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REMARKS ON THE EVALUATION OF THE BESSEL FUNCTIONS FROM THE RECURRENT FORMULA

It is convenient to evaluate the Bessel functions $J_0(x)$, $J_1(x)$, ... (x is a real number) with absolute error less than a given positive number ε by the recurrent relation

$$\frac{2n}{x} J_n(x) = J_{n-1}(x)+J_{n+1}(x). \tag{1}$$

It is necessary (which is done in [1] and in the present note) to fix as accurately as possible the integer N such that $|J_N(x)| \geqslant \varepsilon$ and $|J_n(x)| < \varepsilon$ $(n = N+1, N+2, \ldots)$. One assumes then

$$J_n(x) = \begin{cases} 0 & (n > N), \\ a & (n = N), \\ \dfrac{2(n+1)}{x} J_{n+1}(x) - J_{n+2}(x) & (n = N-1, N-2, \ldots, 0). \end{cases}$$

The unknown quantity $a = J_N(x)$ assumes at the beginning any non-zero value, e.g. $a = 1$; afterwards it is corrected in such a way as to have

$$J_0(x)+2\sum_{k=1}^{[N/2]} J_{2k}(x) = 1$$

(the symbol [ ] denotes the integer part).

Let us determine now the integer N. To do this, in [1] the Neumann functions $Y_0(x)$, $Y_1(x)$, ... are used. In the present note a simpler way is proposed. It is described best by the following Algol-like instruction in which *eps* denotes the absolute error ε , x denotes the argument of the Bessel functions, and N — the wanted index:

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if abs(x) < eps
  then N: = 0
  else begin
          real J, w0, w1, x4;
          N: = 2 × entier(.5 × abs(x) + 1.75);
          J: = (N−1) ↑ (−.333);
          x4: = 4/(x × x);
          w1: = N × (N−1) × x4 − 2;
          for w0: = w1 while J ⩾ eps do
            begin
              N: = N + 2;
              w1: = N × (N−1) × x4 − 2;
              J: = J/(w0 − 1/(w1 − 1/w1))
            end w0;
          N: = N − 2
        end abs(x) ⩾ eps
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Verification. For $|x| < \varepsilon$ we have

$$|J_n(x)| < \varepsilon \max_x |J_n'(x)| < \varepsilon \qquad (n = 1, 2, \ldots);$$

thus consider the case $|x| \geqslant \varepsilon$ only. First, note that if J is such that $|J_m(x)| \leqslant J$ for certain $m \approx |x|$ and if $q_n = J_{n-2}(x)/J_n(x)$, then N may be calculated from $N = m + 2l$, where l is the least natural number satisfying the inequality

$$J/(q_{m+2}\, q_{m+4} \cdots q_{m+2l}) < \varepsilon.$$

This follows from $q_{n+2} > 1$ $(n > |x|)$.

Let $m = 2[\frac{1}{2}|x| + \frac{3}{4}]$; then

$$J = \max_{m-\frac{3}{2} \leqslant x < m+\frac{1}{2}} J_m(x).$$

From [2] (p. 247, formula (7)) for z approximately equal to ν we have

$$J_\nu(z) \approx \frac{\sqrt{3}}{6\pi}\left\{\left(\frac{6}{z}\right)^{1/3}\Gamma\left(\frac{1}{3}\right) + (z-\nu)\left(\frac{6}{z}\right)^{2/3}\Gamma\left(\frac{2}{3}\right) + 0 + \ldots\right\} \tag{2}$$

$$\approx .45 z^{-1/3} + .4(z-\nu) z^{-2/3}$$

and, finally, one may assume that $J \approx (m+1)^{-1/3}$.

Now we shall determine the numbers q_n. From (1) we obtain the identity

(3) $$J_{n-2}(x) = w_n J_n(x) - \frac{n-1}{n+1} J_{n+2}(x),$$

where

(4) $$w_n = \frac{4n(n-1)}{x^2} - \frac{2n}{n+1}.$$

Dividing (3) sidewise by $J_n(x)$ we have

$$q_n = w_n - \frac{n-1}{n+1}\frac{1}{q_{n+2}},$$

i.e.

(5) $$q_n = w_n - \frac{(n-1)/(n+1)}{w_{n+2}-} \frac{(n+1)/(n+3)}{w_{n+4}-} \frac{(n+3)/(n+5)}{w_{n+6}-} \cdots$$

In the above presented algorithm, formulae (4) and (5) were used in the reduced form

$$w_n \approx 4n(n-1)/x^2 - 2,$$

$$q_n \approx w_n - 1/(w_{n+2} - 1/w_{n+2}).$$

The case of Bessel functions with non-integer indices may be treated analogously. And so, for instance, for the functions

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$$

considered in [1] it follows from (2) that $|j_m(x)| \leqslant j$, for $m = 2[\frac{1}{2}|x| + \frac{3}{4}]$, where $j \approx (m+1)^{-5/6}$.

Some numerical examples with $\varepsilon = .00005$ are given in Tables 1 and 2 ($J_n^*(x)$ and $j_n^*(x)$ denote the calculated values of $J_n(x)$ and $j_n(x)$).

TABLE 1

x	n	$J_n^*(x)$	$J_n(x) - J_n^*(x)$
4.4	0	−.3422576	.0000008
	4 (= m)	.3364509	−.0000008
	12 (= N)	.0000178	.0000006
102.4	0	.0369959	−.0000010
	102 (= m)	.1031530	−.0000027
	122 (= N)	.0000114	.0000047
2502.4	0	.0098404	−.0000109
	2502 (= m)	.0338756	−.0000363
	2544 (= N)	.0000360	.0000816

TABLE 2

x	n	$j_n^*(x)$	$j_n(x)-j_n^*(x)$
6.4	0	.0182108	.0000000
	6 (= m)	.1130852	.0000000
	14 (= N)	.0000151	.0000008
102.4	0	.0093345	.0000000
	102 (= m)	.0116074	.0000000
	118 (= N)	.0000109	.0000053
2502.4	0	.0003967	.0000000
	2502 (= m)	.0008090	.0000109
	2524 (= N)	.0000174	.0000577

References

[1] Fr. Mechel, *Improvement in recurrence techniques for the computation of Bessel functions of integral order*, Math. Comp. 22 (1968), p. 202-205.

[2] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge 1966.

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UWAGI O OBLICZANIU WARTOŚCI FUNKCJI BESSELA Z WZORU REKURENCYJNEGO

STRESZCZENIE

Dla danego rzeczywistego x i $\varepsilon > 0$ podaje się algorytm obliczania takiej liczby naturalnej N, że wartości bezwzględne funkcji Bessela $J_{N+1}(x), J_{N+2}(x), \ldots$ są mniejsze od ε.