ANNALES
POLONICI MATHEMATICI
LI (1990)

Mean value theorems for linear and semi-linear rotation invariant operators

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Abstract. We establish a non-linear mean value theorem for bounded rotation invariant solutions to a semi-linear Laplace equation. We also prove some (apparently well known) mean value theorems for smooth solutions of the harmonic, polyharmonic and metaharmonic equations.

1. Introduction. The purpose of this note is to underline the fact that the mean value theorems for solutions to linear rotation invariant equations (e.g. for harmonic, polyharmonic, metaharmonic functions) express very simple properties of the corresponding ordinary *radial part* operators. Those mean value theorems (presented below) are apparently well known, although the author could not find any reference for them (except for the standard mean value theorem for harmonic functions).

The "hard" part of the note is devoted to establishing non-linear mean value theorems for solutions to semi-linear Laplace equations. The methods rely on the Ważewski-type theorems concerning the asymptotic behaviour of solutions of perturbed linear systems. Unfortunately, only the case of bounded rotation invariant solutions has been settled.

2. Linear case. To see how easy things are, we first consider the case of a harmonic function.

THEOREM 1. *Let u be a bounded measurable function satisfying $\Delta u = 0$ in a neighbourhood of the closed ball $\{x\colon \|x-y\| \leqslant R\} \subset \mathbf{R}^n$. Then*

$$u(y) = \frac{1}{\sigma_{n-1} R^{n-1}} \int_{\|x-y\|=R} u(x)\,dS_x, \tag{1}$$

where σ_{n-1} is the measure of the unit sphere S in $\mathbf{R}^n$ and dS_x denotes the surface measure in variable x.

Proof. By translation invariance and rescaling we may assume that $y=0$, $R=1$. We have to prove that

$$u(0)=\frac{1}{\sigma_{n-1}}\int_S u(x)dS_x. \tag{2}$$

As we show below, it is enough to prove (2) only for rotation invariant harmonic functions u. Indeed, for every $A\in SO_0(n)$ (= the identity component of the group of orthogonal transformations in $\boldsymbol{R}^n$) we have

$$\int_S u(x)dS_x=\int_S u\circ A(x)dS_x.$$

Thus, if μ is the normalized Haar measure on $SO_0(n)$, we have

$$\int_S u(x)dS_x=\int_S\Big(\int_{SO_0(n)} u\circ A(x)d\mu\Big)dS_x.$$

Write

$$\tilde{u}(x)=\int_{SO_0(n)} u\circ A(x)d\mu. \tag{3}$$

The function $\tilde{u}$ is harmonic since the Laplace operator commutes with rotations. Further, $\tilde{u}$ is rotation invariant and hence is of the form

$$\tilde{u}(x)=g(\|x\|^2)$$

with $g(s)$ bounded on $[0,1]$, since u was so.

To prove (2), it is enough to show that

$$g(0)=g(1).$$

We have

$$\Delta\tilde{u}(x)=2\left(2s\frac{d^2}{ds^2}+n\frac{d}{ds}\right)g(\|x\|^2).$$

The general solution of the equation

$$\left(2s\frac{d^2}{ds^2}+n\frac{d}{ds}\right)g(s)\doteq 0 \quad \text{for } s>0,$$

is of the form

$$g(s)=c_1s^{-n/2+1}+c_2 \quad \text{if } n>2,$$

$$g(s)=c_1\ln s+c_2 \quad \text{if } n=2.$$

Since in our case g should be bounded at zero, we must have $c_1=0$ in both cases.

Remark 1. Note that we have proved that the only invariant harmonic function in a neighbourhood of zero is the constant function.

As another example we consider the polyharmonic functions.

THEOREM 2. *Suppose u is a (smooth) function in a neighbourhood of the closed ball* $B = \{|x| \leqslant 1\} \subset \mathbf{R}^n$, *fulfilling there the equation*

$$\Delta^m u = 0,$$

where $m \geqslant 1$. *Then the following mean value formula holds for u:*

$$u(0) + \sum_{j=1}^{m-1} \frac{1}{w(1)\ldots w(j)} \Delta^j u(0) = \frac{1}{\sigma_{n-1}} \int_S u(x) dS_x,$$

where $w(k) = 2k(2k+n-2)$.

Proof. As in Theorem 1 we consider the function

$$\tilde{u}(x) = \int_{SO_0(n)} u \circ A(x) d\mu,$$

which solves the equation $\Delta^m \tilde{u} = 0$ and is of the form

$$\tilde{u}(x) = g(\|x\|^2).$$

Write

$$L = 4s\frac{d^2}{ds^2} + 2n\frac{d}{ds}.$$

We have

$$\Delta^m g(\|x\|^2) = \left(L^m g(s)\right)\big|_{s=\|x\|^2}.$$

Let $W(\alpha) = 2\alpha(2\alpha+n-2)$ be the characteristic polynomial for L. Then the characteristic polynomial for L^m equals

$$w_m(\alpha) = w(\alpha)w(\alpha-1)\ldots w(\alpha-m+1).$$

Hence there are m non-negative characteristic roots $\alpha_1 = 0$, $\alpha_2 = 1, \ldots, \alpha_m = m-1$. The remaining roots are negative if $n > 2m$ or they give rise to solutions of the equation

(4) $$L^m g(s) = 0, \quad s > 0,$$

which contain logarithmic terms. Therefore a general smooth solution of equation (4) must be of the form

$$g(s) = c_1 + c_2 s + \ldots + c_m s^{m-1}$$

for some constants $c_1, \ldots, c_m$. Observe that

(5) $$g(0) = c_1,$$
$$(L^j g)(0) = w(1)\ldots w(j)c_{j+1}, \quad j = 1, \ldots, m-1.$$

We have

$$g(1) = c_1 + c_2 + \ldots + c_m$$

and from (5) we get

$$g(1) = g(0) + \sum_{j=1}^{m-1} \frac{1}{w(1)\dots w(j)}(L^j g)(0).$$

This ends the proof, since

$$g(1) = \frac{1}{\sigma_{n-1}} \int_S u(x) dS_x$$

and

$$g(0) = u(0),$$

$$(L^j g)(0) = \Delta^j \tilde{u}(0) = \Delta^j \Big(\int_{SO_0(n)} u \circ A(x) d\mu \Big)\Big|_{x=0} = \int_{SO_0(n)} (\Delta^j u) A(0) d\mu = \Delta^j u(0)$$

for $j = 1, \dots, m-1$.

Remark 2. In a similar way, if we replace (5) by

$$g(1) = c_1 + c_2 + \dots c_n,$$

$$(L^j g)(1) = \sum_{k=j+1}^{m} c_k w(k-1) \dots w(k-j), \quad j = 1, ,\dots, m-1,$$

we get the following mean value formula for a polyharmonic function u:

$$u(0) = \sum_{j=0}^{m-1} \frac{\tilde{c}_j}{\sigma_{n-1}} \int_S \Delta^j u(x) dS_x$$

for some (explicit) constants $\tilde{c}_j$.

THEOREM 3. *For a (smooth) solution u of the metaharmonic equation*

$$\Delta u + \lambda u = 0, \quad \lambda \in \boldsymbol{C},$$

we have

$$\frac{1}{\sigma_{n-1}} \int_S u(x) dS_x = \Big(\sum_{j=0}^{\infty} \frac{(-\lambda)^j}{w(1)\dots w(j)} \Big) u(0).$$

Proof. We see that the function $g(s)$ such that

$$g(\|x\|^2) = \int_{SO_0(n)} u \circ A(x) d\mu$$

satisfies the equation

$$\Big(2s\frac{d^2}{ds^2} + n\frac{d}{ds} + \lambda \Big) g(s) = 0 \quad \text{for } s > 0;$$

and, being smooth, it must be of the form

$$g(s) = c\Big(\sum_{j=0}^{\infty} \frac{(-\lambda s)^j}{w(1)\dots w(j)} \Big),$$

which immediately gives the assertion.

Remark 3. The technique of deriving mean value theorems presented above should also apply to invariant Laplace–Beltrami operators on symmetric spaces and to partially rotation invariant operators.

3. Semi-linear case. We are concerned here with bounded rotation invariant (distributional) solutions of the equation

$$\Delta u + u^{\alpha} = 0$$

for some fixed $\alpha > 0$.

Writing $u(x) = g(\|x\|^2)$, we have for g

$$2\left(2s\frac{d^2}{ds^2} + n\frac{d}{ds}\right)g(s) + g^{\alpha}(s) = 0.$$

Multiplying both sides by s and using the identity

$$s^2\frac{d^2}{ds^2} = \left(s\frac{d}{ds}\right)^2 - s\frac{d}{ds},$$

we get

$$2\left(2\left(s\frac{d}{ds}\right)^2 + (n-2)s\frac{d}{ds}\right)g(s) + sg^{\alpha}(s) = 0.$$

After the substitution $s = e^{-t}$ we get for the function $h(-\ln s) = g(s)$ the equation

$$h'' = \frac{n-2}{2}h' - \frac{e^{-t}}{4}h^{\alpha}$$

which can be written in the matrix form

(6) $$\zeta' = A\zeta + F(t, \zeta), \quad \zeta = (\zeta_1, \zeta_2) = (h, h'),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}, \quad a = \frac{n-2}{2}, \quad F(t, \zeta) = \left(0, -\frac{e^{-t}}{4}\zeta_1^{\alpha}\right).$$

After the change of variables

$$\zeta = B\xi,$$

where

$$B = \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix},$$

the system (6) goes to

$$\xi_1' = \frac{e^{-t}}{4a}(\xi_1 + \xi_2)^{\alpha}, \quad \xi_2' = a\xi_2 - \frac{e^{-t}}{4a}(\xi_1 + \xi_2)^{\alpha}.$$

Adjoining the equation

$$\xi_3' = -\xi_2,$$

we obtain the following system in $\boldsymbol{R}^3$:

$$\xi' = E\xi + G(t, \xi), \tag{7}$$

where

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad G(t, \xi) = \frac{e^{-t}}{4a}((\xi_1+\xi_2)^\alpha, -(\xi_1+\xi_2)^\alpha, 0).$$

We are interested in bounded (at infinity) solutions $\zeta(t)$ of (7) (it is enough to assume that only $\zeta_1(t)$ be bounded, since it can be shown that then $\zeta(t) \to \zeta_w < \infty$ as $t \to \infty$). Hence, only the values of F for bounded ζ (and consequently of G for bounded ξ) are important. Therefore in the sequel we shall assume that, for large $\|\xi\|$, $G(t, \xi) \equiv 0$. Thus we have for some $c > 0$ the estimate

$$\|G(t, \xi)\| \leqslant ce^{-t}\|\xi\|, \qquad \xi \in \boldsymbol{R}^3, \; t \geqslant 0.$$

Writing $\psi_1(t) = ce^{-t}$ we see that for every $\beta > 0$

$$\int_0^\infty t^\beta \psi_1(t)dt < \infty; \tag{8}$$

hence from Theorems 13.1 and 13.2 in [1] we get

PROPOSITION 1. *For every fixed $c_1 \neq 0$ and every $\beta > 0$ there exists a one-parameter family of solutions $\xi(t)$ of* (7) *defined for large t and satisfying as $t \to \infty$ the conditions*

$$\xi_1 = c_1 + o(t^{-\beta}), \qquad \xi_2 = o(t^{-\beta}), \qquad \xi_3 = o(t^{-\beta}). \tag{9}$$

Conversely, if $\xi(t) \not\equiv 0$ is a solution of (7) *such that $t^{-1}\ln\|\xi(t)\| \to 0$, $t \to \infty$, then there exists $c_1 \neq 0$ such that $\xi(t)$ satisfies* (9).

The function $\psi_1(t)$ also fulfils the condition

$$\sup_{s \geqslant t}(1+s-t)^{-1}\int_t^s \psi_1(r)dr \to 0 \qquad \text{as } t \to \infty.$$

Thus from Theorem 11.1 in [1] we get

PROPOSITION 2. *There exist $\delta_1 > 0$ and $T \geqslant 0$ such that if $0 < |\mathring{\xi}_1| < \delta_1$, there exists $\mathring{\xi}_2$ for which the initial value problem*

$$\xi_1(t_0) = \mathring{\xi}_1, \qquad \xi_2(t_0) = \mathring{\xi}_2, \qquad \xi_3(t_0) = 0 \tag{10}$$

for (7) *has for $t \geqslant t_0$ a solution satisfying the conditions $\xi_1(t) \neq 0$ and*

$$|\xi_2(t)| = o(\xi_1(t)) \quad \textit{as } t \to \infty, \; \xi_3(t) = 0, \tag{11}$$

(12) $$\lim_{t\to\infty} t^{-1}\ln\|\xi(t)\| = 0.$$

Moreover, since

$$\|G(t, \xi)-G(t, \tilde{\xi})\| \leqslant \psi_1(t)\|\xi-\tilde{\xi}\|,$$

it follows from Theorem 8.2 *that* $\mathring{\xi}_2 = g(t_0, \mathring{\xi}_1)$ *is uniquely determined by* $\mathring{\zeta}_1$ *and the solution* $\xi(t)$ *satisfying conditions* (10) *is unique. Further, the function* $\mathring{\xi}_2 = g(t_0, \mathring{\xi}_1)$ *is smooth on its domain.*

Theorem 11.2 of [1] implies

PROPOSITION 3. *Let* $\xi(t)$ *be an arbitrary bounded non-zero solution of* (7) *satisfying* (10). *Then conditions* (11) *and* (12) *hold.*

Since for every $\beta > 0$ (8) holds, it follows from Theorem 13.1 in [1] that (11) may be replaced by

(11) $$\xi_1(t) = c_1+o(t^{-\beta}), \qquad \xi_2(t) = o(t^{-\beta}), \qquad \xi_3(t) = 0$$

for some constant $c_1 \neq 0$ and every fixed $\beta > 0$. From Proposition 3 and Theorem 13.2 in [1] we get

PROPOSITION 3′. *Let* $\xi(t)$ *be an arbitrary non-zero solution of* (7) *satisfying* (10). *Then conditions* (11) *and* (12) *hold.*

Combining the above theorems, we get

THEOREM 4. *Let* H *be the function which to a point* $\mathring{\xi}_1$ *assigns* c_1 *such that*

$$\lim_{t\to\infty} \xi_1(t) = c_1,$$

where $\xi(t)$ *is a bounded solution of* (7) *satisfying* (10) *with* $\mathring{\xi}_2 = g(t_0, \xi_1)$. *Then there exist* $t_0 \geqslant 0$ *and open neighbourhoods* U_1, U_2 *of zero in* $\boldsymbol{R}$ *such that* H *is a diffeomorphism.*

Now, from Theorem 4 we obtain in a standard way

THEOREM 5. *There exists* $R > 0$ *such that for every bounded rotation invariant solution* u *of the equation*

$$\Delta u+u^{\alpha} = 0$$

on sufficiently small open set containing the closed ball $\{|x| \leqslant R\}$ *the following non-linear maximum principle holds:*

$$H^{-1}\left(\frac{1}{\sigma_{n-1}R^{n-1}} \int_{\|x\|=R} u(x)dS_x\right) = u(0),$$

where H *is the diffeomorphism from Theorem* 4.

References

[1] P. Hartman, *Ordinary differential equations*, John Wiley and Sons, 1964.
[2] Z. Szmydt, *On homogeneous rotation invariant distributions and the Laplace operator*, Ann. Polon. Math. 36 (1979), 249-259.

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Reçu par la Rédaction le 20.10.1988