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Periodic solutions of differential equations in the cylindrical space

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1. In [3] Mamrilla and the author have proved that if the real-valued function f defined on the real line R is negative, continuous and periodic and if the polynomial $\varphi(\lambda) = \lambda^{n-1} + a_1\lambda^{n-2} + \ldots + a_{n-1}$ has all roots with negative real parts, then the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + f(y) = 0 \tag{1}$$

admits at least one solution $y = y(t)$ having the following property:

(P) there is a $T > 0$ such that $y(t+T) = y(t)+1$ for $t \in R$.

It will be shown that this result holds true under the assumption that $\varphi(\lambda)$ has no roots lying on the imaginary axis. Moreover, in the case $n = 3$, the existence of a solution with property (P) will be proved for the equation

$$y''' + a(y')y'' + b(y)y' + f(y) = 0. \tag{2}$$

Let $x(t)$ be the $(n-1)$-vector with components $y'(t), y''(t), \ldots, y^{(n-1)}(t)$, where $y = y(t)$ is a solution of (1) or (2). It is clear that (P) is equivalent to

(P1) there is $T > 0$ such that $x(t+T) = x(t)$, $y(t+T) = y(t)+1$ for all t.

If R_c^n denotes the set obtained from the n-dimensional Euclidean space R^n by the identification of pairs $(x^1, \ldots, x^{n-1}, y)$, $(x^1, \ldots, x^{n-1}, v)$ with integer $y - v$, the "cylindrical space", then (P1) implies that the map $t \to (x(t), y(t))$ of R into R_c^n is periodic with period T.

2. THEOREM 1. *Let a map $f\colon R \to (-\infty, 0)$ be continuous and let*

$$f(y+1) = f(y) \quad \textit{for } y \in R. \tag{3}$$

If the polynomial $\varphi(\lambda) = \lambda^{n-1} + a_1\lambda^{n-2} + \ldots + a_{n-1}$ $(a_i \in R)$ has no pure imaginary roots, then (1) *has a solution with property* (P).

Remark. The condition $a_{n-1} \neq 0$ is necessary for the existence of solutions having property (P). In fact, let $a_{n-1} = 0$ and let $y = y(t)$

satisfy (P). Then the function $s(t) = y^{(n-1)}(t)+a_1y^{(n-2)}(t)+\ldots+$ $+a_{n-2}y'(t)$ is periodic, whence $s'(t)$ changes the sign. But $s'(t) = -f(y(t))$ > 0, which gives the desired contradiction.

THEOREM 2. *Let f be as in Theorem 1. Let functions $a: R\to R$, $b: R\to R$ be continuous and satisfy*

$$(4) \qquad |a(y)| \geqslant a > 0 \quad \textit{for } y\in R,$$

$$(5) \qquad b(y+1) = b(y) \quad \textit{for } y\in R.$$

Let the solutions of (2) *be uniquely determined by initial conditions and exist for all $t\geqslant 0$.*

Then (2) *has a solution having property* (P).

3. Write (1) or (2) as a system

$$(6) \qquad x' = p(x, y),\ y' = x^1,$$

in which $x = (x^1, \ldots, x^{n-1})$ and $p: R^{n-1}\times R\to R^{n-1}$ is continuous and periodic in y with period 1.

It will be shown that, under the assumptions of Theorem 1 or 2, (6) admits a solution satisfying (P1). For this purpose the following obvious lemma will be used:

LEMMA. *Assume that for an arbitrary $(x_0, y_0)\in R^n$ the initial value problem* (6), *$x(0) = x_0$, $y(0) = y_0$ has a unique solution $x = x(t;\ x_0, y_0)$, $y = y(t;\ x_0, y_0)$ which exists for all $t\geqslant 0$. Let the map $h: R^{n-1}\to R$ be continuous and denote by S_0, S_1 the sets*

$$S_0 = \{(x, y):\ y = h(x),\, x\in R^{n-1}\}, \qquad S_1 = \{(x, y):\ y = h(x)+1,\ x\in R^{n-1}\}.$$

For any $x_0\in R^{n-1}$, there is a unique number $t(x_0)$ such that

$$(7) \qquad \big(x\big(t(x_0);\ x_0, h(x_0)\big), y\big(t(x_0);\ x_0, h(x_0)\big)\big)\in S_1.$$

Then (6) *has a solution satisfying* (P1) *if and only if the map u: $R^{n-1}\to R^{n-1}$ defined by*

$$(8) \qquad u(x_0) = x\big(t(x_0);\ x_0, h(x_0)\big)$$

has a fixed point.

The following notation will be used: If A is $n\times m$ matrix, then $\|A\|$, A^T denote respectively the norm and the matrix transpose to A. $\mathrm{Fr}B$ ($\mathrm{Cl}B$) denotes the boundary (closure) of the set B.

For a real-valued function $W(x, y)$ ($x\in R^{n-1}$, $y\in R$), $W'(x, y)$ denotes its derivative with respect to solutions of (6), i.e. $W'(x, y) = dW\big(x(t),$ $y(t)\big)/dt$, where $\big(x(t), y(t)\big)$ is a solution of (6).

4. Proof of Theorem 1. System (6) has the form

$$(9) \qquad x' = Ax+bf(y), \qquad y' = x^1,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & \dots & \cdot \\ 0 & \cdot & \dots & & 0 & 1 \\ -a_{n-1} & & \dots & & & -a_1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \quad x = \begin{pmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^{n-1} \end{pmatrix}.$$

Let S_0 and S_1 be defined as in the Lemma, with $h(x) = (-a_{n-1})^{-1}d^Tx$, $d^T = (a_{n-2}, \dots, a_1, 1)$. Let $W(x, y) = a_{n-1}y + d^Tx$.

Since f is continuous, (3) implies that solutions of (9) exist for all t. Moreover, by $W' = -f(y) \geqslant a > 0$, the equation

$$W\big(x(t;\ x_0, h(x_0)), y(t;\ x_0, h(x_0))\big) = a_{n-1}$$

has a unique solution $t(x_0)$ for every $x_0 \in R^{n-1}$, i.e. (7) holds. It is easy to see that $t(x_0)$ is bounded (see [3]).

Assume additionally that (9) has the property of uniqueness. Thus the Lemma is applicable and the proof reduces to showing that the mapping (8) has a fixed point. The map u is defined by the formula

$$u(x_0) = P(x_0)x_0 + b(x_0),$$

where

$$P(x_0) = X\big(t(x_0)\big), \qquad b(x_0) = \int_0^{t(x_0)} X\big(t(x_0) - s\big) bf\big(y(s;\ x_0, h(x_0))\big) ds$$

($X(t)$ denotes the fundamental matrix of $x' = Ax$.)

Since A has no eigenvalues on the imaginary axis, $P(x_0) - I$ is nonsingular. By the boundedness of $t(x)$ and $f(y)$, $\lim_{\|x\| \to \infty} \big(\|b(x)\| \cdot \|x\|^{-1}\big) = 0$. Thus, by the finite-dimensional version of Theorem 1 [2] (see also [4], Theorem 6.3), u has a fixed point, which completes the proof under the assumption of uniqueness.

In the general case, observe that the boundedness of $t(x)$ implies the boundedness of $\|P(x) - I\|$ and $\|b(x)\|$. Thus $\{x \in R^{n-1}\colon u(x) = x\} \subset \{x \in R^{n-1}\colon \|x\| < k\}$, where a constant $k > 0$ depends only on A and estimates of f. This permits us to approximate (9) by equations with the property of uniqueness, such that corresponding sets of fixed points are in the ball $\{x\colon \|x\| < k\}$. By the standard limiting argument (see [1], Theorem 2.4), we conclude that Theorem 1 holds also without the assumption of uniqueness.

5. Proof of Theorem 2. The first order system equivalent to (2) is

$$(10) \qquad x^{1\prime} = x^2, \quad x^{2\prime} = -a(x^1)x^2 - b(y)x^1 - f(y), \quad y' = x^1.$$

Let $a(z)$, $b(z)$ be positive (in other cases the proof is similar). Put

$$W(x, y) = x^2 + A(x^1) + B(y), \qquad h(x) = B^{-1}\big(-x^2 - A(x^1)\big),$$

where

$$x = (x^1, x^2), \qquad A(z) = \int_0^z a(s)\,ds, \qquad B(z) = \int_0^z b(s)\,ds$$

and B^{-1} denotes the map inverse to B. Let $x(t;\, x_0, y_0)$, $y(t;\, x_0, y_0)$, S_0, S_1, u be defined as in the Lemma.

Observe that $S_0 = \{(x, y)\colon W(x, y) = 0\}$, $S_1 = \{(x, y)\colon W(x, y) = B(1)\}$. Hence $W' = -f(y) \geqslant \bar{a} > 0$ implies that for any $x_0 \in R^2$, there is a $t(x_0) > 0$ satisfying (7). This shows that u is defined for all x_0. Furthermore u is a homeomorphism preserving the orientation of R^2.

By (3), for any $x_0 \in R^2$, the points $x_{i+1} = u(x_i)$ $(i = 0, 1, \ldots)$ belong to the set $\{x \in R^2\colon x = x(t;\, x_0, h(x_0)), t \geqslant 0\}$.

To complete the proof it is enough to show that (10) has a solution $\big(x(t), y(t)\big)$ with $x(t)$ bounded. In fact, if for some x_0, $x\big(t;\, x_0, h(x_0)\big)$ is bounded for $t > 0$, then the sequence $\{x_i\}$ is bounded and u has a fixed point (see for example [4], Theorem 12.4), which by the Lemma proves the theorem.

Let $D(c)$ be the "half-cylinder" $\{(x, y) \in R^2 \times R\colon V(x) < c,\ W(x, y) \geqslant 0\}$, where $V(x) = \frac{1}{2}\big((x^1)^2 + (x^2)^2\big) - e x^1 x^2$, e is a constant such that $0 < e < 1$.

Replacing, if necessary, e by a smaller number, from the formula

$$V'(x) = x^1 x^2 + (x^2 - e x^1)\big(-a(x^1)x^2 - b(y)x^1 - f(y)\big) - e(x^2)^2$$

we obtain

$$V'(\pm\sqrt{2c}, 0) > 0, \qquad V'(0, \pm\sqrt{2c}) < 0 \tag{11}$$

for c sufficiently large. By (11), the set $E = \big\{(x, y) \in \operatorname{Fr} D(c)\colon \big(x(t;\, x, y), y(t;\, x, y)\big) \notin \operatorname{Cl} D(c)$ for all small $t > 0\big\}$ is non-empty and has at least two components. If D_i is any component of E, then the set

$$K_i = \big\{(x, y) \in D(c) \cap S_0\colon \big(x(s;\, x, y), y(s;\, x, y)\big) \in D_i \text{ for some } s > 0,\ \big(x(t;\, x, y), y(t;\, x, y)\big) \in D(c) \quad \text{for } t \in [0, s)\big\},$$

as can easily be verified, is open in S_0. Obviously, K_i corresponding to different D_i are disjoint.

Since $S_0 \cap D(c)$ is connected and open in S_0, $\big(S_0 \cap D(c)\big) \setminus K$, K being the union of K_i, is non-empty. Hence the solution $\big(x(t), y(t)\big)$ of (10) with $\big(x(0), y(0)\big) \in \big(S_0 \cap D(c)\big) \setminus K$ remains in $D(c)$ for $t \geqslant 0$. Since the form $V(x)$ is positive definite, $x(t)$ is bounded and the proof of Theorem 2 is complete.

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