_1} \int_0^{u_1} \frac{\varphi^{(n)}(x)}{\varphi^{(n)}(0)} dx \right] u_1 du_1 \cdots \right] u_{n-1}^{n-1} du_{n-1} \\ &= \frac{n}{u^n} \int_0^u \gamma_{n-1}(u_{n-1}) u_{n-1}^{n-1} du_{n-1}, \end{aligned}$$

we conclude that $\gamma_n(u)$ is the characteristic function of an α -unimodal distribution function with $\alpha = n$.

THEOREM 3. Consider the characteristic function $\varphi(u)$ defined by

$$(2.8) \quad \varphi(u) = \log(1 - iu)/(-iu)$$

and let $\{\gamma_n(u): n \geq 1\}$ denote the sequence of characteristic functions defined by (2.6), obtained from $\varphi(u)$. Then

$$\lim_{n \rightarrow \infty} \gamma_n(u) = \frac{1}{1 - iu}.$$

Proof. The characteristic function $\gamma_n(u)$ defined by (2.6) with $\varphi(u)$ given by (2.8) can be written in the form

$$\gamma_n(u) = \frac{n+1}{u^{n+1}} \int_0^u \left[\frac{1}{1 - i\omega} \right] \omega^n d\omega = \int_0^1 \left[\frac{1}{1 - iu\omega} \right] dH_n(\omega),$$

where $\{H_n(\omega): n \geq 1\}$ denotes a sequence of distribution functions defined by

$$H_n(\omega) = \int_0^\omega x^n dx / \int_0^1 x^n dx.$$

Let $\lambda_{k,n}$ denote the k -th non-central moment of $H_n(\omega)$. Then since

$$\lim_{n \rightarrow \infty} \lambda_{k,n} = 1 \quad \text{for every } k \geq 1,$$

it follows from the moment convergence theorem that the sequence $\{H_n(\omega): n \geq 1\}$ converges to the distribution function degenerate at $\omega = 1$. Hence

$$\lim_{n \rightarrow \infty} \gamma_n(u) = \frac{1}{1 - iu}$$

which is the characteristic function of the exponential distribution with parameter 1.

In the following two theorems, we use the integral transformation of (1.1) to characterise certain probability distributions.

THEOREM 4. *Let $\gamma(u)$ be the characteristic function of a distribution function $G(x)$ on $[0, \infty)$ with finite n -th non-central moment μ_n . Then*

$$(2.9) \quad \frac{\gamma^{(n-1)}(u)}{\gamma^{(n-1)}(0)} = \frac{n}{u^n} \int_0^u \frac{\gamma^{(n)}(\omega)}{\gamma^{(n)}(0)} \omega^{n-1} d\omega, \quad n = 2, 3, \dots,$$

if and only if $G(x)$ is an exponential distribution function.

Proof. Multiply both sides of (2.9) by u^n and differentiate to obtain the differential equation

$$nu^{n-1} \frac{\gamma^{(n-1)}(u)}{\gamma^{(n-1)}(0)} + u^n \frac{\gamma^{(n)}(u)}{\gamma^{(n-1)}(0)} = nu^{n-1} \frac{\gamma^{(n)}(u)}{\gamma^{(n)}(0)},$$

which can be written in the convenient form

$$(2.10) \quad \frac{d}{du} \log \frac{\gamma^{(n-1)}(u)}{\gamma^{(n-1)}(0)} = \frac{d}{du} \log \left[1 - \frac{1}{u} \frac{\gamma^{(n)}(0)}{\gamma^{(n-1)}(0)} u \right]^{-n},$$

whenever $u \neq 0$ and $\gamma^{(n-1)}(u) \neq 0$.

Since the boundary condition of (2.10) is $\gamma(0) = 1$, it follows that

$$\frac{\gamma^{(n-1)}(u)}{\gamma^{(n-1)}(0)} = \left[1 - \frac{1}{n} \frac{\gamma^{(n)}(0)}{\gamma^{(n-1)}(0)} u \right]^{-n}$$

for all $n \geq 1$. In particular, when $n = 1$, we obtain

$$\gamma(u) = (1 - \gamma^{(1)}(0)u)^{-1},$$

which is readily identified as the characteristic of an exponential distribution.

THEOREM 5. *Let $\gamma(u)$ be the characteristic function of a distribution function $G(x)$ with first moment $\mu_1 = 0$ and finite second (non-central) moment μ_2 . Then*

$$(2.11) \quad \gamma(u) = \frac{\alpha}{u^\alpha} \int_0^u \frac{\gamma^{(1)}(\omega)}{\gamma^{(2)}(0)} \omega^{\alpha-1} d\omega$$

if and only if

$$\gamma(u) = (1 + \mu_2 u^2 / \alpha)^{-\alpha/2}.$$

Proof. Since $\mu_2 < \infty$, it follows that the function

$$\int_{-\infty}^x \omega^2 dG(\omega)/\mu_2$$

is a distribution function and $\gamma^{(2)}(u)/\gamma^{(2)}(0)$ is its characteristic function. Furthermore, because $\mu_1 = 0$, the function

$$\frac{\gamma^{(1)}(u)}{\gamma^{(2)}(0)u} = \frac{1}{u} \int_0^u \frac{\gamma^{(2)}(\omega)}{\gamma^{(2)}(0)} d\omega$$

is also a characteristic function. Hence the right-hand side of (2.11) describes a valid characteristic function.

Multiplying both sides of (2.11) by u^α and differentiating we obtain

$$\alpha u^{\alpha-1} \gamma(u) + u^\alpha \gamma^{(1)}(u) = \alpha \frac{\gamma^{(1)}(u)}{\gamma^{(2)}(0)u} u^{\alpha-1}.$$

This differential equation may be written in the form

$$\frac{d}{du} \log \gamma(u) = \frac{d}{du} \log \left[\frac{\alpha}{\gamma^{(2)}(0)} - u^2 \right]^{-\alpha/2}, \quad u \neq 0.$$

With due regard to the boundary condition, $\gamma(0) = 1$, we may write

$$\gamma(u) = (1 + \mu_2 u^2/\alpha)^{-\alpha/2},$$

If $\alpha = 2$, then $\gamma(u)$ is the characteristic function of the Laplace distribution.

3. Applications. Let C_α denote the class of α -unimodal distribution functions. In this section certain stochastic derivations of distributions related to the class C_α and examples of distributions of the class C_α are given.

(a) Let $\{X(t), t \geq 0\}$ be a homogeneous and continuous-in-probability stochastic process with independent increments and denote the characteristic function of the increment $X(t+1) - X(t)$ by $\exp\{\lambda[\varphi(u) - 1]\}$, $\lambda > 0$. From [4] it follows that the stochastic integral

$$\int_0^1 t^{1/\alpha} dX(t), \quad \alpha > 0,$$

exists in the sense of convergence in probability and its characteristic function is given by

$$\exp \left\{ \lambda \left[\frac{\alpha}{u^\alpha} \int_0^u \varphi(\omega) \omega^{\alpha-1} d\omega - 1 \right] \right\}.$$

(b) Recently, some interest has been expressed in the simplicial convex hull of independent points in R^n and, in particular, within the unit n -ball. It can be shown that each point X_i is isotropic and, with the usual Euclidean

norm, that $|X|$ has a type-one scalar beta density. A description of the problem and early treatment is due to Miles [5] and random selection of points uniformly from the n -ball will lead to the radial distance being distributed according to $H_n(\omega)$ of Theorem 3.

(c) That unimodality of the underlying distribution is transferred to the distribution of any order statistic is well known (see [1]). Note, however, that in the particular case of the maximum observation in a sequence of n pseudo-random numbers generated in any Monte-Carlo study is also distributed as $H_n(\omega)$. This observation will usually result in a catastrophic scenario. It is such situations (see [2]) when "the extremes govern the laws of interest (strength of materials, floods, droughts, air pollution, failure of equipment, effects of food additives, etc.)".

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