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# The numerical method of solving ordinary differential equations with the Cauchy initial condition

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In the present note an attempt is made to show a numerical method of solving ordinary differential equations.

The first part is concerned with the ordinary differential equation with the Cauchy initial condition; the right-hand term $f(x, y)$ of the equation will be given the upper and lower estimation by the linear formulae bound with the function $f$.

In the second part the condition for an upper and lower estimation of the differential equation integral curve will be given. This part of the note could be used in the Czaplygin method [1].

**1.** Let us consider the equation

$$y' = f(x, y), \tag{1.1}$$

where $x_0 \leqslant x \leqslant w$ with the condition

$$y(x_0) = y_0. \tag{1.2}$$

For the right-hand term of equation (1.1) it is assumed that

1° $f$ is of class $C^2$,

2° $f_y \neq 0$,

3° $|f(x, u) - f(x, v)| \leqslant L|u-v|$.

We look for the solution of equation (1.1) in $[x_0, w]$ with condition (1.2). The interval $[x_0, w]$ is divided into $n$ equal parts $h = (w-x_0)/n$.

Let us write $x_i = x_0 + ih$, $i = 0, 1, \ldots, n$, and

$$u(x_0) = u_0 = v_0 = v(x_0) = y_0.$$

Then with (1.1) we associate the continuous function $u(x)$ satisfying the equations

$$u' = a_i x + b_i u + c_i, \tag{1.3}$$

$$u(x_i) = u_i, \quad i = 0, 1, \ldots, n-1, \tag{1.4}$$

where $x \in [x_i, x_{i+1}]$, and the continuous function $v(x)$ satisfying the equations

$$(1.5)\qquad v' = a_i x + \beta_i v + \gamma_i,$$

$$(1.6)\qquad v(x_i) = v_i, \quad i = 0, 1, \ldots, n-1,$$

where $x \in [x_i, x_{i+1}]$.

The coefficients of the above equations are defined as follows:

$$a_i = \frac{\partial f(x_i, u_i)}{\partial x}, \quad b_i = \frac{\partial f(x_i, u_i)}{\partial y}, \quad c_i = f(x_i, u_i) - a_i x_i - b_i u_i,$$

$$a_i = \frac{f(x_i+h, y_i+k) + f(x_i+h, y_i-k) - 2f(x_i, v_i)}{2h},$$

$$\beta_i = \frac{f(x_i+h, v_i+k) - f(x_i+h, v_i-k)}{2k},$$

$$\gamma_i = f(x_i, v_i) - a_i x_i - \beta_i v_i.$$

Solving equations (1.3) (1.5) with conditions (1.4), (1.6) in the $i$-th interval, we get the formulae

$$(1.7)\qquad u(x) = \left(\frac{a_i x_i + c_i}{b_i} + \frac{a_i}{b_i^2} + u_i\right) e^{b_i(x-x_i)} - \frac{a_i x + c_i}{b_i} - \frac{a_i}{b_i^2},$$

$$(1.8)\qquad v(x) = \left(\frac{a_i x_i + \gamma_i}{\beta_i} + \frac{a_i}{\beta_i^2} + v_i\right) e^{\beta_i(x-x_i)} - \frac{a_i x + \gamma_i}{\beta_i} - \frac{a_i}{\beta_i^2}.$$

Let us assume that in every interval we have the inequality

$$(1.9)\qquad a_i x + b_i y + c_i \leqslant f(x, y) \leqslant a_i x + \beta_i y + \gamma_i, \quad i = 0, 1, \ldots, n-1,$$

for $(x, y) \in \Pi_i$, where

$$\Pi_i \quad \begin{cases} x_i \leqslant x \leqslant x_{i+1}, \\ y_i - \dfrac{k}{h}(x - x_i) \leqslant y \leqslant y_i + \dfrac{k}{h}(x - x_i) \end{cases}$$

is satisfied.

Considering the fact that $u_0 = v_0 = y_0$ and inequality (1.9), it is easy to prove that

$$(1.10)\qquad u(x) \leqslant y(x) \leqslant v(x) \quad \text{for } x \in [x_i, x_{i+1}],\ i = 0, 1, \ldots, n-1.$$

From Taylor's formula we shall estimate the difference in the first step:

$$v(x) - u(x) = \tfrac{1}{2} L_0 (x - x_0)^2, \quad x \in [x_0, x_1],$$

$$L_0 = |b_0[f(x_0, y_0) + a_0] e^{b_0(\xi - x_0)} - \beta_0 [f(x_0, y_0) + a_0] e^{\beta_0(\bar{\xi} - x_0)}|,$$

where $\xi, \bar{\xi} \in (x_0, x_1)$.

From formulae (1.7), (1.8) and the assumptions 2°, 3°, we shall obtain in the $i$-th step:

$$v(x)-u(x) = \Big| v_i-u_i+[(a_ix_i+\beta_iv_i+\gamma_i)-(a_ix_i+b_iu_i+c_i)](x-x_i)+ + L_i\frac{(x-x_i)^2}{2}\Big|$$

$$\leqslant |v_i-u_i|+L|v_i-u_i|(x-x_i)+L_i\frac{(x-x_i)^2}{2}$$

$$\leqslant |v_i-u_i|(1+Lh)+L_i\frac{h^2}{2},$$

$$L_i = |b_i[f(x_i, u_i)+a_i]e^{b_i(\xi-x_i)}-\beta_i[f(x_i, v_i)+a_i]e^{\beta_i(\bar{\xi}-x_i)}|,$$

$$\xi, \bar{\xi}\in (x_i, x), \qquad x_i\leqslant x\leqslant x_{i+1};$$

hence

$$v_{i+1}-u_{i+1}\leqslant \frac{h^2}{2}[L_0(1+Lh)^i+L_1(1+Lh)^{i-1}+\ldots+L_i].$$

From Taylor's formula we obtain the formulae for $L_i$:

$$L_i = [b_iL+O(h)](v_i-u_i)+O(k), \qquad L_i = b_ib_{i-1}\ldots b_0L^i\frac{h^{2i}}{2}+O(k).$$

Writing $M_n = \max_i L_i$, we obtain the inequality

$$v_{i+1}-u_{i+1}\leqslant \tfrac{1}{2}h^2M_n[(1+Lh)^i+(1+Lh)^{i-1}+\ldots+1], \qquad i = 0, 1, \ldots, n-1;$$

hence

$$v_n-u_n\leqslant \frac{h}{2L}M_n[(1+Lh)^n-1].$$

Keeping in mind that $h = (w-x_0)/n$, we have for every $n$:

$$v_n-u_n\leqslant \frac{h}{2L}M_n[e^{L(w-x_0)}-1].$$

It is easy to show that when $\max(\partial f/\partial y)$ is limited, $\lim_{n\to\infty} M_n = 0$, where $h\to 0$, $k\to 0$. This means that the functions $v_n$ and $u_n$ tend towards the solution $y(x)$ of equation (1.1).

**2.** Now we shall give a sufficient condition for the inequality

$$ax+by+c\leqslant f(x, y)\leqslant ax+\beta y+\gamma \tag{2.1}$$

for $(x, y) \in \Pi$, where

$$\Pi \begin{cases} x_0 \leqslant x \leqslant x_0 + h, \\ y_0 - \frac{k}{h}(x - x_0) \leqslant y \leqslant y_0 + \frac{k}{h}(x - x_0), \end{cases}$$

$$a = f_x(x_0, y_0), \quad b = f_y(x_0, y_0), \quad c = f(x_0, y_0) - ax_0 - by_0,$$

$$\alpha = \frac{1}{2h}[f(x_0 + h, y_0 + k) + f(x_0 + h, y_0 - k) - 2f(x_0, y_0)],$$

$$\beta = \frac{1}{2k}[f(x_0 + h, y_0 + k) - f(x_0 + h, y_0 - k)],$$

$$\gamma = f(x_0, y_0) - \alpha x_0 - \beta y_0.$$

This condition may be obtained from Taylor's formula for the function $f(x, y)$ in the neighbourhood of $(x_0, y_0)$.

THEOREM. *If the function $f(x, y)$ is of class $c^2$, $h \leqslant \frac{k}{M}$, $|f(x, y)| < M$ [2] and $f_y \neq 0$, the sufficient condition for inequality* (2.1) *is*

$$d^2 f > 0 \quad \textit{for } x \in [x_0, x_0 + h], \ y \in [y_0 - k, y_0 + k].$$

Proof. Now from Taylor's formula we shall obtain

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \\ + \tfrac{1}{2} d^2 f\Big|_{\substack{x_0 + \theta(x - x_0) \\ y_0 + \theta(y - y_0)}}, \quad \text{where } \theta \in [0, 1],$$

– this is the left-hand term of (2.1).

Let $\varphi(t)$ be a function defined on the straight line

$$x = x_0 + ht, \quad y = y_0 + kt,$$

as follows:

$$\varphi(t) = f(x_0 + ht, y_0 + kt) - a_0(x_0 + ht) - \beta(y_0 + kt) - \gamma.$$

From the above definition we have $\varphi(0) = \varphi(1) = 0$ and from the assumption that $d^2 f > 0$ we have also $d^2\varphi = d^2 f > 0$ for $t \in [0, 1]$. From Rolle's theorem it follows that $\varphi(t) \leqslant 0$ for $t \in [0, 1]$.

In a similar way one can prove that $\varphi(t) \leqslant 0$ on every edge of the triangle $\Pi_0$ as well as in the interior of $\Pi_0$. This ends the proof of inequality (2.1). Analogously, we can consider the inequality

$$(2.4) \qquad \alpha x + \beta y + \gamma \leqslant f(x, y) \leqslant ax + by + c$$

for $(x, y) \in \Pi$.

From inequality (2.1) we obtain the estimations

$$u(x) \leqslant y(x) \leqslant v(x) \quad \text{for } x \in [x_0, x_0+h] \tag{2.5}$$

and the initial conditions

$$u(x_0) = v(x_0) = y_0. \tag{2.6}$$

To the functions $u(x)$ and $v(x)$ the Czaplygin Theorem 1 could be applied. If the curves $y = u(x)$ and $y = v(x)$ pass through the point $M_0$ and if the differential inequalities are satisfied,

$$u'(x) - f\big(x, u(x)\big) < 0, \qquad v'(x) - f\big(x, v(x)\big) > 0, \tag{2.7}$$

we get for $x > x_0$

$$u(x) < y(x) < v(x), \tag{2.8}$$

$$v_n(x) - u_n(x) < \frac{C}{2^{2^n}}, \tag{2.9}$$

where $C$ depends neither on $x$ nor on $n$, and the convergence process by the Czaplygin method is very rapid.

**References**

[1] И. С. Березин и Н. П. Жидков, *Методы вычислений*, Т. II, Москва 1959.
[2] W. Nikliborc, *Równania różniczkowe*, cz. 1, Warszawa 1951.

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