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CONSTRUCTION OF A RECURRENCE RELATION OF THE LOWEST ORDER FOR COEFFICIENTS OF THE GEGENBAUER SERIES

0. INTRODUCTION

A function f defined in the interval $\langle -1, 1\rangle$ and satisfying the required conditions may be expanded into a series uniformly convergent in this interval with respect to the Gegenbauer polynomials $C_k^{(\lambda)}(x)$ $(\lambda > -1/2)$, orthogonal with weight $(1-x^2)^{\lambda-1/2}$ (in short: the Gegenbauer series):

$$f(x) = \sum_{k=0}^{\infty} a_k[f] C_k^{(\lambda)}(x).$$

Assume that f satisfies the linear differential equation

$$\sum_{i=0}^{n} p_i f^{(i)} = p \tag{0.1}$$

of order n, where $p_0, p_1, \ldots, p_n \not\equiv 0$ are polynomials in x, and the coefficients of the Gegenbauer series of the function p are known. Assume, moreover, that the derivative $f^{(n)}$ can be expanded into a uniformly convergent Gegenbauer series.

We give a method of constructing in view of equation (0.1) the linear recurrence relation of the lowest order,

$$\sum_{j=0}^{r} \lambda_j(k) a_{k+j}[f] = \mu(k) \qquad \text{for } k \geqslant 0,$$

where $\lambda_0, \lambda_1, \ldots, \lambda_r$ $(\lambda_0 \neq 0, \lambda_r \neq 0)$ are rational functions of the variable k.

1. FORMULATION OF THE PROBLEM

Let us write

$$(a)_0 = 1, \qquad (a)_k = a(a+1)\ldots(a+k-1) \qquad \text{for } k \geqslant 1. \tag{1.1}$$

The *k-th Gegenbauer polynomial* $C_k^{(\lambda)}(x)$ of order λ, where $k \geqslant 0$, and λ is a real number greater than $-1/2$, is for $\lambda \neq 0$ defined by the formula

$$C_k^{(\lambda)}(x) = (-1)^k \frac{(2\lambda)_k}{2^k k!(\lambda+1/2)_k}(1-x^2)^{1/2-\lambda}\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-1/2} \tag{1.2}$$

and for $\lambda = 0$ by the formula

$$C_k^{(0)}(x) = \lim_{\lambda\to 0}\lambda^{-1}C_k^{(\lambda)}(x) \tag{1.3}$$

(see Erdélyi [3], vol. II, § 10.9). Polynomials $C_k^{(\lambda)}(x)$ (called also *ultraspherical*) form for a fixed λ an orthogonal complete system in the interval $\langle -1,1\rangle$ with weight $(1-x^2)^{\lambda-1/2}$.

Typical orthogonal polynomials — the *Chebyshev polynomials of the first order* $T_k(x)$ and *of the second order* $U_k(x)$ and the *Legendre polynomials* $P_k(x)$ — are particular forms of the Gegenbauer polynomials. Namely, we have

$$\begin{gathered} T_0(x) = C_0^{(0)}(x), \quad T_k(x) = \frac{k}{2}C_k^{(0)}(x) \quad \text{for } k \geqslant 1, \\ U_k(x) = C_k^{(1)}(x) \quad \text{and} \quad P_k(x) = C_k^{(1/2)}(x) \quad \text{for } k \geqslant 0 \end{gathered} \tag{1.4}$$

(see [3], vol. II, §§ 10.10 and 10.11).

A function f which is continuous in the interval $\langle -1,1\rangle$ and satisfies required conditions (see, e.g., [3], vol. II, § 10.19, and [5], vol. I, § 8.5) can be expanded into a series uniformly convergent in this interval with respect to the Gegenbauer polynomials, i.e. into the *Gegenbauer series*

$$f(x) = \sum_{k=0}^{\infty} a_k[f]C_k^{(\lambda)}(x). \tag{1.5}$$

The coefficients $a_k[f]$ of this series are defined by the formula

$$a_k[f] = \frac{\int_{-1}^{1}(1-x^2)^{\lambda-1/2}C_k^{(\lambda)}(x)f(x)\,dx}{\int_{-1}^{1}(1-x^2)^{\lambda-1/2}[C_k^{(\lambda)}(x)]^2\,dx} \quad \text{for } k \geqslant 0. \tag{1.6}$$

We know (see [7], § 8) that in many cases amog the Gegenbauer series of a given function f, obtained for different values of the parameter λ, the most rapidly convergent series is that with respect to the polynomials $C_k^{(0)}(x)$, hence — in view of the two first equations (1.4) — the series with respect to the polynomials $T_k(x)$, called in the sequel the *Chebyshev series*:

$$f(x) = \frac{1}{2}b_0[f]T_0(x) + \sum_{k=1}^{\infty} b_k[f]T_k(x). \tag{1.7}$$

The *Chebyshev coefficients* $b_k[f]$ of the function f are defined by the formula

(1.8) $$b_k[f] = \frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{-1/2}T_k(x)f(x)\,dx \quad \text{for } k \geqslant 0.$$

Let for $k \geqslant 0$

(1.9) $$c_k[f] = \begin{cases} (k+\lambda)^{-1}a_k[f] & \text{for } \lambda \neq 0, \\ b_k[f] & \text{for } \lambda = 0, \end{cases}$$

where $a_k[f]$ and $b_k[f]$ are defined by (1.6) and (1.8), respectively.

Numbers $c_k[f]$ will be called the *Gegenbauer coefficients* of the function f.

In the sequel it will be convenient to use coefficients with negative indices. We assume that if $2\lambda = m$, where m is a non-negative integer, then (see [2])

(1.10) $$c_{-k}[f] = \begin{cases} 0 & \text{for } k = 1, 2, \ldots, m-1, \\ c_{k-m}[f] & \text{for } k \geqslant m, \end{cases}$$

and if 2λ is not an integer, we take

(1.11) $$c_{-k}[f] = 0 \quad \text{for } k \geqslant 1.$$

Assume that the first derivative of the function f can be expanded into a uniformly convergent Gegenbauer series. In view of equations

(1.12) $$\begin{aligned} 2(k+\lambda)C_k^{(\lambda)}(x) &= \frac{d}{dx}[C_{k+1}^{(\lambda)}(x) - C_{k-1}^{(\lambda)}(x)] \quad \text{for } k \geqslant 1, \\ 2\lambda C_0^{(\lambda)}(x) &= \frac{d}{dx}C_1^{(\lambda)}(x), \\ 2T_k(x) &= \frac{d}{dx}\left[\frac{1}{k+1}T_{k+1}(x) - \frac{1}{k-1}T_{k-1}(x)\right] \quad \text{for } k \geqslant 2, \\ 4T_1(x) &= \frac{d}{dx}T_2(x), \qquad T_0(x) = \frac{d}{dx}T_1(x) \end{aligned}$$

(see [3], vol. II, §§ 10.9 and 10.11) it is easy to show that

(1.13) $$2(k+\lambda)c_k[f] = c_{k-1}[f'] - c_{k+1}[f'] \quad \text{for } k \geqslant 1$$

(see, e.g., [1] and [2]).

Similarly, in view of equations

(1.14)

$$xC_k^{(\lambda)}(x) = \frac{1}{2(k+\lambda)}[(k+1)C_{k+1}^{(\lambda)}(x)+(k+2\lambda-1)C_{k-1}^{(\lambda)}(x)] \quad \text{for } k \geqslant 1,$$

$$xC_0^{(\lambda)}(x) = \frac{1}{2\lambda}C_1^{(\lambda)}(x),$$

$$xT_k(x) = \frac{1}{2}[T_{k+1}(x)+T_{k-1}(x)] \quad \text{for } k \geqslant 1,$$

$$xT_0(x) = T_1(x)$$

(see [3], vol. II, §§ 10.9 and 10.11) we have

(1.15) $$c_k[xf] = \frac{1}{2}\{a(k)c_{k-1}[f]+\beta(k)c_{k+1}[f]\} \quad \text{for } k \geqslant 1,$$

where

(1.16) $$a(k) = \begin{cases} k/(k+\lambda) & \text{for } \lambda \neq 0, \\ 1 & \text{for } \lambda = 0, \end{cases} \qquad \beta(k) = 2-a(k)$$

(see [1] and [2]).

Let us generalize relation (1.15). For an arbitrary function μ of the variable k we define

(1.17) $$\mu^+(k) = \mu(k+1) \quad \text{and} \quad \mu^-(k) = \mu(k-1).$$

By induction with respect to l it can be shown that

(1.18) $$c_k[x^l f] = 2^{-l}\sum_{j=0}^{l} a_{lj}(k)c_{k-l+2j}[f] \quad \text{for } k, l \geqslant 0,$$

where

$$a_{lj}(k) = \binom{l}{j} \quad \text{for } \lambda = 0,$$

and

$$a_{00}(k) \equiv 1,$$

(1.19) $$a_{lj} = \begin{cases} a a_{l-1,0}^- & \text{for } j = 0, \\ a a_{l-1,j}^- + \beta a_{l-1,j-1}^+ & \text{for } j = 1, 2, \ldots, l-1 \text{ and } l \geqslant 1, \\ \beta a_{l-1,l-1}^+ & \text{for } j = l \end{cases}$$

for $\lambda \neq 0$.

In general, it is very complicated to find the values of integrals in formulae (1.6) and (1.8). Explicit forms of the Gegenbauer coefficients resulting from these formulae were obtained for many functions (see, e.g.,

[5], vol. II, §§ 9.2 and 9.3, or [7], §§ 10-12). The expressions obtained are often very intricate and contain, for example, symbols of special functions, which makes difficulty when they are used.

It is relatively easy to find the values of the Gegenbauer coefficients if they satisfy a recurrence relation of the form

$$\sum_{j=0}^{r} \lambda_j(k) c_{k+j}[f] = \mu(k) \quad \text{for } k \geqslant 0, \tag{1.20}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_r$ and μ are known functions of the variable k. A particular solution of (1.18), satisfying additionally the so-called normalizing condition

$$\sum_{k=0}^{\infty} u_k c_k[f] = v,$$

where the constants u_k ($k \geqslant 0$) and $v \not\equiv 0$ are given, and the series on the left-hand side is convergent, can be obtained by using the well-known methods of Miller and Olver as well as modifications and generalizations of those methods (see [5], vol. II, § 12.5; [7], § 15; [6] and [9]).

Among methods of constructing the relations of form (1.20) for the Gegenbauer coefficients of a given function f, we can distinguish a very simple and universal method which may be applied in the case where this function satisfies the linear differential equation

$$\sum_{i=0}^{n} p_i f^{(i)} = p \tag{1.21}$$

of order n, where $p_0, p_1, \ldots, p_n \not\equiv 0$ are polynomials, and the Gegenbauer coefficients of the function p are known.

Let us assume that the derivative $f^{(n)}$ can be expanded into a uniformly convergent Gegenbauer series; it is then possible to expand into such a series each of the functions $f, f', \ldots, f^{(n-1)}$.

Let us write

$$c_k^{(i)} = c_k[f^{(i)}] \quad \text{for } i = 0, 1, \ldots, n.$$

We also write c_k, c_k', c_k'' instead of $c_k^{(0)}, c_k^{(1)}, c_k^{(2)}$, respectively.

The k-th Gegenbauer coefficients of both sides of equation (1.21) are equal, which implies the formula

$$\sum_{i=0}^{n} c_k[p_i f^{(i)}] = c_k[p] \quad \text{for } k \geqslant 0. \tag{1.22}$$

This relation is true also for any negative integer k, but (1.10) and (1.11) imply that in this case relation (1.22) is either 1° for $2\lambda = m$

(m — non-negative integer) equivalent to a relation obtained from (1.22) by substituting $-(k+m)\geqslant 0$ for k; or 2° trivial.

Every term of the sum on the left-hand side of (1.22) may be — in view of (1.18) — expressed in the form

$$c_k[p_i f^{(i)}] = L_0^{(i)} c_k^{(i)} \quad \text{for } i = 0, 1, \ldots, n, \tag{1.23}$$

where

$$L_0^{(i)} c_k^{(i)} = \sum_{j=-d_i}^{d_i} \lambda_j^{(i)}(k) c_{k+j}^{(i)}, \tag{1.24}$$

d_i denoting the degree of the polynomial p_i, and $\lambda_j^{(i)}$ ($j = -d_i, -d_i+ +1, \ldots, d_i$) being rational functions of the variable k.

Let $p_{i0}x^{d_i}$ be the monomial of the highest degree of the polynomial $p_i(x)$. It is easily seen that for $j = -d_i, d_i$ the coefficients $\lambda_j^{(i)}$ depend on this monomial and do not depend on the monomials of lower degrees of the polynomial $p_i(x)$.

If $\lambda \neq 0$, then by (1.18), (1.19) and (1.16), in view of the above remark, we have

$$\begin{aligned}\lambda_{-d_i}^{(i)}(k) &= p_{i0}\alpha(k)\alpha(k-1)\ldots\alpha(k-d_i+1) = p_{i0}\frac{(k-d_i+1)_{d_i}}{(k+\lambda-d_i+1)_{d_i}},\\ \lambda_{d_i}^{(i)}(k) &= p_{i0}\beta(k)\beta(k+1)\ldots\beta(k+d_i-1) = p_{i0}\frac{(k+2\lambda)_{d_i}}{(k+\lambda)_{d_i}}.\end{aligned} \tag{1.25}$$

If $\lambda = 0$ (according to formula (1.9), expressions of the form $\{c_k^{(i)}\}$ denote in this case the Chebyshev coefficients of the function $f^{(i)}$), then $\lambda_j^{(i)}$ in formula (1.24) are constant and such that

$$\begin{aligned}\lambda_j^{(i)} &= \lambda_{-j}^{(i)} \quad \text{for } j = 1, 2, \ldots, d_i,\\ \lambda_{-d_i}^{(i)} &= \lambda_{d_i}^{(i)} = p_{i0}.\end{aligned} \tag{1.26}$$

Equations (1.22) and (1.23) imply

$$\sum_{i=0}^{n} L_0^{(i)} c_k^{(i)} = c_k[p] \quad \text{for } k \geqslant 0. \tag{1.27}$$

This equation and following from (1.13) relations

$$2(k+\lambda)c_k^{(i-1)} = c_{k-1}^{(i)} - c_{k+1}^{(i)} \quad \text{for } i = 1, 2, \ldots, n, \tag{1.28}$$

considered for $k \geqslant 0$, form a system of $n+1$ linear difference equations with unknown sequences $\{c_k^{(i)}\}$ ($i = 0, 1, \ldots, n$).

Clenshaw [1] has proposed an approximate method of solving this system for $\lambda = 0$, which was applied by Elliott [2] for the case of $\lambda \neq 0$.

In applications to particular differential equations the Gegenbauer coefficients (exclusively in the case of $\lambda = 0$) of all derivatives of the function f were sometimes eliminated, which finally implied a single recurrence relation of type (1.20) (see, e.g., [5], vol. II, § 16.2).

A *general algorithm* for the construction of relation (1.20) for the Chebyshev coefficients, based on the differential equation (1.21) and equations (1.12) and (1.14), was given by Paszkowski [7], § 13; the order of this relation is equal to

$$(1.29)\qquad 2 \max_{\substack{0\leqslant i\leqslant n\\ p_{n-i}\neq 0}} (d_{n-i}+i).$$

Paszkowski [7] observed that system (1.27), (1.28) sometimes implies a recurrence relation of the order lower than (1.29).

We present in this paper an *optimum algorithm*, i.e. a method of constructing the recurrence relation (1.20) for the Gegenbauer coefficients with the *lowest order* among relations following from (1.27), (1.28). (Let us observe that in [7] there is given an optimum algorithm but only for the simplest case of $\lambda = 0$ and $n = 1$.)

In Section 2 we give the formulation and the proof of the correctness of the optimum algorithm.

In Section 3 we describe an algorithm which can be considered as the generalization of the first of two above-mentioned Paszkowski algorithms.

In Section 4 we show that if the coefficient p_n of the differential equation (1.21) has no zeros equal to 1 or -1, then the optimum algorithm described in Section 2 leads to a recurrence relation for the Chebyshev coefficients ($\lambda = 0$) with order (1.29). This means that in this case the Paszkowski algorithm is the optimum one.

2. OPTIMUM ALGORITHM

We give now definitions and fundamental properties of difference operators (Section 2.1), then introduce the set $A(L)$ and describe the construction of minimum operator in this set (Section 2.2). This operator plays an important role in the proposed method of constructing the recurrence relation of the lowest order for the Gegenbauer coefficients of the function satisfying the differential equation (1.21). After having defined auxiliary operators in Section 2.3, this method is presented in Section 2.4; main results are contained in Theorem 2.1. Having in view the practical realization of the procedure there described, we finally give its compact formulation which will be called *Algorithm I* (Section 2.5).

2.1. Difference operators.

Definition 2.1. Let S denote a linear space of complex number sequences with addition of sequences and multiplication of a sequence by a constant defined as usually. *Difference operator* is an operator L mapping the space $\boldsymbol{S}$ into itself and such that

$$L\{z_k\} = \Big\{\sum_{j=0}^{r} \lambda_j(k) z_{k+u+j}\Big\} \quad \text{for } \{z_k\} \in \boldsymbol{S},$$

where $r = r(L) \geqslant 0$ and $u = u(L)$ are integers, and the coefficients $\lambda_0, \lambda_1, \ldots, \lambda_r$ ($\lambda_0 \not\equiv 0$, $\lambda_r \not\equiv 0$) are known functions. The number r is called the *order of the operator L*.

Obviously, every difference operator is linear. Let us emphasize that in this paper only those operators are used the coefficients of which are rational functions of the variable k. The set of all these operators will be denoted by $\boldsymbol{L}$. By "operator" we always mean a difference operator.

Let E^m (m — an integer) and I be operators with the following properties:

(2.1) $$E^m\{z_k\} = \{z_{k+m}\}, \quad I\{z_k\} = \{z_k\} \quad \text{for } \{z_k\} \in \boldsymbol{S}.$$

I will be called the *identity operator*. It follows from (2.1) that $E^0 = I$.

The *zero operator* Θ, associating with any sequence $\{z_k\} \in \boldsymbol{S}$ the sequence $\{0\}$ composed of mere zeros, belongs also to difference operators although it does not satisfy conditions of Definition 2.1. Remark that the symbols $r(\Theta)$ and $u(\Theta)$ have no definite meaning. It is convenient to assume that $r(\Theta) = -1$.

Two operators L_1 and L_2 are *equal* if $L_1\{z_k\} = L_2\{z_k\}$ for every $\{z_k\} \in \boldsymbol{S}$.

Addition and multiplication by a number are for difference operators defined in a natural manner.

Definition 2.2. Let L and M be difference operators. The operator $N = c_1 L + c_2 M$ (c_1 and c_2 — complex numbers) is defined by

$$N\{z_k\} = c_1 L\{z_k\} + c_2 M\{z_k\} \quad \text{for } \{z_k\} \in \boldsymbol{S}.$$

This definition implies the linearity of the space $\boldsymbol{L}$.

Let us also introduce the composition of operators.

Definition 2.3. Let L and M belong to $\boldsymbol{L}$. An operator $P = LM$ is such a one for which $P\{z_k\} = L\{y_k\}$, where $\{y_k\} = M\{z_k\}$ ($\{z_k\} \in \boldsymbol{S}$).

Composition of operators is associative and distributive (from the left and from the right) with respect to addition of operators. (The space $\boldsymbol{L}$ is, therefore, a ring with unity formed by the operator I.) Since it is possible that $LM \neq ML$, we see that, in general, the composition of operators is not commutative. If $LM = ML$, we say that operators L and M are *commutative*. In particular, any two operators with constant coefficients are commutative.

It follows from (2.1) that

$$E^l E^m = E^{l+m}, \quad \text{where } l \text{ and } m \text{ are integers.}$$

If a is a rational function and $M \epsilon \boldsymbol{L}$, then we write

$$a(k)M = LM, \quad \text{where } L\{z_k\} = \{a(k)z_k\}.$$

It is easy to see that for any non-zero operators L and M we have

$$r(L+M) \leqslant \max\{r(L)+u(L), r(M)+u(M)\} - \min\{u(L), u(M)\} \quad (L \neq -M), \tag{2.2}$$

$$u(L+M) \geqslant \min\{u(L), u(M)\} \tag{2.3}$$

$$r(LM) = r(L)+r(M), \quad u(LM) = u(L)+u(M), \tag{2.4}$$

where we use notation of Definition 2.1.

Take $l = r(L)$, $m = r(M)$, and let $\lambda_i(k)$ $(i = 0, 1, \ldots, l)$ and $\mu_i(k)$ $(i = 0, 1, \ldots, m)$ be coefficients of the operators L and M, respectively. We have the sharp inequality in (2.2) if and only if

$$u(L) = u(M), \quad \lambda_0(k)+\mu_0(k) \equiv 0 \tag{2.5}$$

or

$$l+u(L) = m+u(M), \quad \lambda_l(k)+\mu_m(k) \equiv 0, \tag{2.6}$$

whereas the sharp inequality in (2.3) occurs only in the case where equalities (2.5) are satisfied.

Assume that Lz_k denotes the k-th element of the sequence $L\{z_k\}$ $(L \epsilon \boldsymbol{L};\ \{z_k\} \epsilon \boldsymbol{S})$.

Definition 2.4. Let $L \epsilon \boldsymbol{L}$, and let μ be a given function and $\{z_k\}$ an unknown sequence from $\boldsymbol{S}$. The relation

$$Lz_k = \mu(k) \quad \text{for } k \geqslant m \ (m \text{ — an integer})$$

will be called the (*linear*) *recurrence relation* or the *difference equation*; the *order of a recurrence relation* is the order of the operator L.

2.2. Minimum operator of the set $\boldsymbol{A}(L)$. Let D be the operator defined by the formula

$$D = E^{-1} - E. \tag{2.7}$$

Definition 2.5. Let L be a given operator and assume that for a non-zero operator A there exists an operator Q such that $AL = QD$.

The set of all operators having this property will be denoted by $\boldsymbol{A}(L)$.

$A^* \epsilon \boldsymbol{A}(L)$ will be called a *minimum operator* if for every $A \epsilon \boldsymbol{A}(L)$ we have $r(A^*) \leqslant r(A)$.

It is easy to show the following

LEMMA 2.1. *If $A \in \boldsymbol{A}(L)$ and C is a non-zero operator, then $CA \in \boldsymbol{A}(L)$. If $A_1, A_2 \in \boldsymbol{A}(L)$ and $A = A_1 + A_2$ is a non-zero operator, then $A \in \boldsymbol{A}(L)$.*

We prove now

LEMMA 2.2. *If $A, A^* \in \boldsymbol{A}(L)$ and A^* is a minimum operator in the set $\boldsymbol{A}(L)$, then there exists a non-zero operator C such that*

$$A = CA^*. \tag{2.8}$$

In particular, if A is also a minimum operator in the set $\boldsymbol{A}(L)$, then (2.8) *holds for $C = \varrho(k)E^m$, where ϱ is a rational function, $\varrho \not\equiv 0$, and m — an integer.*

Proof. We define a sequence of operators $A_0, A_1, \ldots$ Let $A_0 = A$. For $i = 0, 1, \ldots$ such that the operator A_i is defined and different from the operator Θ we write

$$A_{i+1} = A_i - C_i A^*, \tag{2.9}$$

where C_i is a non-zero operator for which we have

$$\begin{gathered} r(C_i) = r(A_i) - r(A^*), \qquad u(C_i) = u(A_i) - u(A^*), \\ \gamma_{i0}(k) = a_{i0}(k)/a_0^*\big(k + u(C_i)\big), \end{gathered} \tag{2.10}$$

while γ_{i0}, a_{i0} and a_0^* denote the zero coefficients of the operators C_i, A_i and A^*, respectively.

If A_{i+1} $(i \geqslant 0)$ is a non-zero operator, then in view of Lemma 2.1 we infer that $A_{i+1} \in \boldsymbol{A}(L)$. Moreover, from (2.9) it follows then — by virtue of (2.2), (2.4), (2.10) and (2.5) — that $r(A_{i+1}) < r(A_i)$. Therefore, after a finite number of steps we obtain the operator $A_p \in \boldsymbol{A}(L)$ such that

$$r(A_p) = r(A^*), \tag{2.11}$$

or

$$r(A_p) > r(A^*), \qquad A_{p+1} = \Theta.$$

We are going to show that also in the first case we have $A_{p+1} = \Theta$. Indeed, if we assume that $A_{p+1} \neq \Theta$, then the preceding remarks imply that $A_{p+1} \in \boldsymbol{A}(L)$ and that $r(A_{p+1}) < r(A_p)$, which is a contradiction since the operator A_p is (by (2.11)) the minimum one in the set $\boldsymbol{A}(L)$.

Thus in any case formula (2.9) implies

$$\begin{aligned} A_0 &= C_0 A^* + A_1 = C_0 A^* + C_1 A^* + A_2 = \ldots \\ &= C_0 A^* + C_1 A^* + \ldots + C_p A^* + A_{p+1} \\ &= (C_0 + C_1 + \ldots + C_p) A^*, \end{aligned}$$

which means that equality (2.8) holds for $C = C_0 + C_1 + \ldots + C_p$.

If A is a minimum operator in the set $\boldsymbol{A}(L)$, then $r(A) = r(A^*)$ and, therefore, equation (2.8) and the first relation of (2.4) imply that C is an operator of the zero order, hence it is of the form $\varrho(k)E^m$, where ϱ is a rational function, $\varrho \not\equiv 0$, and m is an integer, q.e.d.

We now show that if L is an operator of order not greater than 1, then the set $\boldsymbol{A}(L)$ is not empty, and next we give for this case a formula defining the minimum operator in $\boldsymbol{A}(L)$ (Lemma 2.3). We also show that for any operator L the minimum operator in the set $\boldsymbol{A}(W)$, where W is an operator of order not greater than 1 connected with L (and defined in Lemma 2.4), belongs to the set $\boldsymbol{A}(L)$ and is in this set the minimum operator (Lemma 2.5).

Let λ_i and μ_i $(i = 1, 2)$ be polynomials and let $a_i = \lambda_i/\mu_i$. Let us write

$$\{a_1, a_2\} = \frac{\gcd(\lambda_1, \lambda_2)}{\gcd(\mu_1, \mu_2)},$$

where the symbol $\gcd(\varphi_1, \varphi_2)$ denotes the greatest common divisor of polynomials φ_1 and φ_2. The symbol $\{a_1, a_2, a_3\}$ has the analogical meaning.

LEMMA 2.3. *Let W be an operator defined by the formula*

$$W = \eta(k)I + \vartheta(k)E, \tag{2.12}$$

where η and ϑ are rational functions. Define the operators A and R in the following manner:

$$A = \frac{1}{\omega(k)} \cdot \begin{cases} I & \text{for } \eta = \vartheta \equiv 0 \ (\text{case I}), \\ \eta^+(k)I - \eta(k)E & \text{for } \eta = \vartheta \not\equiv 0 \ (\text{case II}), \\ \eta^+(k)I + \eta(k)E & \text{for } \eta = -\vartheta \not\equiv 0 \ (\text{case III}), \\ \eta^+(k)E^{-1} - \eta^-(k)E & \text{for } \eta \not\equiv 0,\ \vartheta = c\eta,\ c = \text{const}, \\ & \qquad c^2 \neq 1 \ (\text{case IV}), \\ \vartheta^+(k)E^{-1} - \vartheta^-(k)E & \text{for } \vartheta \not\equiv 0,\ \eta = c_1\vartheta,\ c_1 = \text{const}, \\ & \qquad c_1^2 \neq 1 \ (\text{case V}), \\ a(k)E^{-1} + \beta(k)I - a^-(k)E & \text{for } \eta/\vartheta \not\equiv \text{const} \ (\text{case VI}), \end{cases} \tag{2.13}$$

$$R = \frac{1}{\omega(k)} \cdot \begin{cases} \Theta & (\text{case I}), \\ \eta(k)\eta^+(k)E & (\text{case II or III}), \\ \eta^-(k)\eta^+(k)[I + cE] & (\text{case IV}), \\ \vartheta^-(k)\vartheta^+(k)[c_1 I + E] & (\text{case V}), \\ a(k)\eta^-(k)I + a^-(k)\vartheta^+(k)E & (\text{case VI}), \end{cases} \tag{2.14}$$

where

$$
\omega = \begin{cases} 1 & (\textit{case } \mathrm{I}), \\ \{\eta^+, \eta\} & (\textit{case } \mathrm{II} \textit{ or } \mathrm{III}), \\ \{\eta^-, \eta^+\} & (\textit{case } \mathrm{IV}), \\ \{\vartheta^-, \vartheta^+\} & (\textit{case } \mathrm{V}), \\ \{a, \beta, a^-\} & (\textit{case } \mathrm{VI}), \end{cases}
$$

(2.15)
$$a = \eta\eta^+ - \vartheta\vartheta^+, \qquad \beta = \eta^-\vartheta^+ - \eta^+\vartheta^-.$$

(Notation in these formulae is that of (1.17).)

So in any case I-VI *the operator A belongs to the set $\boldsymbol{A}(W)$ and is in this set the minimum one. We also have*

(2.16)
$$AW = RD.$$

Proof. For $\eta = \vartheta \equiv 0$ (case I) we have $W = \Theta$ and relation (2.16) holds for

(2.17)
$$A = I, \quad R = \Theta.$$

Let $\eta \not\equiv 0$ or $\vartheta \not\equiv 0$. Assume that

(2.18)
$$A = E^m[a(k)E^{-1} + \beta(k)I + \gamma(k)E],$$

(2.19)
$$R = E^m[\varphi(k)I + \psi(k)E],$$

where m is an integer and $a, \beta, \gamma, \varphi$ and ψ denote rational functions. Let us substitute for the symbols A, W, R and D the right-hand sides of formulae (2.18), (2.12), (2.19) and (2.7), respectively, next perform the corresponding operations and equal coefficients of the operators on both sides of the relation obtained. We arrive at 4 equations

(2.20)
$$a\eta^- = \varphi, \quad a\vartheta^- + \beta\eta = \psi, \quad \beta\vartheta + \gamma\eta^+ = -\varphi, \quad \gamma\vartheta^+ = -\psi,$$

where we use the relationship $E^{\pm 1}\eta(k) = \eta(k \pm 1) = \eta^{\pm}(k)$ etc. (see (2.1) and (1.17)). By eliminating φ and ψ we get the system

(2.21)
$$a\eta^- + \beta\vartheta + \gamma\eta^+ = 0, \quad a\vartheta^- + \beta\eta + \gamma\vartheta^+ = 0.$$

It is easily seen that if any two of functions a, β and γ are vanishing identically, then the third of them vanishes also identically. This means that the set $\boldsymbol{A}(W)$ does not contain operators of the zero order.

Let us successively examine the cases defined in (2.13). Assume that case VI occurs, i.e. $\eta \not\equiv 0$, $\vartheta \not\equiv 0$ and there exists no constant c such that $\vartheta = c\eta$. We will show that then the set $\boldsymbol{A}(W)$ does not contain operators of the first order. Remark to this end that none of functions

$$\Delta_1 = \begin{vmatrix} \vartheta & \eta^+ \\ \eta & \vartheta^+ \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \eta^- & \eta^+ \\ \vartheta^- & \vartheta^+ \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \eta^- & \vartheta \\ \vartheta^- & \eta \end{vmatrix}$$

vanishes identically. Indeed, if, e.g., $\Delta_2 \equiv 0$, then $\vartheta^-/\eta^- = \vartheta^+/\eta^+$, i.e. the rational function ϑ/η has the period equal to 2. The only rational function with this property is the constant function, hence $\vartheta = c\eta$ (c = const), contrary to the assumption. If we assume that $\Delta_1 \equiv 0$ or $\Delta_3 \equiv 0$, the conclusion will be the same. It is thus clear that every non-trivial solution α, β, γ of system (2.21) is such that $\alpha \not\equiv 0$, $\beta \not\equiv 0$ and $\gamma \not\equiv 0$. Therefore, if $A \in \boldsymbol{A}(W)$, the order of A cannot be lower than 2. An operator of the second order having the form of (2.18) belongs to the set $\boldsymbol{A}(W)$ (and is in this set a minimum operator) if $\alpha/\beta = -\Delta_1/\Delta_2$, $\gamma/\beta = -\Delta_3/\Delta_2$. For instance, we can put

$$\alpha = -\Delta_1 = \eta\eta^+ - \vartheta\vartheta^+, \quad \beta = \Delta_2 = \eta^-\vartheta^+ - \eta^+\vartheta^-, \quad \gamma = -\Delta_3 = -\alpha^-, \tag{2.22}$$

which together with the first and the last of equations (2.20) implies

$$\varphi = \alpha\eta^-, \quad \psi = \alpha^-\vartheta^+. \tag{2.23}$$

Assume now that $\eta \not\equiv 0$, $\vartheta = c\eta$, where c = const. Relations (2.21) take the forms

$$\alpha\eta^- + c\beta\eta + \gamma\eta^+ = 0, \quad c\alpha\eta^- + \beta\eta + c\gamma\eta^+ = 0. \tag{2.24}$$

Let us consider 3 cases according to values of the constant c.

If $c^2 \neq 1$ (case IV), then from (2.24) we have

$$\beta \equiv 0, \quad \alpha\eta^- + \gamma\eta^+ = 0.$$

Therefore, if $A \in \boldsymbol{A}(W)$, then the order of operator A cannot be lower than 2. The second order operator will be obtained, e.g., for

$$\alpha = \eta^+, \quad \beta \equiv 0, \quad \gamma = -\eta^-. \tag{2.25}$$

In this case, the first and the last of equations (2.20) imply

$$\varphi = \eta^-\eta^+, \quad \psi = c\varphi. \tag{2.26}$$

If $c = \pm 1$ (case II or III), then system (2.24) will be reduced to the single relation

$$\alpha\eta^- + c\beta\eta + \gamma\eta^+ = 0.$$

Operator (2.18), belonging to the set $\boldsymbol{A}(W)$, is, therefore, of the first order if $\alpha \equiv 0$, $c\beta\eta + \gamma\eta^+ = 0$ or if $\gamma \equiv 0$, $\alpha\eta^- + c\beta\eta = 0$. In particular, we can assume that

$$\alpha \equiv 0, \quad \beta = \eta^+, \quad \gamma = -c\eta. \tag{2.27}$$

In this case (see (2.20)) we have

$$\varphi \equiv 0, \quad \psi = \eta\eta^{+}. \tag{2.28}$$

If $\vartheta \not\equiv 0$, $\eta = c_1\vartheta$, $c_1 = \text{const}$, $c_1^2 \neq 1$ (case V), then similarly as in case IV we state that in the set $\boldsymbol{A}(W)$ there exist no operators of the order lower than 2. Operator (2.18) belongs to this set and it is of the second order (and thus is the minimum operator), if

$$\alpha = \vartheta^{+}, \quad \beta \equiv 0, \quad \gamma = -\vartheta^{-}. \tag{2.29}$$

In formula (2.19) we have then to put

$$\psi = \vartheta^{-}\vartheta^{+}, \quad \varphi = c_1\psi. \tag{2.30}$$

Formulae (2.13)-(2.15) follow from (2.18) and (2.19) for $m = 0$, and from (2.22), (2.23) and (2.25)-(2.30). The factor $1/\omega$ in (2.13) and (2.14) is introduced in order to simplify the forms of operators A and R, which by Lemma 2.2 does not change minimality of the operator A, q.e.d.

Let L be a non-zero operator of order r with coefficients $\lambda_0, \lambda_1, \ldots, \lambda_r$. Let us write

$$\sigma_1(L; k) = \sum_{j=0}^{[r/2]} \lambda_{2j}(k), \quad \sigma_2(L; k) = \sum_{j=0}^{[(r-1)/2]} \lambda_{2j+1}(k). \tag{2.31}$$

(By $[a]$ we denote the integer part of the number a.) We assume, moreover, that $\sigma_h(\Theta; k) \equiv 0$ $(h = 1, 2)$. Functions $\sigma_h(L; k)$ are obviously rational in k.

It is easy to see that for m being an integer we have

$$\sigma_h(E^m L; k) = E^m \sigma_h(L; k), \quad \sigma_h(LE^m; k) = \sigma_h(L; k) \quad (h = 1, 2). \tag{2.32}$$

If ϱ is a rational function, then

$$\sigma_h\big(\varrho(k)L; k\big) = \varrho(k)\sigma_h(L; k) \quad (h = 1, 2). \tag{2.33}$$

Let $L_1, L_2 \in \boldsymbol{L}$ and $L = L_1 + L_2$. We consider four cases.

Case I. $L_1 = \Theta$ or $L_2 = \Theta$ or $L = \Theta$.

Case II. Numbers $u(L_1)$ and $u(L_2)$ are different and either both even or both odd.

Case III. $L \neq \Theta$, $u(L_1) = u(L_2) \neq u(L)$; a. $u(L)$ is even, b. $u(L)$ is odd.

Case IV. $u(L_1) - u(L_2)$ is odd.

In cases II and IV we obviously have $L \neq \Theta$. We easily get the relation

$$\sigma_h(L; k) = \sigma_{h_1}(L_1; k) + \sigma_{h_2}(L_2; k) \quad \text{for } h = 1, 2, \tag{2.34}$$

where

$$h_1 = h_2 = \begin{cases} h & \text{(case I, II or IIIa)}, \\ 3-h & \text{(case IIIb)} \end{cases}$$

or

$$h_i = h, \quad h_{3-i} = 3-h \quad \text{(case IV)},$$

whereas i ($i = 1$ or 2) is such that $u(L_i)$ is equal to the less of the numbers $u(L_1)$ and $u(L_2)$.

It is easy to show the following

LEMMA 2.4. *Let L be a non-zero operator of order r such that*

$$Lz_k = \sum_{j=0}^{r} \lambda_j(k) z_{k+u+j}.$$

Let N be an operator such that $N = \Theta$ for $r \leqslant 1$, and

$$Nz_k = \sum_{j=0}^{r-2} \nu_j(k) z_{k+u+1+j} \quad \textit{for } r > 1,$$

where the coefficients $\nu_0, \nu_1, \ldots, \nu_{r-2}$ are defined by the recurrence formula

$$\nu_j = \nu_{j+2} - \lambda_{j+2} \quad \textit{for } j = r-2, r-3, \ldots, 0; \ \nu_r = \nu_{r-1} \equiv 0.$$

Then

$$L = ND + WE^u, \tag{2.35}$$

where D is defined by (2.7) *and W by the formula*

$$W = \sigma_1(L; k) I + \sigma_2(L; k) E.$$

Lemma 2.5 contains the most important result of this section.

LEMMA 2.5. *Let L be the operator defined in Lemma* 2.4. *The operator A, defined by* (2.13) *for $\eta = \sigma_1(L; k)$ and $\vartheta = \sigma_2(L; k)$, belongs to the set $\boldsymbol{A}(L)$ and it is the minimum operator in this set.*

Proof. Let N and W be two operators defined as in Lemma 2.4. We will show that

$$\boldsymbol{A}(L) = \boldsymbol{A}(W), \tag{2.36}$$

which together with Lemma 2.3 will prove Lemma 2.5.

1° If $C \in \boldsymbol{A}(L)$, then according to Definition 2.5 there exists an operator Q such that $CL = QD$, which together with (2.35) implies the relation $CW = (Q - CN)E^{-u}D$ meaning that $C \in \boldsymbol{A}(W)$.

2° Let $C \in \boldsymbol{A}(W)$. Applying the operator C to both sides of (2.35) and taking into account that there exists an operator S such that $CW = SD$, we have $CL = (CN + SE^u)D$. Hence $C \in \boldsymbol{A}(L)$.

Thus we have proved relation (2.36).

2.3. Auxiliary operators and their properties. Let B_i $(i = 0, 1, \ldots)$ be an operator defined by the formula

$$(2.37)\qquad B_i = \begin{cases} D & \text{for } i = 0, \\ (k+\lambda)^{-1}[a_i(k)E^{-1} - \beta_i(k)E] & \text{for } i = 1, 2, \ldots, \end{cases}$$

where

$$(2.38)\qquad a_i(k) = (k+\lambda+i-1)_2, \quad \beta_i(k) = (k+\lambda-i)_2,$$

notation being that of Section 1.

Define operators S_{ij} and P_i by the formulae

$$(2.39)\qquad S_{ij} = \begin{cases} I & \text{for } i < j, \\ B_i B_{i-1} \ldots B_j & \text{for } i \geqslant j \geqslant 0, \end{cases}$$

and

$$(2.40)\qquad P_i = S_{i-1,0} \quad \text{for } i = 0, 1, \ldots$$

It is easy to see that

$$(2.41)\qquad S_{ij} = B_i S_{i-1,j} = S_{i,j+1} B_j \quad \text{for } i \geqslant j \geqslant 0,$$

$$(2.42)\qquad P_i = \begin{cases} S_{i-1,j} P_j & \text{for } i \geqslant j \geqslant 0, \\ B_{i-1} P_{i-1} & \text{for } i = 1, 2, \ldots \end{cases}$$

Let $\gamma_0, \gamma_1, \ldots$ be polynomials in k defined by the formulae

$$(2.43)\qquad \gamma_0(k) \equiv 1, \quad \gamma_i(k) = (k+\lambda-i+1)_{2i-1} \quad \text{for } i = 1, 2, \ldots$$

Equations (2.38) and (2.43) imply

$$(2.44)\qquad \gamma_{i+1} = a_i \gamma_i^- = \beta_i \gamma_i^+ \quad \text{for } i = 1, 2, \ldots,$$

notation being that of (1.17).

Using the symbol defined in (2.7), relation (1.28) can be rewritten in the form

$$(2.45)\qquad Dc_k^{(i)} = 2(k+\lambda)c_k^{(i-1)} \quad \text{for } i > 0,$$

which allows us to obtain the relationship between c_k and the terms of the sequence $\{c_k^{(i)}\}$ for $i > 0$. Namely, we prove

LEMMA 2.6. *For any* $i = 0, 1, \ldots$ *we have*

$$(2.46)\qquad P_i c_k^{(i)} = 2^i \gamma_i(k) c_k.$$

Proof (by induction). According to (2.40) and (2.43) we have $P_0 = I$ and $\gamma_0 \equiv 1$, which implies that for $i = 0$ equation (2.46) holds trivially true.

Since $P_1 = B_0 = D$ and $\gamma_1(k) = k+\lambda$ (see (2.40), (2.37) and (2.43)), we see that for $i = 1$ relation (2.46) is of the form

$$Dc'_k = 2(k+\lambda)c_k$$

and, therefore, by virtue of (2.45), it is true.

Assume that equality (2.46) holds for a certain $i \geqslant 1$. Relations (2.42), (2.37), (2.44) and the foregoing equality imply

$$\begin{aligned} P_{i+1}c_k^{(i+1)} &= B_iP_ic_k^{(i+1)} = 2^iB_i\gamma_i(k)c'_k \\ &= 2^i(k+\lambda)^{-1}[a_i(k)\gamma_i^-(k)c'_{k-1}-\beta_i(k)\gamma_i^+(k)c'_{k+1}] \\ &= 2^i(k+\lambda)^{-1}\gamma_{i+1}(k)Dc'_k = 2^{i+1}\gamma_{i+1}(k)c_k. \end{aligned}$$

Identity (2.46) is, therefore, true for any $i \geqslant 0$, q.e.d.

The second of two lemmata, which we shall prove, defines coefficients of the operator $S_{ij}(\gamma_j(k)I)$ which will immediately imply the form of the operator P_i. These informations will be used in Sections 2.4 and 3.2.

Let

$$(2.47)\qquad \varrho_{ijm}(k) = (-1)^m\binom{i-j}{m}(k+\lambda-i)_{j+m}(k+\lambda-i+j+2m)\times \\ \times(k+\lambda+m+1)_{i-m}[(k+\lambda)^2-i^2]^{-1}$$

for $i = 1, 2, \ldots; j = 0, 1, \ldots, i$ and $m = 0, 1, \ldots, i-j$.

It is easily seen that ϱ_{ijm} is a polynomial in k of degree $i+j-1$. Moreover, let us remark that putting $\lambda = 0$ into the right-hand side of (2.47) and substituting $i-j$ for j we obtain a polynomial denoted in [7], § 13, by $b_{ijm}(k)$ (see also Section 3.2).

LEMMA 2.7. *For every* $i = 1, 2, \ldots$ *we have*

$$(2.48)\qquad \begin{cases} \varrho_{i+1,j0} = (k+\lambda)^{-1}a_i\varrho_{ij0}^-, \\ \varrho_{i+1,jm} = (k+\lambda)^{-1}(a_i\varrho_{ijm}^- - \beta_i\varrho_{ij,m-1}^+) \quad \textit{for } m = 1, 2, \ldots, i-j, \\ \varrho_{i+1,j,i+1-j} = -(k+\lambda)^{-1}\beta_i\varrho_{ij,i-j}^+, \end{cases}$$

where $j = 0, 1, \ldots, i$ *and notation is that of* (1.17) *and* (2.38).

Proof. We introduce an auxiliary variable $\varkappa = k+\lambda$.

First we prove the first equality of system (2.48). According to (2.38), (2.47) and (1.17) the right-hand side of this equality may be rewritten in the form

$$\varkappa^{-1}(\varkappa+i-1)_2(\varkappa-i-1)_j(\varkappa-i+j-1)(\varkappa)_i[(\varkappa-1)^2-i^2]^{-1}.$$

Divide this expression by $\varkappa+i-1$ and then multiply and divide it by $\varkappa+i+1$ and use the equality

$$\varkappa^{-1}(\varkappa)_i(\varkappa+i)(\varkappa+i+1) = (\varkappa+1)_{i+1}.$$

We get then

$$(\varkappa-i-1)_j(\varkappa-i+j-1)(\varkappa+1)_{i+1}[\varkappa^2-(i+1)^2]^{-1}.$$

This is exactly the expression for $\varrho_{i+1,j0}$ which is implied by (2.47). We can similarly prove the last relation of system (2.48).

We will show that the second equality of this system holds. Its right-hand side, by (2.38), (2.47) and (1.17), is of the form

$$\varkappa^{-1}\Big\{(\varkappa+i-1)_2(-1)^m\binom{i-j}{m}(\varkappa-i-1)_{j+m}(\varkappa-i+j+2m-1)(\varkappa+m)_{i-m}\times$$
$$\times[(\varkappa-1)^2-i^2]^{-1}-(\varkappa-i)_2(-1)^{m-1}\binom{i-j}{m-1}(\varkappa-i+1)_{j+m-1}\times$$
$$\times(\varkappa-i+j+2m-1)(\varkappa+m+1)_{i-m+1}[(\varkappa+1)^2-i^2]^{-1}\Big\}.$$

Let us divide the minuend in brackets by $\varkappa+i-1$, and the subtrahend by $\varkappa-i+1$. Multiplying and dividing the minuend by $\varkappa+i+1$, and the subtrahend by $\varkappa-i-1$ and next using the equalities

$$(\varkappa+m)_{i-m}(\varkappa+i)(\varkappa+i+1) = (\varkappa+m)(\varkappa+m+1)_{i+1-m},$$
$$(\varkappa-i-1)(\varkappa-i)(\varkappa-i+1)_{j+m-1} = (\varkappa-i-1)_{j+m}(\varkappa-i+j+m-1),$$

we get the expression

$$\varkappa^{-1}\Big\{\binom{i-j}{m}(\varkappa+m)+\binom{i-j}{m-1}(\varkappa-i+j+m-1)\Big\}(-1)^m(\varkappa-i-1)_{j+m}\times$$
$$\times(\varkappa-i-1+j+2m)(\varkappa+m+1)_{i+1-m}[\varkappa^2-(i+1)^2]^{-1}.$$

Since the sum in brackets is equal to $\binom{i+1-j}{m}\varkappa$, this expression is identical with that which may be obtained for $\varrho_{i+1,j,m}$ from formula (2.47), q.e.d.

LEMMA 2.8. *For every $i = 1, 2, \ldots$ and $j = 0, 1, \ldots, i$ we have*

(2.49) $$S_{i-1,j}\big(\gamma_j(k)z_k\big) = \sum_{m=0}^{i-j}\varrho_{ijm}(k)z_{k-i+j+2m} \quad \textit{for } \{z_k\}\in\mathbf{S},$$

where the notation used is that of (2.39), (2.43) *and* (2.47).

Proof. Let us first examine the case of $i = j$. Since

$$S_{i-1,i} = I,$$
$$\varrho_{ii0}(k) = (k+\lambda-i)_i(k+\lambda)(k+\lambda+1)_i[(k+\lambda)^2-i^2]^{-1}$$
$$= (k+\lambda-i+1)_{2i-1} = \gamma_i(k),$$

we have

$$S_{i-1,}\big(\gamma_i(k)z_k\big) = \varrho_{ii0}(k)z_k \quad \text{for } i = 1, 2, \dots$$

Let us now assume that $j < i$ and proceed by induction on i. Let $i = 1$. The left-hand side of (2.49) assumes then for $j = 0$ the form

$$S_{00}\big(\gamma_0(k)z_k\big) = B_0 z_k = z_{k-1} - z_{k+1},$$

and the right-hand side the form

$$\sum_{m=0}^{1} \varrho_{10m}(k) z_{k-1+2m} = z_{k-1} - z_{k+1},$$

because $\varrho_{100}(k) = -\varrho_{101}(k) \equiv 1$. Hence the equality is true in the considered case.

Assume that it is true for an $i \geqslant 1$ and for $j = 0, 1, \dots, i$. Equality (2.41) implies

$$S_{ij}\big(\gamma_j(k)z_k\big) = B_i S_{i-1,j}\big(\gamma_j(k)z_k\big).$$

By virtue of (2.49) and definition (2.37) of the operator B_i and having applied Lemma 2.7, we obtain for the right-hand side of the foregoing relation the form

$$B_i \sum_{m=0}^{i-j} \varrho_{ijm}(k) z_{k-i+j+2m} = (k+\lambda)^{-1} \Big\{ a_i(k)\, \varrho_{ij0}^{-}(k) z_{k-i-1+j} +$$

$$+ \sum_{m=1}^{i-j} [a_i(k)\, \varrho_{ijm}^{-}(k) - \beta_i(k)\, \varrho_{ij,m-1}^{+}(k)] z_{k-i-1+j+2m} - \beta_i(k)\, \varrho_{ij,i-j}^{+}(k) z_{k+i+1-j} \Big\}$$

$$= \sum_{m=0}^{i+1-j} \varrho_{i+1,jm}(k) z_{k-i-1+j+2m}.$$

This completes the proof.

Putting $j = 0$ into (2.49) we get in view of (2.40) the following

COROLLARY 2.1. *For $i = 1, 2, \dots$ we have*

$$P_i z_k = \sum_{m=0}^{i} \varrho_{i0m}(k) z_{k-i+2m} \quad \textit{for } \{z_k\} \in \mathbf{S}.$$

Since $\varrho_{i00} \not\equiv 0$ and $\varrho_{i0i} \not\equiv 0$ (see (2.47)), this corollary implies

(2.50) $\qquad r(P_i) = 2i, \quad u(P_i) = -i \quad \text{for } i = 1, 2, \dots,$

where the notation used is that of Definition 2.1 (Section 2.1).

2.4. Recurrence relation of the lowest order for Gegenbauer coefficients. Let us now come back to the problem formulated in Section 1. For Gegenbauer coefficients $\{c_k^{(i)}\}$ $(i = 0, 1, \dots, n)$ of function f and its derivatives $f', f'', \ldots, f^{(n)}$ we have there obtained the system of relations (1.27), (1.28); it follows from the differential equation (1.21) and from equalities (1.12), (1.14). Theorem 2.1 (which will be preceded by some lemmata) defines the recurrence relation of the lowest order for the coefficients $\{c_k\}$.

Definition 2.6. Let V be a non-zero operator with the property

$$r(V) = -2u(V) \tag{2.51}$$

and such that if $\tau_0, \tau_1, \ldots, \tau_v$ $(v = r(V))$ are its coefficients, then

$$\tau_0 = \varphi/\psi, \quad \tau_v = \hat{\varphi}/\hat{\psi}, \tag{2.52}$$

where the polynomials $\varphi, \psi, \hat{\varphi}, \hat{\psi}$ satisfy the relation

$$\deg(\varphi) - \deg(\psi) = \deg(\hat{\varphi}) - \deg(\hat{\psi}), \tag{2.53}$$

$\deg(a)$ denoting the degree of the polynomial a.

The common value of differences of both sides of (2.53) will be denoted by $l(V)$. By $\boldsymbol{V}$ we denote the set of all operators V having the above-given properties.

Note that for an i $(0 \leqslant i \leqslant n)$ such that $p_i \not\equiv 0$ the operator $L_0^{(i)}$ satisfying relation (1.23) belongs to the set $\boldsymbol{V}$, for by (1.24), (1.25) and the second line of (1.26) we have

$$r(L_0^{(i)}) = 2d_i, \quad u(L_0^{(i)}) = -d_i \tag{2.54}$$

and conditions (2.52) and (2.53) are satisfied, whereas

$$l(L_0^{(i)}) = 0. \tag{2.55}$$

To the set $\boldsymbol{V}$ there also belong operators $P_0, P_1, \ldots$ defined by (2.40), which follows from (2.50) and from Corollary 2.1. It is easy to see that

$$l(P_0) = 0, \quad l(P_i) = i-1 \quad \text{for } i = 1, 2, \ldots \tag{2.56}$$

Lemma 2.9. *If $V_1, V_2 \in \boldsymbol{V}$, $l(V_1) \neq l(V_2)$, then $V = V_1 + V_2 \in \boldsymbol{V}$ and we have*

$$r(V) = \max\{r(V_1), r(V_2)\}, \tag{2.57}$$

$$l(V) = \begin{cases} l(V_m) & \text{if } r(V_m) > r(V_{3-m}) \text{ for } m = 1 \text{ or } 2, \\ \max\{l(V_1), l(V_2)\} & \text{if } r(V_1) = r(V_2). \end{cases} \tag{2.58}$$

Proof. For $i = 1, 2$ let $u_i = u(V_i)$, $v_i = r(V_i)$, and let $\tau_{i0}, \tau_{i1}, \ldots, \tau_{i,v_i}$ denote coefficients of the operator V_i. Since $V_i \in \boldsymbol{V}$, we have $v_i = -2u_i$.

The rest of the proof will be divided into two parts.

I. If $u_1 \neq u_2$, also $v_1 + u_1 \neq v_2 + u_2$. Hence $V \neq \Theta$ and, therefore, in view of the fact that none of systems (2.5), (2.6) is satisfied, there are excluded sharp inequalities in the following from (2.2), (2.3) relations

$$r(V) \leqslant \max\{v_1+u_1,\, v_2+u_2\} - \min\{u_1,\, u_2\},$$
$$u(V) \geqslant \min\{u_1,\, u_2\}. \tag{2.59}$$

Hence $r(V) = \max\{v_1, v_2\} = -2u(V)$ and we infer that equality (2.57) holds and V satisfies condition (2.51).

Let $\tau_0, \tau_1, \ldots, \tau_v$ ($v = r(V)$) denote coefficients of the operator V. It is easy to see that $\tau_0 = \tau_{m0}$, $\tau_v = \tau_{m,v_m}$, where m is such that $v_m = \max\{v_1, v_2\}$, which means that V satisfies conditions (2.52) and (2.53), whereas $l(V) = l(V_m)$. Hence $V \in \boldsymbol{V}$.

II. Let $u_1 = u_2$. We then have $v_1+u_1 = v_2+u_2$. For $i = 1, 2$ let

$$\tau_{i0} = \varphi_i/\psi_i, \qquad \tau_{i,v_i} = \hat{\varphi}_i/\hat{\psi}_i,$$
$$l(V_i) = \deg(\varphi_i) - \deg(\psi_i) = \deg(\hat{\varphi}_i) - \deg(\hat{\psi}_i).$$

It is easy to verify that for $l(V_1) \neq l(V_2)$ we have

$$\tau_{10}+\tau_{20} = \varphi/\psi \not\equiv 0, \qquad \tau_{1,v_1}+\tau_{2,v_2} = \hat{\varphi}/\hat{\psi} \not\equiv 0, \tag{2.60}$$

where the polynomials $\varphi, \psi, \hat{\varphi}, \hat{\psi}$ are such that

$$\deg(\varphi) - \deg(\psi) = \deg(\hat{\varphi}) - \deg(\hat{\psi}) = \max\{l(V_1), l(V_2)\}. \tag{2.61}$$

So we have $V \neq \Theta$ and none of systems (2.5), (2.6) being satisfied, we infer that sharp inequalities are excluded in (2.59) and we find — as in the foregoing case — that equalities (2.57) and (2.51) are true. Coefficients τ_0 and τ_v of the operator V are equal to the first and the second sums in (2.60), respectively, and so, by virtue of (2.60) and (2.61), they satisfy conditions (2.52) and (2.54) of Definition 2.6; it is easy to verify that we also have $l(V) = \max\{l(V_1), l(V_2)\}$. Hence we have $V \in \boldsymbol{V}$ and the second part of formula (2.58) is true, q.e.d.

It is easy to prove the following

LEMMA 2.10. *If $V_1, V_2 \in \boldsymbol{V}$, then $V = V_1V_2 \in \boldsymbol{V}$ and*

$$l(V) = l(V_1)+l(V_2).$$

COROLLARY 2.2. *If $V \in \boldsymbol{V}$, $\varrho = \varphi/\psi$, where φ and ψ are polynomials in k, $\varphi \not\equiv 0$, $\psi \not\equiv 0$, then $\varrho(k)V \in \boldsymbol{V}$ and*

$$l\big(\varrho(k)V\big) = l(V) + \deg(\varphi) - \deg(\psi).$$

Proof. Note that the operator U such that $Uz_k = \varrho(k)z_k$ ($\{z_k\} \in \boldsymbol{S}$) belongs to the set $\boldsymbol{V}$ and $l(U) = \deg(\varphi) - \deg(\psi)$. This and Lemma 2.10 imply the corollary, for $\varrho(k)V = UV$ according to the definition in Section 2.1, q.e.d.

LEMMA 2.11. *Let us take*

$$V = \sum_{i=0}^{n} 2^i L_0^{(i)}[\gamma_{n-i}^{-1}(k)P_{n-i}], \tag{2.62}$$

where $L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)}$ *are operators satisfying relations* (1.23), *and operators* $P_0, P_1, \ldots, P_n$ *and polynomials* $\gamma_0, \gamma_1, \ldots, \gamma_n$ *are defined by formulae* (2.40) *and* (2.43), *respectively. Then* $V \in \boldsymbol{V}$ *and we have*

$$r(V) = 2 \max_{\substack{0 \leqslant i \leqslant n \\ p_{n-i} \neq 0}} (d_{n-i}+i). \tag{2.63}$$

Proof. Assume

$$V_i = 2^i L_0^{(i)}[\gamma_{n-i}^{-1}(k)P_{n-i}] \quad \text{for } i = 0, 1, \ldots, n. \tag{2.64}$$

Since $\gamma_{n-i}^{-1}(k)P_{n-i} \neq \Theta$, we infer that the equality $V_i = \Theta$ is equivalent to the equality $L_0^{(i)} = \Theta$, whence to the equality $p_i \equiv 0$ (see (1.23)). If $p_i \not\equiv 0$ (by assumption this is the case for $i = n$), then it follows from Lemma 2.10 and Corollary 2.2 that $V_i \in \boldsymbol{V}$ and (by virtue of (2.55), (2.56) and (2.43)) that

$$l(V_i) = i-n \text{ for } 0 \leqslant i \leqslant n-1, \quad l(V_n) = 0. \tag{2.65}$$

From (2.64) by virtue of the first relation of (2.4) it follows that

$$r(V_i) = r(L_0^{(i)}) + r(P_{n-i}),$$

which in view of (2.50) and (2.54) implies

$$r(V_i) = 2(d_i+n-i). \tag{2.66}$$

Put

$$U_j = \sum_{i=n-j}^{n} V_i \quad \text{for } j = 0, 1, \ldots, n. \tag{2.67}$$

We will prove by induction on j that $U_j \in \boldsymbol{V}$ and that the formulae

$$r(U_j) = \max_{\substack{n-j \leqslant i \leqslant n \\ p_i \neq 0}} r(V_i), \tag{2.68}$$

$$l(U_j) = l(V_{m_j}) \quad \text{for } n-j \leqslant m_j \leqslant n \tag{2.69}$$

are true for any $j = 0, 1, \ldots, n$.

For $j = 0$ we have $U_0 = V_n \in \boldsymbol{V}$ and, therefore, (2.68) and (2.69) are trivially true. Assume that $U_j \in \boldsymbol{V}$ and that formulae (2.68), (2.69) hold for a j such that $1 \leqslant j \leqslant n-1$. According to (2.67) we have

$$U_{j+1} = U_j + V_{n-j-1}.$$

For $p_{n-j-1} \equiv 0$ we have $V_{n-j-1} = \Theta$, $U_{j+1} = U_j \in \boldsymbol{V}$ and formulae obtained from (2.68) and (2.69) by replacing j by $j+1$ are obviously true. For $p_{n-j-1} \not\equiv 0$ we have $V_{n-j-1} \neq \Theta$ and by (2.65) and (2.69) we get

$$l(V_{n-j-1}) = -j-1 < -j \leqslant l(U_j).$$

By Lemma 2.9 we infer that U_{j+1} belongs to the set $\boldsymbol{V}$,

$$r(U_{j+1}) = \max\{r(U_j), r(V_{n-j-1})\} = \max_{\substack{n-j-1\leqslant i\leqslant n\\ p_i\not\equiv 0}} r(V_i),$$

and $l(U_{j+1})$ is equal to $l(U_j)$ or to $l(V_{n-j-1})$, whence $l(U_{j+1}) = l(V_{m_{j+1}})$, where $n-j-1 \leqslant m_{j+1} \leqslant n$.

So for every $j = 0, 1, \ldots, n$ the operator U_j belongs to the set $\boldsymbol{V}$ and formulae (2.68) and (2.69) hold.

Since the operator V, defined by (2.62), is equal to U_n (see (2.67) and (2.64)), we infer that (2.63) follows from (2.68) and (2.66).

Definition 2.7. Suppose that for a non-zero operator P there exists an operator L such that

$$P\sum_{i=0}^{n} L_0^{(i)} c_k^{(i)} = Lc_k, \tag{2.70}$$

where $L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)}$ are operators satisfying relations (1.23). The set of all operators having this property will be denoted by $\boldsymbol{P}$.

LEMMA 2.12. *If $P \in \boldsymbol{P}$ and $L \in \boldsymbol{L}$ satisfy relation* (2.70), *then*

$$r(L) = r(P) + 2\max_{\substack{0\leqslant i\leqslant n\\ p_i\not\equiv 0}}(d_i - i). \tag{2.71}$$

Proof. By Lemma 2.6 it follows from (2.70) that

$$P\sum_{i=0}^{n} 2^{i-n} L_0^{(i)}[\gamma_{n-i}^{-1}(k) P_{n-i} c_k^{(n)}] = 2^{-n} L[\gamma_n^{-1}(k) P_n c_k^{(n)}],$$

which implies the equality

$$PV = L[\gamma_n^{-1}(k) P_n],$$

where V is operator (2.62). Using the first of equalities (2.4) we get

$$r(P) + r(V) = r(L) + r(P_n),$$

whence, by virtue of the first formula (2.50) and formula (2.63), we have formula (2.71).

Definition 2.8. Let operators $L_0^{(i)}$ $(i = 0, 1, \ldots, n)$ satisfy relations (1.23) and the function π_0 be such that $\pi_0(k) = c_k[p]$, where p denotes the right-hand side of the differential equation (1.21). Define for any $t = 1, 2, \ldots, n$ the operators $L_t^{(0)}, L_t^{(1)}, \ldots, L_t^{(n-t)}$ and the function π_t

by recurrence formulae

(2.72) $$L_t^{(i)} = A_{t-1}L_{t-1}^{(i)} \quad \text{for } i = 0, 1, \ldots, n-t-1,$$

(2.73) $$L_t^{(n-t)} = A_{t-1}L_{t-1}^{(n-t)} + M_{t-1},$$

(2.74) $$\pi_t(k) = A_{t-1}\pi_{t-1}(k),$$

where

1° A_{t-1} belongs to the set $A(L_{t-1}^{(n-t+1)})$ and is the minimum operator in this set,

2° M_{t-1} is an operator such that

(2.75) $$M_{t-1}z_k = 2Q_{t-1}[(k+\lambda)z_k] \quad \text{for } \{z_k\} \in S$$

and the operator Q_{t-1} satisfies the relation

(2.76) $$A_{t-1}L_{t-1}^{(n-t+1)} = Q_{t-1}D.$$

Lemma 2.13. *For $t = 0, 1, \ldots, n$ we have*

(2.77) $$L_t^{(i)} = \Pi_t L_0^{(i)} \quad \text{for } i = 0, 1, \ldots, n-t-1,$$

(2.78) $$L_t^{(n-t)} c_k^{(n-t)} = \Pi_t \sum_{j=n-t}^{n} L_0^{(j)} c_k^{(j)},$$

(2.79) $$\pi_t(k) = \Pi_t c_k[p],$$

where Π_t is an operator defined by

(2.80) $$\Pi_t = \begin{cases} I & \text{for } t = 0, \\ A_{t-1}\Pi_{t-1} & \text{for } t = 1, 2, \ldots, n. \end{cases}$$

Proof. Relations (2.77) and (2.79) follow easily from equalities (2.72) and (2.74), respectively.

We will show by induction on t that identity (2.78) is true. For $t = 0$ it is obviously true. Assume that it is true for a t such that $0 \leqslant t \leqslant n-1$. By formula (2.73) we have

(2.81) $$L_{t+1}^{(n-t-1)} c_k^{(n-t-1)} = (A_t L_t^{(n-t-1)} + M_t) c_k^{(n-t-1)}.$$

Using (2.45), (2.75) and (2.76) we obtain

$$M_t c_k^{(n-t-1)} = 2Q_t[(k+\lambda)c_k^{(n-t-1)}] = Q_t D c_k^{(n-t)} = A_t L_t^{(n-t)} c_k^{(n-t)}.$$

This formula, the induction assumption and formula (2.77) allow us to transform the right-hand side of (2.81) and obtain

$$A_t\Pi_t\Big(L_0^{(n-t-1)}c_k^{(n-t-1)} + \sum_{j=n-t}^{n} L_0^{(j)}c_k^{(j)}\Big) = \Pi_{t+1}\sum_{j=n-t-1}^{n} L_0^{(j)}c_k^{(j)}.$$

So we see that identity (2.78) is true for any $t = 0, 1, \ldots, n$, q.e.d.

Obviously we have

COROLLARY 2.3. *Operators Π_n and $L_n^{(0)}$ satisfy the relation*

$$(2.82)\qquad \Pi_n \sum_{i=0}^{n} L_0^{(i)} c_k^{(i)} = L_n^{(0)} c_k$$

and, therefore, Π_n belongs to the set $\boldsymbol{P}$.

LEMMA 2.14. *If $P \in \boldsymbol{P}$ and $L \in \boldsymbol{L}$ satisfy relation* (2.70), *then there exists a non-zero operator Φ such that*

$$(2.83)\qquad P = \Phi \Pi_n,$$

$$(2.84)\qquad L = \Phi L_n^{(0)},$$

where Π_n is defined by (2.80) *and the operator $L_n^{(0)}$ is defined according to Definition* 2.8.

Proof. Since by virtue of the first equality of (2.42), by (2.40), (2.39) and (2.37) we have $P_i = S_{i-1,1} D$ for $i \geqslant 1$, we infer by (2.46) that

$$(2.85)\qquad c_k = 2^{-i} \gamma_i^{-1}(k) S_{i-1,1} D c_k^{(i)} \quad \text{for } i \geqslant 1.$$

We will show by induction that

$$(2.86)\qquad P = \Phi_t \Pi_t \quad \text{for } t = 1, 2, \ldots, n,$$

where Φ_t is a non-zero operator and Π_t an operator defined by (2.80).

Let $t = 1$. Using (2.85), we transform (2.70) into the form

$$P\Big\{L_0^{(n)} c_k^{(n)} + \sum_{j=1}^{n} 2^{-j} L_0^{(n-j)} [\gamma_j^{-1}(k) S_{j-1,1} D c_k^{(n)}]\Big\} = 2^{-n} L[\gamma_n^{-1}(k) S_{n-1,1} D c_k^{(n)}],$$

which implies the equality

$$P L_0^{(n)} = \Big\{2^{-n} L[\gamma_n^{-1}(k) S_{n-1,1}] - P \sum_{j=1}^{n} 2^{-j} L_0^{(n-j)} [\gamma_j^{-1}(k) S_{j-1,1}]\Big\} D.$$

Therefore $P \in \boldsymbol{A}(L_0^{(n)})$. By virtue of Lemma 2.2 there exists an operator $\Phi_1 \neq \Theta$ such that $P = \Phi_1 A_0$, where A_0 is the minimum operator in the set $\boldsymbol{A}(L_0^{(n)})$.

Assume that for a t such that $1 \leqslant t \leqslant n-1$ there exists an operator $\Phi_t \neq \Theta$ which satisfies relation (2.86). By virtue of (2.77) and (2.78) we obtain

$$P \sum_{i=0}^{n} L_0^{(i)} c_k^{(i)} = \Phi_t \Pi_t \sum_{i=0}^{n} L_0^{(i)} c_k^{(i)} = \Phi_t \sum_{i=0}^{n-t} L_t^{(i)} c_k^{(i)}.$$

Taking into account (2.70) we therefore have the identity

$$\Phi_t \sum_{i=0}^{n-t} L_t^{(i)} c_k^{(i)} = L c_k$$

which by (2.85) can be transformed into the form

$$\Phi_t\Big\{L_t^{(n-t)}c_k^{(n-t)}+\sum_{j=1}^{n-t}2^{-j}L_t^{(n-t-j)}[\gamma_j^{-1}(k)S_{j-1,1}Dc_k^{(n-t)}]\Big\}$$
$$= 2^{t-n}L[\gamma_{n-t}^{-1}(k)S_{n-t-1,1}Dc_k^{(n-t)}]$$

implying the equality

$$\Phi_t L_t^{(n-t)} = \Big\{2^{t-n}L[\gamma_{n-t}^{-1}(k)S_{n-t-1,1}]-\Phi_t\sum_{j=1}^{n-t}2^{-j}L_t^{(n-t-j)}[\gamma_j^{-1}(k)S_{j-1,1}]\Big\}D,$$

i.e.

$$\Phi_t\in A(L_t^{(n-t)}).$$

There exists, by Lemma 2.2, a non-zero operator Φ_{t+1} such that $\Phi_t = \Phi_{t+1}A_t$, where A_t is the minimum operator in the set $A(L_t^{(n-t)})$, which implies the equality $P = \Phi_{t+1}\Pi_{t+1}$.

Thus we come to the conclusion that (2.86) is true for every $t = 1, 2, \ldots, n$. For $t = n$ it is of the form $P = \Phi_n\Pi_n$, where $\Phi_n \neq \Theta$. Assuming $\Phi = \Phi_n$, we get equality (2.83).

Relation (2.84) is a consequence of this equality and of identity (2.82), q.e.d.

THEOREM 2.1. *Let f be a function satisfying the differential equation* (1.21) *of order n and such that its n-th derivative can be expanded into the uniformly convergent Gegenbauer series. We then have the recurrence relation*

$$L_n^{(0)}c_k = \pi_n(k) \quad \text{for } k \geqslant 0, \tag{2.87}$$

where $L_n^{(0)}$ and π_n are an operator and a function, respectively, formed in the manner given in Definition 2.8, *and $\{c_k\}$ are Gegenbauer coefficients of the function f defined by* (1.9). *The order $l_n^{(0)}$ of this relation is expressed by the formula*

$$l_n^{(0)} = \sum_{j=0}^{n-1}r(A_j)+2\max_{\substack{0\leqslant i\leqslant n\\ p_i\neq 0}}(d_i-i), \tag{2.88}$$

where $A_0, A_1, \ldots, A_{n-1}$ are operators given by Definition 2.8, *$r(A_j)$ denotes the order of the operator A_j ($j = 0, 1, \ldots, n-1$), polynomials $p_0, p_1, \ldots, p_n$ are coefficients of equation* (1.21), *and d_i denotes the degree of the polynomial $p_i \not\equiv 0$ ($0 \leqslant i \leqslant n$). Among recurrence relations for the Gegenbauer coefficients of the function f, which were obtained by virtue of the differential equation* (1.21) *and relations* (1.12), (1.14), *relation* (2.87) *is that of the lowest order.*

It is easy to see that coefficients of the operator $L_n^{(0)}$ are rational functions and, $\{c_k\}$ being the Chebyshev coefficients of the function f, they are polynomials in k.

Proof. Let the operator Π_n, defined by (2.80), be acting on both sides of equation (1.27). Applying (2.82) and (2.79) we get (2.87).

Formula (2.88) is true in view of Corollary 2.3 and Lemma 2.12.

Let us show the last statement of the theorem. From equation (1.21) and equalities (1.12), (1.14) we obtain system (1.27), (1.28). The elimination process of terms of the sequences $\{c_k'\}, \{c_k''\}, \ldots, \{c_k^{(n)}\}$ from that system, leading to the relation of form $Lc_k = \mu(k)$, where $L \in \boldsymbol{L}$ and μ is a function, is equivalent to the action on both sides of equation (1.27) of the operator $P \in \boldsymbol{P}$. By Lemmata 2.12 and 2.14 the operator L is of the lowest order which is equal to the order $l_n^{(0)}$ of the operator $L_n^{(0)}$ in the case where

$$r(P) = r(\Pi_n) = \sum_{j=0}^{n-1} r(A_j).$$

COROLLARY 2.4. *We have*

$$2 \max_{\substack{0 \leqslant i \leqslant n \\ p_i \neq 0}} (d_i - i) \leqslant l_n^{(0)} \leqslant 2 \max_{\substack{0 \leqslant i \leqslant n \\ p_{n-i} \neq 0}} (d_{n-i} + i).$$

The corollary follows from (2.88) and from the fact that according to Definition 2.8 and Lemma 2.5 we have $0 \leqslant r(A_j) \leqslant 2$ for $j = 0, 1, \ldots \ldots, n-1$.

2.5. Algorithm I. We now formulate an optimum algorithm, called *Algorithm I*, which allows us to construct a recurrence relation used in Theorem 2.1 (Section 2.4).

ALGORITHM I. Define the system of operators $L_t^{(i)}$ $(t = 0, 1, \ldots, n;$ $i = 0, 1, \ldots, n-t)$ and the system of functions $\pi_0, \pi_1, \ldots, \pi_n$ in the following way.

(a) Define (using (1.18)) operators $L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)}$ to satisfy relations (1.23). Assume that $\pi_0(k) = c_k[p]$, where p stands for the right-hand side of the differential equation (1.21).

(b) Let us now perform for every $t = 1, 2, \ldots, n$ the following operations.

1° Let

$$L_{t-1}^{(n-t+1)} z_k = \sum_{j=0}^{l} \lambda_j(k) z_{k+u+j} \quad \text{for } \{z_k\} \in \boldsymbol{S}, \tag{2.89}$$

where $l = r(L_{t-1}^{(n-t+1)})$, and $u = u(L_{t-1}^{(n-t+1)})$.

Define an operator N_{t-1} such that $N_{t-1} = \Theta$ for $l \leqslant 1$ or such that

$$N_{t-1} z_k = \sum_{j=0}^{l-2} \nu_j(k) z_{k+u+1+j} \; (\{z_k\} \in \boldsymbol{S}) \quad \text{for } l > 1. \tag{2.90}$$

We assume, moreover, that

$$(2.91)\qquad \eta = \begin{cases} \lambda_0 & \text{for } l \leqslant 1, \\ \lambda_0 - \nu_0 & \text{for } l > 1, \end{cases} \qquad \vartheta = \begin{cases} 0 & \text{for } l = 0, \\ \lambda_1 & \text{for } l = 1, \\ \lambda_1 - \nu_1 & \text{for } l > 1, \end{cases}$$

where the functions $\nu_0, \nu_1, \ldots, \nu_{l-2}$ $(l > 1)$ are defined by the recurrence formula

$$(2.92)\qquad \nu_j = \nu_{j+2} - \lambda_{j+2}$$

for $j = l-2, l-3, \ldots, 0$ and $\nu_l = \nu_{l-1} \equiv 0$.

2° Let us form operators A and R according to formulae (2.13) and (2.14) for η and ϑ defined by (2.91), and next assume that $A_{t-1} = A$, $R_{t-1} = R$.

3° Let us form operators $Q_{t-1}, M_{t-1}, L_t^{(0)}, L_t^{(1)}, \ldots, L_t^{(n-t)}$ applying the formulae

$$(2.93)\qquad Q_{t-1} = A_{t-1}N_{t-1} + R_{t-1}E^u,$$

$$(2.94)\qquad M_{t-1}z_k = 2Q_{t-1}[(k+\lambda)z_k] \quad \text{for } \{z_k\} \in S,$$

$$(2.95)\qquad L_t^{(i)} = A_{t-1}L_{t-1}^{(i)} \quad \text{for } i = 0, 1, \ldots, n-t-1,$$

$$(2.96)\qquad L_t^{(n-t)} = A_{t-1}L_{t-1}^{(n-t)} + M_{t-1}.$$

4° We assume that

$$(2.97)\qquad \pi_t(k) = A_{t-1}\pi_{t-1}(k).$$

The equation

$$L_n^{(0)}c_k = \pi_n(k)$$

is a recurrence relation of the lowest order — in the sense of Theorem 2.1 (Section 2.4) — for the Gegenbauer coefficients $\{c_k\}$ of the function f satisfying assumptions of this theorem.

Proof. We will show that operators $L_t^{(i)}$ $(t = 0, 1, \ldots, n;\ i = 0, 1, \ldots \ldots, n-t)$ and functions $\pi_0, \pi_1, \ldots, \pi_n$, formed in the above-given way, satisfy conditions of Definition 2.8. This obviously holds in case of the operators $L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)}$ and the function π_0.

Let t be such that $1 \leqslant t \leqslant n$. From (2.91) and (2.92) it follows that

$$\eta = \sum_{j=0}^{[l/2]} \lambda_{2j}, \qquad \vartheta = \sum_{j=0}^{[(l-1)/2]} \lambda_{2j+1}$$

and, therefore, we have

$$\eta = \sigma_1(L_{t-1}^{(n-t+1)}; k), \qquad \vartheta = \sigma_2(L_{t-1}^{(n-t+1)}; k)$$

(see (2.89) and (2.31)).

By Lemma 2.4 we have

$$L_{t-1}^{(n-t+1)} = N_{t-1}D + W_{t-1}E^u, \tag{2.98}$$

where N_{t-1} is an operator defined in part 1b of the proved algorithm, and $W_{t-1} = \eta(k)I + \vartheta(k)E$.

Operators A_{t-1} and R_{t-1}, defined in part 2b, are such that

$$A_{t-1}W_{t-1} = R_{t-1}D$$

and, moreover, A_{t-1} belongs to the set $\boldsymbol{A}(L_{t-1}^{(n-t+1)})$ and is a minimum operator in this set (see Lemmata 2.3 and 2.5 in Section 2.3). These statements and equalities (2.98) imply

$$A_{t-1}L_{t-1}^{(n-t+1)} = Q_{t-1}D,$$

where Q_{t-1} is an operator defined by (2.93) and, therefore, it satisfies (2.76). Hence, in view of the formal identity of the corresponding formulae (2.75) and (2.94) as well as of (2.72)-(2.74) and (2.95)-(2.97), we state that the operators $L_t^{(0)}, L_t^{(1)}, \ldots, L_t^{(n-t)}$ and the function π_t are formed in accordance to Definition 2.8, which together with Theorem 2.1 implies the correctness of the algorithm.

Example 2.1. The complete elliptic integral of the second kind

$$f(x) = \int_0^{\pi/2} (1-x^2\sin^2 t)^{1/2}dt \qquad \text{for } -1 \leqslant x \leqslant 1 \tag{2.99}$$

satisfies the differential equation

$$x(x^2-1)f'' + (x^2-1)f' - xf = 0$$

(see [4], p. 437). We have here $n = 2$, $p_0(x) = -x$, $p_1(x) = x^2-1$, $p_2(x) = x(x^2-1)$, $p \equiv 0$.

Let us now apply Algorithm I to the construction of a recurrence relation for the Chebyshev coefficients of the function f. We, therefore, assume that $\lambda = 0$.

(a) By virtue of formula (1.18) we define operators $L_0^{(0)}$, $L_0^{(1)}$ and $L_0^{(2)}$ such that

$$L_0^{(0)}c_k = c_k[p_0f] = -\frac{1}{2}(c_{k-1}+c_{k+1}),$$

$$L_0^{(1)}c_k' = c_k[p_1f'] = \frac{1}{4}(c_{k-2}'-2c_k'+c_{k+2}'),$$

$$L_0^{(2)}c_k'' = c_k[p_2f''] = \frac{1}{8}(c_{k-3}''-c_{k-1}''-c_{k+1}''+c_{k+3}'').$$

Assume, moreover, that $\pi_0(k) = c_k[p] = 0$.

(b) By formulae (2.89)-(2.92) for $t = 1$ we have $l = 6$, $u = -3$, and so

$$N_0 z_k = \frac{1}{8}(z_{k-2} - z_{k+2}), \qquad \eta = \vartheta = 0.$$

Formulae (2.13) and (2.14) define in this case the operators $A_0 = I$ and $R_0 = \Theta$.

According to formulae (2.93)-(2.97) we have

$$Q_0 = N_0,$$

$$M_0 z_k = \frac{1}{4}[(k-2)z_{k-2} - (k+2)z_{k+2}],$$

$$L_1^{(0)} = L_0^{(0)},$$

$$L_1^{(1)} z_k = \frac{1}{4}[(k-1)z_{k-2} - 2z_k - (k+1)z_{k+2}],$$

$$\pi_1 \equiv 0.$$

Let $t = 2$. Applying formulae (2.89)-(2.92) we get $l = 4$, $u = -2$, and so

$$N_1 z_k = \frac{1}{4}[(k+3)z_{k-1} + (k+1)z_{k+1}],$$

$$\eta \equiv -1, \qquad \vartheta \equiv 0.$$

Hence formulae (2.13) and (2.14), for which we have case IV, imply (for $\omega \equiv -1$) that $A_1 = D$ and $R_1 = -I$. From this statement and from (2.93)-(2.97) it follows that

$$Q_1 z_k = \frac{1}{4}[(k-2)z_{k-2} - 4z_k - (k+2)z_{k+2}],$$

$$M_1 z_k = \frac{1}{2}[(k-2)^2 z_{k-2} - 4kz_k - (k+2)^2 z_{k+2}],$$

$$L_2^{(0)} z_k = \frac{1}{2}[(k-1)(k-3)z_{k-2} - 4kz_k - (k+1)(k+3)z_{k+2}],$$

$$\pi_2 \equiv 0.$$

For the Chebyshev coefficients $\{b_k[f]\}$ of function (2.99) we obtain the following recurrence relation of the fourth order:

$$(k-1)(k-3)b_{k-2}[f] - 4kb_k[f] - (k+1)(k+3)b_{k+2}[f] = 0 \quad \text{for } k \geqslant 0.$$

This function being even, we infer — in view of $T_k(-x) = (-1)^k T_k(x)$ for $k \geqslant 0$ — that coefficients $b_{2k+1}[f]$ $(k \geqslant 0)$ are equal to zero and coefficients $\{w_k\}$, where $w_k = b_{2k}[f]$, satisfy the second order relation

$$(2k-1)(2k-3)w_{k-1} - 8kw_k - (2k+1)(2k+3)w_{k+1} = 0 \quad \text{for } k \geqslant 0.$$

Example 2.2. Let us construct a recurrence relation for the Gegenbauer coefficients $\{c_k\}$ of function (2.99), which are defined by formula (1.9) for $\lambda \neq 0$. Let n, p_0, p_1, p_2 and p have the same meaning as in Example 2.1.

(a) Applying (1.19) for $l = 0, 1, 2$ and 3 we get:

$$a_{00} \equiv 1,$$

$$a_{10}(k) = \frac{k}{k+\lambda}, \quad a_{11}(k) = \frac{k+2\lambda}{k+\lambda}$$

$$a_{20}(k) = \frac{(k-1)_2}{(k+\lambda-1)_2}, \quad a_{21}(k) = 2\,\frac{k^2+2\lambda k+\lambda-1}{(k+\lambda)^2-1}, \quad a_{22}(k) = \frac{(k+2\lambda)_2}{(k+\lambda)_2},$$

$$a_{30}(k) = \frac{(k-2)_3}{(k+\lambda-2)_3}, \quad a_{31}(k) = 3k\,\frac{k^2+(2\lambda-1)k-2}{(k+\lambda-2)(k+\lambda)_2},$$

$$a_{32}(k) = 3(k+2\lambda)\,\frac{k^2+(2\lambda+1)k+2(\lambda-1)}{(k+\lambda-1)_2(k+\lambda+2)}, \quad a_{33}(k) = \frac{(k+2\lambda)_3}{(k+\lambda)_3}.$$

Using (1.18) we define operators $L_0^{(0)}$, $L_0^{(1)}$ and $L_0^{(2)}$ such that

$$L_0^{(0)}c_k = c_k[p_0 f] = -\frac{1}{2(k+\lambda)}[kc_{k-1}+(k+2\lambda)c_{k+1}],$$

$$L_0^{(1)}c_k' = c_k[p_1 f'] = \frac{1}{4}\left\{\frac{(k-1)_2}{(k+\lambda-1)_2}c_{k-2}' - 2\left[1+\frac{\lambda(\lambda-1)}{(k+\lambda)^2-1}\right]c_k' + \right.$$
$$\left. + \frac{(k+2\lambda)_2}{(k+\lambda)_2}c_{k+2}'\right\},$$

$$L_0^{(2)}c_k'' = c_k[p_2 f'']$$
$$= \frac{1}{8}\left\{\frac{(k-2)_3}{(k+\lambda-2)_3}c_{k-3}'' - \frac{k}{k+\lambda}\left[1+\frac{3\lambda(\lambda-1)}{(k+\lambda-2)(k+\lambda+1)}\right]c_{k-1}'' - \right.$$
$$\left. - \frac{k+2\lambda}{k+\lambda}\left[1+\frac{3\lambda(\lambda-1)}{(k+\lambda-1)(k+\lambda+2)}\right]c_{k+1}'' + \frac{(k+2\lambda)_3}{(k+\lambda)_3}c_{k+3}''\right\}.$$

We assume that $\pi_0(k) = c_k[p] = 0$.

(b) Performing the operations 1°-4° for $t = 1$ of part (b) of Algorithm I we obtain the operator N_0 and find functions η and ϑ, and next construct operators $A_0, R_0, Q_0, M_0, L_1^{(0)}, L_1^{(1)}$ and the function π_1. Consequently,

$$N_0 z_k = \frac{1}{8}\left\{\frac{(k-2)_3}{(k+\lambda-2)_3} z_{k-2} - \frac{2\lambda k(k+2\lambda)}{(k+\lambda-1)_3} z_k - \frac{(k+2\lambda)_3}{(k+\lambda)_3} z_{k+2}\right\},$$

$$\eta = \vartheta \equiv 0, \quad A_0 = I, \quad R_0 = \Theta, \quad Q_0 = N_0,$$

$$M_0 z_k = \frac{1}{4}\left\{\frac{(k-2)_3}{(k+\lambda-1)_2} z_{k-2} - \frac{2\lambda k(k+2\lambda)}{(k+\lambda)^2-1} z_k - \frac{(k+2\lambda)_3}{(k+\lambda)_2} z_{k+2}\right\},$$

$$L_1^{(0)} = L_0^{(0)},$$

$$L_1^{(1)} z_k = \frac{1}{4}\left\{\frac{k(k-1)^2}{(k+\lambda-1)_2} z_{k-2} - 2\,\frac{(\lambda+1)k^2+2\lambda(\lambda+1)k+(2\lambda+1)(\lambda-1)}{(k+\lambda)^2-1} z_k - \frac{(k+2\lambda)(k+2\lambda+1)^2}{(k+\lambda)_2} z_{k+2}\right\},$$

$$\pi_1 \equiv 0.$$

Repeating this procedure for $t = 2$ we get the operator N_1 and the functions η and ϑ as well as the operators $A_1, R_1, Q_1, M_1, L_2^{(0)}$ and the function π_2. We have

$$N_1 z_k = \frac{1}{4}\left[\frac{k^3+2(4\lambda+1)k^2+(16\lambda^2-3)k+4\lambda(\lambda-1)(2\lambda+1)}{(k+\lambda-1)_2} z_{k-1} + \frac{(k+2\lambda)(k+2\lambda+1)^2}{(k+\lambda)_2} z_{k+1}\right],$$

$$\eta = -(2\lambda+1), \quad \vartheta \equiv 0, \quad A_1 = D, \quad R_1 = -(2\lambda+1)I,$$

$$Q_1 z_k = \frac{1}{4}\left[\frac{(k-1)(k-2)^2}{(k+\lambda-2)_2} z_{k-2} - 2\,\frac{(\lambda+2)k^2+2\lambda(\lambda+2)k-2(\lambda+1)}{(k+\lambda)^2-1} z_k - \frac{(k+2\lambda+1)(k+2\lambda+2)^2}{(k+\lambda+1)_2} z_{k+2}\right],$$

$$M_1 z_k = \frac{1}{2}\left[\frac{(k-1)(k-2)^2}{k+\lambda-1} z_{k-2} - 2(k+\lambda)\frac{(\lambda+2)k^2+2\lambda(\lambda+2)k-2(\lambda+1)}{(k+\lambda)^2-1} z_k - \frac{(k+2\lambda+1)(k+2\lambda+2)^2}{k+\lambda+1} z_{k+2}\right],$$

$$L_2^{(0)} z_k = \frac{1}{2}\left[\frac{(k-3)(k-1)^2}{k+\lambda-1} z_{k-2} - 2(\lambda+2)(k+\lambda)\frac{k^2+2\lambda k-1}{(k+\lambda)^2-1} z_k - \frac{(k+2\lambda+1)^2(k+2\lambda+3)}{k+\lambda+1} z_{k+2}\right],$$

$$\pi_2 \equiv 0.$$

The Gegenbauer coefficients $\{c_k\}$ of function (2.99) satisfy, therefore, the recurrence relation

$$L_2^{(0)} c_k = 0 \quad \text{for } k \geqslant 0$$

of the fourth order, with $L_2^{(0)}$ already calculated. Since this function is even and, by a well-known property of the Gegenbauer polynomials,

$$C_k^{(\lambda)}(-x) = (-1)^k C_k^{(\lambda)}(x) \quad \text{for } k \geqslant 0$$

(see [3], vol. II, § 10.9), we have $c_{2k+1} = 0$ $(k \geqslant 0)$, and the coefficients $\{w_k\}$, $w_k = c_{2k}$, satisfy the following recurrence relation of the second order:

$$(2k+\lambda+1)(2k-3)(2k-1)^2 w_{k-1} - 2(\lambda+2)(2k+\lambda)(4k^2+4\lambda k-1) w_k -$$
$$-(2k+\lambda-1)(2k+2\lambda+1)^2(2k+2\lambda+3) w_{k+1} = 0 \quad \text{for } k \geqslant 0.$$

Algorithm I as well as algorithms described in Sections 3 and 4 are easy and simple in applications. If, however, the order n of the differential equation (1.21) or degrees of polynomials $p_0, p_1, \ldots, p_n$, which are coefficients of this equation, are large, then the corresponding calculations may be tiresome. It is to note that there is a possibility of automatic realization of these algorithms in one of programming languages if it admits formula manipulation as, e.g., ABC ALGOL (see [8]).

Let us finally remark that the results of the present section and those of Sections 3 and 4 may be easily applied to the case of the ***shifted coefficients of Gegenbauer and of Chebyshev***, i.e. to the coefficients of series with respect to polynomials $\{C_k^{*(\lambda)}(x)\}$ and $\{T_k^*(x)\}$, respectively, where

$$C_k^{*(\lambda)}(x) = C_k^{(\lambda)}(2x-1), \qquad T_k^*(x) = T_k(2x-1) \quad \text{for } 0 \leqslant x \leqslant 1.$$

3. GENERALIZATION OF THE PASZKOWSKI METHOD

First we describe an algorithm (called by us *Algorithm II*) which leads to a recurrence relation for the Gegenbauer coefficients of a function satisfying the differential equation (1.21). The order of this relation is equal to number (1.29), which, in view of results of Section 2.4 (see Theorem 2.1 and Corollary 2.4), implies that Algorithm II, generally, is not an optimum one.

Next we will show how to generalize the results of Paszkowski [7], § 13, on the Gegenbauer coefficients and to get in this way an algorithm called the generalized Paszkowski algorithm (GPA). Moreover, it will be shown that algorithms II and GPA are equivalent in the sense that they lead to the same recurrence relation (Section 3.2).

3.1. Algorithm II. It may be shown that the differential equation (1.21) is equivalent to the equation

$$\sum_{i=0}^{n} (q_i f)^{(i)} = p, \tag{3.1}$$

where

$$q_i = \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{j-i} p_j^{(j-i)} \quad \text{for } i = 0, 1, \ldots, n \tag{3.2}$$

(see, e.g., [7], § 13). Obviously, both sides of (3.1) have the same Gegenbauer coefficients, whence

$$\sum_{i=0}^{n} c_k^{(i)}[q_i f] = c_k[p],$$

where, according to the assumptions of Section 1, we have

$$c_k^{(i)}[q_i f] = c_k[(q_i f)^{(i)}].$$

Let the operator P_n, defined in Section 2.3 by formula (2.40), act on both sides of this relation and take into account that $P_n = S_{n-1,i} P_i$ for $i = 0, 1, \ldots, n$ (see (2.42)). We get

$$\sum_{i=0}^{n} S_{n-1,i} P_i c_k^{(i)}[q_i f] = P_n c_k[p],$$

which by Lemma 2.6 (Section 2.3) implies the identity

$$\sum_{i=0}^{n} 2^i S_{n-1,i}\big(\gamma_i(k) c_k[q_i f]\big) = P_n c_k[p]. \tag{3.3}$$

Functions $q_0, q_1, \ldots, q_n$ being polynomials, we can in view of relation (1.18) define operators $L_0, L_1, \ldots, L_n$ such that

$$L_i c_k[f] = \gamma_i(k) c_k[q_i f] \quad \text{for } i = 0, 1, \ldots, n, \tag{3.4}$$

which together with (3.3) implies the following

THEOREM 3.1. *If the function f satisfies conditions of Theorem 2.1 (Section 2.4), then*

$$L c_k = \pi_n(k) \quad \textit{for } k \geqslant 0, \tag{3.5}$$

where

$$L = \sum_{i=0}^{n} 2^i S_{n-1,i} L_i, \tag{3.6}$$

$$\pi_n(k) = P_n c_k[p], \tag{3.7}$$

$c_k = c_k[f]$ *being the Gegenbauer coefficients of the function f.*

Remark that it follows from Theorem 3.2 (see Section 3.2) that the order of relation (3.5) is equal to number (1.29) which limits from above the order of the recurrence relation constructed according to Algorithm I (Section 2.5).

Formalized description of the method of the construction of relation (3.5) leads to the following Algorithm II, which in view of the above remark is not, in general, the optimum algorithm.

ALGORITHM II. The recurrence relation (3.5) can be constructed in the following way:

(a) Define operators $L_0, L_1, \ldots, L_n$ to satisfy identities (3.4).

(b) Form operators $S_{n-1,i}$ $(i = 0, 1, \ldots, n)$ according to the formulae

$$(3.8)\quad S_{n-1,n} = I, \qquad S_{n-1,i} = S_{n-1,i+1}B_i \quad \text{for } i = n-1, n-2, \ldots, 0,$$

where B_i is an operator defined by (2.37), and next determine the operator L by (3.6).

(c) Define functions $\pi_0, \pi_1, \ldots, \pi_n$ by the formulae

$$(3.9)\quad \pi_0(k) = c_k[p], \qquad \pi_i(k) = B_{i-1}\pi_{i-1}(k) \quad \text{for } i = 1, 2, \ldots, n.$$

The function $\pi_n(k)$ forms the right-hand side of equation (3.5).

Proof follows from Theorem 3.1 and relations (2.41) and (2.42).

Remark. Part (c) of Algorithm II may be modified if instead of formulae (3.9) we use formula (3.7) and take into account that $P_n = S_{n-1,0}$.

Example 3.1. The function

$$(3.10)\qquad f(x) = (ax)^{1-\mu}s_{\mu\nu}(ax) \quad \text{for } -1 \leqslant x \leqslant 1 \text{ and } a \neq 0,$$

where μ and ν are real numbers such that $\mu+\nu$ and $\mu-\nu$ are not odd integers and $s_{\mu\nu}$ denotes the Lommel function, satisfies the differential equation

$$(3.11)\qquad x^2f'' + bxf' + (a^2x^2 + c)f = a^2x^2,$$

where $b = 2\mu-1$ and $c = (\mu-1)^2-\nu^2$ (see [3], vol. II, § 7.5.5).

We are going to construct a recurrence relation for the Chebyshev coefficients $\{b_k[f]\}$ (i.e. for the Gegenbauer coefficients defined by (1.9) for $\lambda = 0$) of function (3.10). Applying (3.2) for $n = 2, p_0(x) = a^2x^2+c$, $p_1(x) = bx$, $p_2(x) = x^2$, we obtain

$$q_0(x) = a^2x^2+d, \qquad q_1(x) = (b-4)x, \qquad q_2(x) = x^2,$$

where $d = 2-b+c$.

(a) Define operators L_0, L_1 and L_2 such that (see (3.4))

$$L_0 b_k = \gamma_0(k) b_k[q_0 f] = \frac{1}{4}\big(a^2 b_{k-2} + 2(a^2+2d) b_k + a^2 b_{k+2}\big),$$

$$L_1 b_k = \gamma_1(k) b_k[q_1 f] = \frac{1}{2}(b-4)k(b_{k-1}+b_{k+1}),$$

$$L_2 b_k = \gamma_2(k) b_k[q_2 f] = \frac{1}{4}(k-1)_3(b_{k-2}+2b_k+b_{k+2}),$$

where $b_k = b_k[f]$ and $\gamma_i(k)$ $(i = 0, 1, 2)$ are polynomials defined by formulae (2.43) for $\lambda = 0$.

(b) According to formula (2.37) for $\lambda = 0$ we have

$$B_0 = D, \qquad B_1 = (k+1)E^{-1} - (k-1)E.$$

Applying successively formulae (3.8) and (3.6) we get

$$S_{12} = I, \qquad S_{11} = B_1, \qquad S_{10} = (k+1)E^{-2} - 2kI + (k-1)E^2,$$

$$L = \frac{1}{4}a^2(k+1)E^{-4} + \left((k+1)[(k+b-4)(k-1)+d] + \frac{1}{2}a^2\right)E^{-2} +$$
$$+ 2k\left(k^2 - \frac{1}{4}a^2 - d - 1\right)I + \left((k-1)[(k-b+4)(k+1)+d] - \frac{1}{2}a^2\right)E^2 +$$
$$+ \frac{1}{4}a^2(k-1)E^4.$$

(c) The function $p(x) = a^2x^2$ on the right-hand side of (3.11) may be written, in view of fact that $T_0(x) = 1$ and $T_2(x) = 2x^2-1$, in the form

$$p(x) = \frac{1}{2}a^2\big(T_0(x) + T_2(x)\big),$$

which implies that

$$\pi_0(k) = b_k[p] = \begin{cases} a^2 & \text{for } k = 0, \\ \frac{1}{2}a^2 & \text{for } |k| = 2, \\ 0 & \text{for } |k| = 1, 3, 4, \ldots, \end{cases}$$

where we made use of the following equality resulting from formula (1.10) for $\lambda = m = 0$:

$$b_{-k}[f] = b_k[f] \quad \text{for } k > 0.$$

The second formula of (3.9) defines for $i = 1, 2$ successively the functions

$$\pi_1(k) = \begin{cases} \frac{1}{2}a^2 & \text{for } |k| = 1, 3, \\ 0 & \text{for } |k| = 0, 2, 4, 5, \ldots, \end{cases}$$

$$\pi_2(k) = \begin{cases} a^2 & \text{for } |k| = 0, 2, \\ \frac{5}{2}a^2 & \text{for } |k| = 4, \\ 0 & \text{for } |k| = 1, 3, 5, 6, \ldots \end{cases}$$

The Chebyshev coefficients $\{b_k\}$ of function (3.10) satisfy the following recurrence relation of the eighth order:

$$Lb_k = \pi_2(k) \quad \text{for } k \geqslant 0;$$

L and π_2 being calculated above.

Since this function is even, $b_{2k+1} = 0$ $(k \geqslant 0)$ and the coefficients $w_k = b_{2k}$ satisfy the following relation of the fourth order:

$$a^2(2k+1)w_{k-2} + 2\big(2(2k+1)[(2k+b-4)(2k-1)+d] + a^2\big)w_{k-1} +$$
$$+4k[4(4k^2-d-1)-a^2]w_k +$$
$$+2\big(2(2k-1)[(2k-b+4)(2k+1)+d] - a^2\big)w_{k+1} + a^2(2k-1)w_{k+2}$$
$$= \begin{cases} 4a^2 & \text{for } k = 0, 2, \\ 10a^2 & \text{for } k = 4, \\ 0 & \text{for } k = 1, 3, 5, 6, \ldots \end{cases}$$

Example 3.2. Using Algorithm II, we are going to construct a recurrence relation for the Gegenbauer coefficients $\{c_k[f]\}$ of function (3.10), which are defined by formula (1.9) for $\lambda \neq 0$.

(a) Using (1.18) we define operators L_0, L_1 and L_2:

$$L_0 c_k = \gamma_0(k) c_k[q_0 f]$$

$$= \frac{1}{4}\left(a^2 \frac{(k-1)_2}{(k+\lambda-1)_2} c_{k-2} + 2\left[a^2 \frac{k^2+2\lambda k+\lambda-1}{(k+\lambda)^2-1} + 2d\right] c_k + a^2 \frac{(k+2\lambda)_2}{(k+\lambda)_2} c_{k+2}\right),$$

$$L_1 c_k = \gamma_1(k) c_k[q_1 f] = \frac{1}{2}(b-4)[k c_{k-1} + (k+2\lambda) c_{k+1}],$$

$$L_2 c_k = \gamma_2(k) c_k[q_2 f]$$

$$= \frac{1}{4}[(k+\lambda+1)(k-1)_2 c_{k-2} + 2(k+\lambda)(k^2+2\lambda k+\lambda-1) c_k +$$

$$+ (k+\lambda-1)(k+2\lambda)_2 c_{k+2}],$$

where the constants a, b, d and the functions q_0, q_1, q_2 are such as in Example 3.1, and the polynomials $\gamma_i(k)$ ($i = 0, 1, 2$) are defined by formulae (2.43).

(b) By virtue of (2.37) we have

$$B_0 = D, \quad B_1 = (k+\lambda+1)E^{-1} - (k+\lambda-1)E.$$

Define operators S_{1i} ($i = 0, 1, 2$) according to (3.8) and next the operator L by (3.6):

$$S_{12} = I, \quad S_{11} = B_1, \quad S_{10} = (k+\lambda+1)E^{-2} - 2(k+\lambda)I + (k+\lambda-1)E^2;$$

$$L = \frac{1}{4} a^2 \frac{(k+\lambda+1)(k-3)_2}{(k+\lambda-3)_2} E^{-4} +$$

$$+ \left[(k+\lambda+1)[(k+b-4)(k-1)+d] + a^2 \frac{(2\lambda+1)k-3(\lambda+1)}{2(k+\lambda-3)}\right] E^{-2} +$$

$$+ 2(k+\lambda)\left[k^2+2\lambda k + (b-3)\lambda - d - 1 - a^2 \frac{k^2+2\lambda k-2\lambda^2-3\lambda-4}{4[(k+\lambda)^2-4]}\right] I +$$

$$+ \left[(k+\lambda-1)[(k+2\lambda-b+4)(k+2\lambda+1)+d] -\right.$$

$$\left. - a^2 \frac{(2\lambda+1)k+4\lambda^2+5\lambda+3}{2(k+\lambda+3)}\right] E^2 + \frac{1}{4} a^2 \frac{(k+\lambda-1)(k+2\lambda+2)_2}{(k+\lambda+2)_2} E^4.$$

(c) Since $C_0^{(\lambda)}(x) = 1$ and $C_2^{(\lambda)}(x) = 2(\lambda)_2 x^2 - \lambda$ (see formula (1.2)), we have

$$p(x) = a^2 x^2 = \frac{1}{2(\lambda)_2} a^2 [\lambda C_0^{(\lambda)}(x) + C_2^{(\lambda)}(x)].$$

This formula and formulae (1.10) and (1.11) imply

$$\pi_0(k) = c_k[p] = \begin{cases} \dfrac{1}{2(\lambda+1)}a^2 & \text{for } k = 0, \\ \dfrac{1}{2(\lambda)_2}a^2 & \text{for } k = 2, \\ 0 & \text{for } k = -2, -1, 1, 3, 4, \ldots \end{cases}$$

Using the second formula of (3.9) we successively define the functions π_1 and π_2:

$$\pi_1(k) = \begin{cases} -\dfrac{1}{2(\lambda+1)}a^2 & \text{for } k = -1, \\ \dfrac{\lambda-1}{2(\lambda)_2}a^2 & \text{for } k = 1, \\ \dfrac{1}{2(\lambda)_2}a^2 & \text{for } k = 3, \\ 0 & \text{for } k = 0, 2, 4, 5, \ldots, \end{cases}$$

$$\pi_2(k) = \begin{cases} -\dfrac{(\lambda+1)(2\lambda-1)}{2(\lambda)_2}a^2 & \text{for } k = 0, \\ \dfrac{\lambda^2+\lambda-4}{2(\lambda)_2}a^2 & \text{for } k = 2, \\ \dfrac{\lambda+5}{2(\lambda)_2}a^2 & \text{for } k = 4, \\ 0 & \text{for } k = 1, 3, 5, 6, \ldots \end{cases}$$

The Gegenbauer coefficients $\{c_k\}$ of function (3.10) satisfy, therefore, the recurrence relation of the eighth order,

$$Lc_k = \pi_2(k) \quad \text{for } k \geqslant 0,$$

with the above-calculated L and π_2. Since this function is even, we have $c_{2k+1} = 0$ $(k \geqslant 0)$, and for the coefficients $\{w_k\}$, $w_k = c_{2k}$, this equation gives a recurrence relation of the fourth order.

3.2. The generalized Paszkowski algorithm. The results of Paszkowski [7], § 13, for the Chebyshev coefficients will now be generalized for the case of the Gegenbauer coefficients.

From equality (3.3) of Section 3.1 it follows by Lemma 2.8 and Co rollary 2.1 (Section 2.3) that

$$(3.12)\qquad \sum_{i=0}^{n} 2^i \sum_{m=0}^{n-i} \varrho_{nim}(k)c_{k-n+i+2m}[q_i f] = \sum_{m=0}^{n} \varrho_{n0m}(k)c_{k-n+2m}[p].$$

An analogous relation was obtained in the book [7], § 13, for $\lambda =$ (Theorem 13.1). The following theorem may be considered as a general zation of Theorem 13.2 of this book.

THEOREM 3.2. *Equality* (3.12) *implies the recurrence relation*

$$(3.13)\qquad \sum_{i=-d}^{d} \tau_i(k)c_{k+i}[f] = \sum_{m=0}^{n} \varrho_{n0m}(k)c_{k-n+2m}[p] \quad \text{for } k \geqslant 0,$$

where

$$d = \max_{\substack{0\leqslant j\leqslant n \\ p_{n-j}\neq 0}} (d_{n-j}+j),$$

p stands for the right-hand side of the differential equation (1.21), *and poly nomials* $p_0, p_1, \ldots, p_n$ *are the coefficients of this equation,* d_l *denotes th degree of the polynomial* $p_l \not\equiv 0$ $(0 \leqslant l \leqslant n)$. *The coefficients* $\tau_{-d}, \tau_{-d+1}, \ldots,$ *are:* 1° *rational functions for* $\lambda \neq 0$ *or* 2° *for* $\lambda = 0$ — *polynomials in mostly of the degree* $2n-1$, *such that* $\tau_{-i}(k) = -\tau_i(-k)$ $(i = 0, 1, \ldots, d$ *moreover,* $\tau_{-d} \not\equiv 0$ *and* $\tau_d \not\equiv 0$.

If $p_0, p_1, \ldots, p_n$ *are alternately even and odd functions, then* (3.1 *contains only* $d+1$ *coefficients* $c_{k-d}[f], c_{k-d+2}[f], \ldots, c_{k+d}[f]$.

Proof is analogous to that given in [7] for the above-mentione theorem.

The method using the equality (3.12) in order to construct the recu rence relation (3.13) is called the *generalized Paszkowski algorithm* (GPA For $\lambda = 0$, GPA is identical with the algorithm given in [7], § 13, whic we call *Paszkowski's algorithm.*

The equivalence of identities (3.3) and (3.12) implies the identit of relations (3.5) and (3.13) and, therefore, the equivalence of Algorithm and GPA.

As already noticed, Algorithm II (and so GPA) is not, in genera an optimum algorithm. We formulate the following

HYPOTHESIS. *If the coefficient of equation* (1.21), *appearing at t derivative of the highest order, has zeros different from* 1 *and* -1, *then Alg rithm* II (*GPA*) *leads to a recurrence relation for the Gegenbauer coefficien of a function satisfying the assumptions of Theorem* 2.1 (*Section* 2.4), *an the order of this relation is the lowest one in the sense defined in this theore*

We infer from Theorem 4.2, which will be proved in Section 4, that this hypothesis is true in the case of the Chebyshev coefficients ($\lambda = $

4. MODIFICATION OF THE OPTIMUM ALGORITHM FOR THE CHEBYSHEV COEFFICIENTS

Algorithm I, described in Section 2.5, may be applied, in view of the second part of (1.9), to constructing the recurrence relation of the Chebyshev coefficients $\{b_k[f]\}$ of the function f satisfying assumptions of Theorem 2.1 (Section 2.4); to this end it suffices to take in (1.18) and (2.94) the value $\lambda = 0$ (see Example 2.1 in Section 2.5).

We are now going to show that if the coefficient p_n of the differential equation (1.21) of order n has no zeros equal to 1 or -1, then it is possible to modify Algorithm I (without loss of optimality) which mainly consists in the substitution of operators $A_0, A_1, \ldots, A_{n-1}$ by operators $B_0, B_1, \ldots \ldots, B_{n-1}$, respectively, defined by (2.37) for $\lambda = 0$. This follows from Theorem 4.1 which will be proved in Section 4.2.

Let us emphasize that by B_i, S_{ij}, P_i and γ_j we denote in this section operators and functions defined in Section 2.3 by formulae (2.37), (2.39), (2.40) and (2.43), respectively, where the parameter λ is equal to zero. Thus

$$B_i = \begin{cases} D & \text{for } i = 0, \\ k^{-1}[\alpha_i(k)E^{-1} - \beta_i(k)E] & \text{for } i = 1, 2, \ldots, \end{cases} \tag{4.1}$$

$$\alpha_i(k) = (k+i-1)_2, \qquad \beta_i(k) = (k-i)_2, \tag{4.2}$$

$$S_{ij} = \begin{cases} I & \text{for } i < j, \\ B_i B_{i-1} \ldots B_j & \text{for } i \geqslant j \geqslant 0, \end{cases} \tag{4.3}$$

$$P_i = S_{i-1,0} \quad \text{for } i = 0, 1, \ldots, \tag{4.4}$$

$$\gamma_0(k) \equiv 1, \qquad \gamma_i(k) = (k-i+1)_{2i-1} \quad \text{for } i = 1, 2, \ldots \tag{4.5}$$

We also have

$$\begin{aligned} B_j \gamma_j(k) &\equiv 0 \quad \text{for } j = 0, 1, \ldots, \\ S_{ij} \gamma_j(k) &\equiv 0 \quad \text{for } i \geqslant j \geqslant 0. \end{aligned} \tag{4.6}$$

The first identity of (4.6) follows from (4.1), (4.5) and (2.44), and the second one from the first and from definition (4.3) of the operator S_{ij}.

It can also be shown that for any operator L we have

$$\sigma_t(S_{ij}L; k) = S_{ij}\sigma_t(L; k) \quad \text{for } t = 1, 2. \tag{4.7}$$

This formula follows from properties (2.32)-(2.34) of $\sigma_t(L; k)$ and from the definition of the operator S_{ij}.

4.1. The set T of operators. We begin with

Definition 4.1. Let $\boldsymbol{T}$ be the set of difference operators with co: stant coefficients such that

$$(4.8)\qquad Lz_k = \sum_{j=-d}^{d} \lambda_j z_{k+j} \quad \text{for } \{z_k\} \in \boldsymbol{S},$$

where

$$(4.9)\qquad \lambda_j = \lambda_{-j} \quad \text{for } j = 0, 1, \ldots, d.$$

Note that the operators I and Θ belong to the set $\boldsymbol{T}$ as well as — l (1.26) — the operators $L_0^{(i)}$ ($i = 0, 1, \ldots, n$) which satisfy (1.23) and (1.2 for $\lambda = 0$.

LEMMA 4.1. *If $L \in \boldsymbol{T}$ and the operator L^* is such that*

$$(4.10)\qquad L^* z_k = L(kz_k) \quad \textit{for } \{z_k\} \in \boldsymbol{S},$$

then

$$(4.11)\qquad L^* = kL + L_1 D,$$

where $L_1 \in \boldsymbol{T}$ and $D = E^{-1} - E$.

Proof. Let the operator L be such that identity (4.8) holds. Then

$$(4.12)\qquad L^* = kL + M,$$

where the operator M is such that

$$Mz_k = \sum_{j=-d}^{d} j\lambda_j z_{k+j} \quad \text{for } \{z_k\} \in \boldsymbol{S}.$$

If $d = 0$, then $M = \Theta$ and from (4.12) we obtain (4.11) for $L_1 =$ (Let $d \geqslant 1$. Remark that

$$(4.13)\qquad \begin{aligned} \sigma_1(M; k) &= \sum_{j=0}^{d} (2j-d)\lambda_{2j-d} = 0, \\ \sigma_2(M; k) &= \sum_{j=0}^{d-1} (2j-d+1)\lambda_{2j-d+1} = 0, \end{aligned}$$

where the notation is that of (2.31).

Lemma 2.4 and the above-given formulae imply

$$(4.14)\qquad M = L_1 D,$$

the operator L_1 being such that

$$L_1 z_k = \sum_{j=1-d}^{d-1} \tilde{\lambda}_j z_{k+j},$$

where the coefficients $\tilde{\lambda}_j$ are defined by the following formulae:

$$\tilde{\lambda}_{2j-d-1} = -\sum_{i=j}^{d}(2i-d)\lambda_{2i-d} \quad \text{for } j = 1, 2, \ldots, d,$$

$$\tilde{\lambda}_{2j-d} = -\sum_{i=j}^{d-1}(2i-d+1)\lambda_{2i-d+1} \quad \text{for } j = 1, 2, \ldots, d-1.$$

Since

$$\tilde{\lambda}_{d-2j+1} - \tilde{\lambda}_{2j-d-1} = \sigma_1(M; k) \quad \text{for } j = 1, 2, \ldots, d,$$

$$\tilde{\lambda}_{d-2j} - \tilde{\lambda}_{2j-d} = \sigma_2(M; k) \quad \text{for } j = 1, 2, \ldots, d-1,$$

we infer in view of (4.13) that L_1 belongs to the set $\boldsymbol{T}$. Equality (4.11) follows from (4.12) and (4.14).

Assume that

(4.15) $$b_k^{(i)} = b_k[f^{(i)}] \quad \text{for } i = 0, 1, \ldots, n$$

(see (1.8)). Instead of $b_k^{(0)}$, $b_k^{(1)}$ and $b_k^{(2)}$ we will write b_k, b_k' and b_k'', respectively.

LEMMA 4.2. *Let $L_0 \in \boldsymbol{T}$. For every $i = 0, 1, \ldots, n$ we have*

(4.16) $$P_i L_0 b_k^{(i)} = 2^i N_i b_k,$$

where P_i is operator (4.4), *and N_i — an operator defined by*

(4.17) $$N_i = \sum_{j=0}^{i}\binom{i}{j} S_{i-1,j}[\gamma_j(k) L_{i-j}],$$

$S_{i-1,j}$ being the operator (4.3), *γ_j — the function defined by formulae* (4.5), *and the operators $L_1, L_2, \ldots, L_n$ satisfy the relations*

(4.18) $$L_{t-1}(kz_k) = kL_{t-1}z_k + L_t Dz_k$$

for $t = 1, 2, \ldots, n$ and $\{z_k\} \in \mathbf{S}$.

Moreover, we see that

(4.19) $$\sigma_h(N_i; k) = \sigma_{h_i}(L_0; k)\gamma_i(k) \quad \textit{for } h = 1, 2,$$

where $h_i = h_i(h)$ is equal to 1 or 2 and $h_i(1) + h_i(2) = 3$.

Proof (by induction). Since $P_0 = S_{-1,0} = I$ and $\gamma_0 \equiv 1$, we have $N_0 = L_0$; equality (4.16) is for $i = 0$ trivially true.

Assume that (4.16) and (4.17) are true for a certain $i = 0$. From formulae (2.42), (4.17) and (2.41) it follows that

(4.20) $$P_{i+1} L_0 b_k^{(i+1)} = B_i P_i L_0 b_k^{(i+1)} = 2^i B_i N_i b_k'$$
$$= 2^i \sum_{j=0}^{i}\binom{i}{j} S_{ij}[\gamma_j(k) L_{i-j} b_k'].$$

Since $L_0 \in \boldsymbol{T}$, we infer by Lemma 4.1 that the operators $L_1, L_2, \ldots, L_n$, defined by identities (4.18), exist and belong to the set $\boldsymbol{T}$. This means, in particular, that they are operators with constant coefficients. The identity $Db_k' = 2kb_k$ (see (1.13) and (1.9)) and the equality

$$B_j\big(\gamma_j(k)I\big) = k^{-1}\gamma_{j+1}(k)D,$$

following from (2.44), as well as (4.18) and (2.41) imply

$$\begin{aligned}
&S_{ij}\big(\gamma_j(k)L_{i-j}b_k'\big) = S_{i,j+1}B_j\big(\gamma_j(k)L_{i-j}b_k'\big) = S_{i,j+1}\big(k^{-1}\gamma_{j+1}(k)DL_{i-j}b_k'\big)\\
&= S_{i,j+1}\big(k^{-1}\gamma_{j+1}(k)L_{i-j}Db_k'\big) = 2S_{i,j+1}\big(k^{-1}\gamma_{j+1}(k)L_{i-j}(kb_k)\big)\\
&= 2S_{i,j+1}\big(\gamma_{j+1}(k)L_{i-j}b_k + k^{-1}\gamma_{j+1}(k)DL_{i+1-j}b_k\big)\\
&= 2S_{i,j+1}\big[\gamma_{j+1}(k)L_{i-j}b_k + B_j\big(\gamma_j(k)L_{i+1-j}b_k\big)\big]\\
&= 2\big\{S_{i,j+1}\big(\gamma_{j+1}(k)L_{i-j}b_k\big) + S_{ij}\big(\gamma_j(k)L_{i+1-j}b_k\big)\big\} \qquad \text{for } j = 0, 1, \ldots, i.
\end{aligned}$$

The last expression in (4.20) may be, therefore, transformed to the form

$$2^{i+1}\sum_{j=0}^{i+1}\binom{i+1}{j}S_{ij}\big(\gamma_j(k)L_{i+1-j}b_k\big).$$

Hence identity (4.16) holds for every $i = 0, 1, \ldots, n$.

We now prove formula (4.19). By virtue of (2.33), (2.34) and (4.7) it follows from (4.17) that

$$\sigma_h(N_i; k) = \sum_{j=0}^{i}\binom{i}{j}S_{i-1,j}\big(\gamma_j(k)\sigma_{h_j}(L_{i-j}; k)\big) \qquad \text{for } h = 1, 2,$$

where $h_j = h_j(h)$ is equal to 1 or 2 and $h_j(1) + h_j(2) = 3$ ($j = 0, 1, \ldots, i$) Since $\sigma_h(L_t; k) = \text{const}$ ($t = 0, 1, \ldots, n$ and $h = 1, 2$), we infer in view of the second equality of (4.6) that terms of the sum on the right-hand side of the above-given formula, obtained for $j = 0, 1, \ldots, i-1$, are identically equal to zero. The last term of this sum equals $\gamma_i(k)\sigma_{h_i}(L_0; k)$, which implies formula (4.19).

4.2. Modification of Algorithm I. First we show

LEMMA 4.3. *Let f be a function developable into the Chebyshev series* (1.7), *p — a polynomial and L — a difference operator satisfying the identity $Lb_k = b_k[pf]$, where $b_k = b_k[f]$.*

Then the inequality

$$p(-1)p(1) \neq 0$$

is equivalent to the inequality

$$|\sigma_1(L; k)| \neq |\sigma_2(L; k)|.$$

Proof. Let q and r_1x+r_2 be the quotient and the rest, respectively, from the division of the polynomial p by x^2-1:

$$p(x) = (x^2-1)q(x)+r_1x+r_2. \tag{4.21}$$

Define operators M and N by

$$Mb_k = b_k[qf] \quad \text{and} \quad Nb_k = b_k[(r_1x+r_2)f].$$

By (1.18) for $\lambda = 0$ we get

$$b_k[(x^2-1)qf] = \frac{1}{4}(b_{k-2}[qf]-2b_k[qf]+b_{k+2}[qf]) = \frac{1}{4}D^2Mb_k,$$

$$Nb_k = \frac{1}{2}r_1(b_{k-1}+b_{k+1})+r_2b_k,$$

where $D = E^{-1}-E$.

We have

$$Lb_k = b_k[pf] = b_k[(x^2-1)qf]+b_k[(r_1x+r_2)f]$$
$$= \frac{1}{4}D^2Mb_k+Nb_k,$$

which by (2.34) implies

$$\sigma_h(L;k) = \sigma_{h_1}\left(\frac{1}{4}D^2M;k\right)+\sigma_{h_2}(N;k) \quad \text{for } h = 1,2, \tag{4.22}$$

where $h_i = h_i(h) = 1$ or 2 and $h_i(1)+h_i(2) = 3$ $(i = 1,2)$.

Since M is an operator with constant coefficients (which follows from the definition of M and from (1.18) for $\lambda = 0$), we have $\sigma_h(M;k) = \text{const}$ $(h = 1,2)$. Using (2.32)-(2.34) we verify that

$$\sigma_h\left(\frac{1}{4}D^2M;k\right) = \frac{1}{4}D^2\sigma_h(M;k) \equiv 0 \quad \text{for } h = 1,2.$$

It is easy to see that

$$\sigma_h(N;k) = r_j \quad \text{for } h = 1,2,$$

where

$$j = j(h) = \begin{cases} h & \text{for } r_1 \neq 0 \text{ or } r_2 = 0, \\ 3-h & \text{for } r_1 = 0 \text{ and } r_2 \neq 0. \end{cases}$$

Consequently, it follows from (4.22) that

$$\sigma_h(L;k) = r_l \quad \text{for } h = 1,2,$$

where $l = l(h) = 1$ or 2, whereas $l(1)+l(2) = 3$. This result together with equality (4.21) proves our lemma.

LEMMA 4.4. *Let*

$$\hat{L}_0^{(i)} = L_0^{(i)} \quad \text{for } i = 0, 1, \ldots, n, \tag{4.23}$$

where $L_0^{(0)}, L_0^{(1)}, \ldots, L_0^{(n)}$ *are operators satisfying* (1.23) *for* $\lambda = 0$. *Let* t *be such that* $1 \leqslant t \leqslant n$. *Suppose that the operator* B_{t-1} *defined by* (4.1) *belongs to the set* $\boldsymbol{A}(\hat{L}_{t-1}^{(n-t+1)})$ *and, therefore, that there exists an operator* $\hat{Q}_{t-1}$ *for which*

$$B_{t-1}\hat{L}_{t-1}^{(n-t+1)} = \hat{Q}_{t-1} D \tag{4.24}$$

and define operators $\hat{L}_t^{(0)}, \hat{L}_t^{(1)}, \ldots, \hat{L}_t^{(n-t)}$ *by the formulae*

$$\hat{L}_t^{(i)} = B_{t-1}\hat{L}_{t-1}^{(i)} \quad \text{for } i = 0, 1, \ldots, n-t-1, \tag{4.25}$$

$$\hat{L}_t^{(n-t)} = B_{t-1}\hat{L}_{t-1}^{(n-t)} + \hat{M}_{t-1}, \tag{4.26}$$

where $\hat{M}_{t-1}$ *is an operator for which*

$$\hat{M}_{t-1} z_k = 2\hat{Q}_{t-1}(kz_k) \quad \text{for } \{z_k\} \in \boldsymbol{S}. \tag{4.27}$$

We then have

$$\sigma_h(\hat{L}_t^{(n-t)}; k) = 2^t \sigma_{h_t}(\hat{L}_0^{(n)}; k)\gamma_t(k) \quad \text{for } h = 1, 2, \tag{4.28}$$

where γ_t *is the polynomial defined by formulae* (4.5) *and* $h_t = h_t(h) = 1$ *or* 2, *whereas* $h_t(1) + h_t(2) = 3$.

From Lemma 4.5, which will be proved in the sequel, it follows that the assumption on the operator B_{t-1} in Lemma 4.4 is satisfied for every $t = 1, 2, \ldots, n$.

Proof. Note that equalities (2.72), (2.73), (2.75) and (2.76), which define the operators $L_t^{(i)}$ for $\lambda = 0$ (Section 2.4), and relations (4.25)-(4.27), (4.24), respectively, are from the formal point of view the same up to the notation. It may be shown that analogously to (2.78) we have

$$\hat{L}_t^{(n-t)} b_k^{(n-t)} = P_t \sum_{i=n-t}^{n} \hat{L}_0^{(i)} b_k^{(i)}, \tag{4.29}$$

where P_t is the operator defined by (4.4).

Applying the equality

$$P_t = S_{t-1,j} P_j \quad \text{for } j = 0, 1, \ldots, t,$$

corresponding to (4.3) and (4.4), we will transform the right-hand side of (4.29) to the form

$$\sum_{j=0}^{t} S_{t-1,j} P_j \hat{L}_0^{(n-t+j)} b_k^{(n-t+j)}.$$

In view of (4.23) and (1.24), (1.26), the operators $\hat{L}_0^{(j)}$ $(j = 0, 1, \ldots, n)$ belong to the set $\boldsymbol{T}$, which by Lemma 4.2 implies that there exist operators $N_{0t}, N_{1t}, \ldots, N_{tt}$ such that

$$P_j \hat{L}_0^{(n-t+j)} b_k^{(n-t+j)} = 2^j N_{jt} b_k^{(n-t)} \quad \text{for } j = 0, 1, \ldots, t$$

and

$$\text{(4.30)} \quad \sigma_h(N_{jt}; k) = \sigma_{h_j}(\hat{L}_0^{(n-t+j)}; k)\gamma_j(k) \quad \text{for } j = 0, 1, \ldots, t \text{ and } h = 1, 2,$$

where $h_j = h_j(h) = 1$ or 2 and $h_j(1) + h_j(2) = 3$.

We thus get the equality

$$\hat{L}_t^{(n-t)} = \sum_{j=0}^{t} 2^j S_{t-1,j} N_{jt}$$

which, by virtue of (2.33), (2.34), (4.7) and (4.30), implies

$$\sigma_h(\hat{L}_t^{(n-t)}; k) = \sum_{j=0}^{t} 2^j \sigma_{h_j}(\hat{L}_0^{(n-t+j)}; k) S_{t-1,j}\gamma_j(k) \quad \text{for } h = 1, 2.$$

Using the second equality of (4.6) we state that the terms of the sum on the right-hand side, obtained for $j = 0, 1, \ldots, t-1$, are identically equal to zero. The last term of this sum is equal to $2^t \sigma_{h_t}(\hat{L}_0^{(n)}; k)\gamma_t(k)$, for $S_{t-1,t} = I$, which implies relation (4.28), q.e.d.

LEMMA 4.5. *For every $t = 0, 1, \ldots, n-1$, the operator B_t, defined by* (4.1), *belongs to the set $\boldsymbol{A}(\hat{L}_t^{(n-t)})$, where $\hat{L}_t^{(n-t)}$ is defined in Lemma* 4.4.

Proof. Since $B_0 = D$, and $\hat{L}_0^{(n)}$ is the operator with constant coefficients, we have

$$B_0 \hat{L}_0^{(n)} = \hat{L}_0^{(n)} D,$$

which means that $B_0 \in \boldsymbol{A}(\hat{L}_0^{(n)})$.

Suppose that $1 \leqslant t \leqslant n-1$ and that the operator B_{t-1} belongs to the set $\boldsymbol{A}(\hat{L}_{t-1}^{(n-t+1)})$. Applying formula (4.7), Lemma 4.4 (formula (4.28)) and the first equality of (4.6) we infer that

$$\sigma_h(B_t \hat{L}_t^{(n-t)}; k) = B_t \sigma_h(\hat{L}_t^{(n-t)}; k) = 2^t \sigma_{h_t}(\hat{L}_0^{(n)}; k) B_t \gamma_t(k) \equiv 0 \quad \text{for } h = 1, 2.$$

Consequently, by virtue of Lemma 2.4 we see that there exists an operator $\hat{Q}_t$ such that

$$B_t \hat{L}^{(n-t)} = \hat{Q}_t D$$

and, therefore, $B_t \in \boldsymbol{A}(\hat{L}_t^{(n-t)})$, q.e.d.

It follows from the last lemma that every stage of the construction described in Lemma 4.4 is realizable when we form the system of operators $\hat{L}_t^{(i)}$ $(t = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, n-t)$.

Assume that

$$(4.31)\qquad \hat{\pi}_0(k) = b_k[p], \quad \hat{\pi}_t(k) = B_{t-1}\hat{\pi}_{t-1}(k) \text{ for } t = 1, 2, \ldots, n,$$

where $p = p(x)$ denotes the right-hand side of (1.21).

Determine, moreover, polynomials $\delta_0, \delta_1, \ldots, \delta_{n+2}$ of the variable k by

$$(4.32)\qquad \delta_0 = \delta_1 \equiv 1, \quad \delta_{t+2} = \gamma_t \delta_t \text{ for } t = 0, 1, \ldots, n,$$

where γ_t is the polynomial defined by (4.5). It is easy to see that

$$(4.33)\qquad \delta_{t+2}^{+} = (k+1)_t \delta_{t+1}, \quad \delta_{t+2}^{-} = (k-t)_t \delta_{t+1} \quad \text{for } t = 0, 1, \ldots, n,$$

the notation being that of (1.1) and (1.17).

Lemma 4.6. *If the coefficient p_n of the derivative of the highest order in the differential equation* (1.21) *is such that*

$$(4.34)\qquad p_n(-1)p_n(1) \neq 0,$$

then for every $t = 0, 1, \ldots, n$ we have

$$(4.35)\qquad L_t^{(i)} = \delta_t(k)\hat{L}_t^{(i)} \quad \textit{for } i = 0, 1, \ldots, n-t,$$

$$(4.36)\qquad \pi_t = \delta_t \hat{\pi}_t,$$

where $L_t^{(0)}, L_t^{(1)}, \ldots, L_t^{(n-t)}$ are operators and π_t are functions formed in Algorithm I (*Section* 2.5), *$\hat{L}_t^{(0)}, \hat{L}_t^{(1)}, \ldots, \hat{L}_t^{(n-t)}$ denote operators defined in Lemma* 4.4, *$\hat{\pi}_t$ and δ_t are functions defined by* (4.31) *and* (4.32), *respectively.*

Moreover, operators $A_0, A_1, \ldots, A_{n-1}$, defined in Algorithm I, *are such that*

$$(4.37)\qquad A_t = (k+1)_t E^{-1} - (k-t)_t E \quad \textit{for } t = 0, 1, \ldots, n-1.$$

Proof. The operator $L_0^{(n)}$ satisfies by definition the identity

$$b_k[p_n f^{(n)}] = L_0^{(n)} b_k[f^{(n)}],$$

hence by Lemma 4.3 inequality (4.34) implies the alternative

$$(4.38)\qquad \sigma_1 \neq 0, \quad \sigma_2 = c_1\sigma_1 \quad \text{for } c_1^2 \neq 1$$

or

$$(4.39)\qquad \sigma_2 \neq 0, \quad \sigma_1 = c_2\sigma_2 \quad \text{for } c_2^2 \neq 1,$$

where

$$(4.40)\qquad \sigma_h = \sigma_h(L_0^{(n)}; k) \quad \text{for } h = 1, 2.$$

It is easily seen that equalities (4.35) and (4.36) are true for $t = 0$. It may be verified that, according to Algorithm I, the operator A_0 is in view of (4.38)-(4.40) of form (4.37) for $t = 0$.

Suppose that (4.35) and (4.36) hold for a t such that $0 \leqslant t \leqslant n-1$. By virtue of (2.33), (2.34) and (4.28) we then have

$$\sigma_h(L_t^{(n-t)}; k) = 2^t \sigma_{h_t}(\hat{L}_0^{(n)}; k)\gamma_t(k)\,\delta_t(k) \quad \text{for } h = 1, 2,$$

where $h_t = 1$ or 2 and, therefore, by (4.23), (4.40) and (4.32), we obtain

$$\sigma_h(L_t^{(n-t)}; k) = 2^t \sigma_{h_t}\delta_{t+2}(k) \quad \text{for } h = 1, 2.$$

So, if relations (4.38) hold, then

$$\eta(k) = \sigma_1(L_t^{(n-t)}; k) \not\equiv 0 \qquad \vartheta(k) = \sigma_2(L_t^{(n-t)}; k) = c_1\eta(k),$$

where c_1 is the constant occurring in (4.38); according to Algorithm I, the operator A_t is defined by (2.13) for the above-given values η and ϑ. Since formula (2.13) is connected with case IV we have

$$A_t = \frac{1}{\omega_t(k)}[\delta_{t+2}^{+}(k)E^{-1} - \delta_{t+2}^{-}(k)E],$$

where $\omega_t = \gcd(\delta_{t+2}^{+}, \delta_{t+2}^{-})$, which in view of (4.33) implies equality (4.37).

It may be similarly proved that this equality occurs also in the case where relations (4.39) hold.

It is easy to verify that if the operators A_t and B_t are defined by (4.37) and (4.1) respectively, then we have

$$A_t[\delta_t(k)I] = \delta_{t+1}(k)B_t,$$

which by (4.35) and (4.36) implies the relations

$$A_t L_t^{(i)} = \delta_{t+1}(k)B_t\hat{L}_t^{(i)} \quad \text{for } i = 0, 1, \ldots, n-t,$$

$$A_t\pi_t(k) = \delta_{t+1}(k)B_t\hat{\pi}_t(k).$$

By virtue of (2.72)-(2.76), (4.24)-(4.27) and the second formula of (4.31) we thus get

$$L_{t+1}^{(i)} = \delta_{t+1}(k)\hat{L}_{t+1}^{(i)} \quad \text{for } i = 0, 1, \ldots, n-t-1,$$

$$\pi_{t+1} = \delta_{t+1}\hat{\pi}_{t+1},$$

which completes the proof of Lemma 4.6.

THEOREM 4.1. *If the function f satisfies the differential equation* (1.21) *of order n and the coefficient p_n of this equation has no zeros equal to* 1 *or to* -1 *and if the derivative $f^{(n)}$ may be expanded into a uniformly convergent series of Chebyshev, then the recurrence relation*

$$\hat{L}_n^{(0)}b_k[f] = \hat{\pi}_n(k) \quad \textit{for } k \geqslant 0 \tag{4.41}$$

is true and has the lowest order — in the sense defined in Theorem 2.1 (*Section* 2.4) *— which is equal to number* (1.29).

Proof. Equation (4.41) follows from the first part of Theorem 2.1 and from equalities (4.35) and (4.36). It is obvious that this equation and equation (2.87) have the same and, therefore, the lowest order expressed by formula (2.88). In virtue of Lemma 4.6 all operators $A_0, A_1, \ldots, A_{n-1}$ are of the second order, hence the right-hand side of (2.88) and expression (1.29) are equal, q.e.d.

Therefore, if the assumptions of Theorem 4.1 are satisfied, it is possible to modify part (b) of Algorithm I (for $\lambda = 0$) in the following manner:

For $t = 1, 2, \ldots, n$:

1° Form the operator B_{t-1} according to formula (4.1).

2° Form the operator S_{t-1} according to the formula

$$S_{t-1} = B_{t-1} L_{t-1}^{(n-t+1)}$$

and then the operator Q_{t-1} such that $S_{t-1} = Q_{t-1} D$.

3° Construct operators $M_{t-1}, L_t^{(0)}, L_t^{(1)}, \ldots, L_t^{(n-t)}$ and a function π_t using formulae (2.94)-(2.97), where the symbol A_{t-1} was replaced by the symbol B_{t-1}.

THEOREM 4.2. *Assume that the function f satisfies the differential equation* (1.21) *of order n and that the derivative $f^{(n)}$ can be expanded into a uniformly convergent Chebyshev series. The Paszkowski Algorithm* (*Section* 3.2) *and, therefore, the equivalent for $\lambda = 0$ Algorithm* II (*Section* 3.1), *lead to a recurrence relation for the Gegenbauer coefficients of the function f, and this relation is of the lowest order — in the sense assumed in Theorem* 2.1 (*Section* 2.4) *— if and only if the coefficient p_n of equation* (1.21) *has no zeros equal to* 1 *or to* -1.

Proof. 1° Suppose that the polynomial p_n has no zeros equal to 1 or -1. Let us recall that Paszkowski's algorithm gives a recurrence relation for the Chebyshev coefficients of the function f and the order of this relation is equal to number (1.29). By virtue of Theorem 4.1 this is the relation of the lowest order.

2° If $p_n(1) = 0$ or $p_n(-1) = 0$, then by Lemma 4.3 we get

$$|\sigma_1(L_0^{(n)}; k)| = |\sigma_2(L_0^{(n)}; k)|.$$

The operator A_0, formed in Algorithm I, is then defined by formula (2.13) for which we have case I, II or III and, therefore, the order of this operator is not greater than 1. This and formula (2.88) imply (in view of the fact that for $j = 1, 2, \ldots, n-1$ the order of the operator A_j is not greater than 2) that equation (2.87) for $\lambda = 0$, being satisfied by the Chebyshev coefficients of the function f, has the order lower than number (1.29).

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References

[1] C. W. Clenshaw, *The numerical solution of linear differential equations in Chebyshev series*, Proc. Cam. Phil. Soc. 53 (1957), p. 134-149.

[2] D. Elliott, *The expansion of functions in ultraspherical polynomials*, J. Austral. Math. Soc. 1 (1959-1960), p. 428-438.

[3] A. Erdélyi et al., *Higher transcendental functions*, New York 1953.

[4] E. Kamke (Э. Камке), *Справочник по обыкновенным дифференциальным уравнениям*, Москва 1971.

[5] Y. L. Luke, *The special functions and their approximations*, New York - London 1969.

[6] J. Oliver, *The numerical solution of linear recurrence relations*, Numer. Math. 11 (1968), p. 349-360.

[7] S. Paszkowski, *Zastosowania numeryczne wielomianów i szeregów Czebyszewa*, Warszawa 1975.

[8] R. P. van de Riet, *ABC ALGOL*, Mathematical Centre Tracts 46-47, Amsterdam 1973.

[9] R. E. Scraton, *A modification of Miller's recurrence algorithm*, BIT 12 (1972), p. 242-251.

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KONSTRUKCJA ZWIĄZKU REKURENCYJNEGO NAJNIŻSZEGO RZĘDU DLA WSPÓŁCZYNNIKÓW SZEREGU GEGENBAUERA

STRESZCZENIE

Funkcję f, określoną w przedziale $\langle -1, 1\rangle$ i spełniającą odpowiednie warunki, można rozwinąć w jednostajnie zbieżny w tym przedziale *szereg Gegenbauera* (1.5), tj. szereg względem wielomianów Gegenbauera (1.2) ortogonalnych z wagą $(1-x^2)^{\lambda-1/2}$ $(\lambda > -\frac{1}{2})$. Szczególnie ważne rozwinięcie otrzymujemy w wypadku $\lambda = 0$; wobec (1.4) jest ono prosto związane z szeregiem Czebyszewa (1.7) funkcji f. Współczynniki tych szeregów wyrażają się odpowiednio wzorami (1.6) i (1.8). Będziemy rozważać łącznie szeregi Gegenbauera dla $\lambda \neq 0$ i szereg Czebyszewa i określamy w tym celu za pomocą wzoru (1.9) ciąg *współczynników Gegenbauera*.

Obliczanie całek występujących w (1.6) lub (1.8) jest ogólnie biorąc zadaniem złożonym. Jawne wyrażenia dla współczynników Gegenbauera otrzymano dla wielu funkcji (zobacz np. [5], tom II, §§ 9.2 i 9.3, lub [7], §§ 10-12). Jednak takie wyrażenia są często skomplikowane, np. zawierają symbole funkcji specjalnych, co utrudnia ich stosowanie.

Wartości współczynników Gegenbauera można znaleźć stosunkowo łatwo, jeśli współczynniki te spełniają związek rekurencyjny postaci (1.20). Przegląd przybliżonych metod rozwiązywania takich równań można znaleźć np. w [7], § 15.

Wśród metod konstrukcji związków typu (1.20) dla współczynników Gegenbauera danej funkcji f prostotą i uniwersalnością wyróżnia się metoda stosowana wówczas, gdy funkcja ta spełnia równanie różniczkowe liniowe rzędu n (1.21), w którym $p_0, p_1, \ldots, p_n \not\equiv 0$ są wielomianami, a funkcja p ma znane współczynniki Gegenbauera. Paszkowski [7], § 13, podał algorytm konstrukcji związku (1.20) dla współczynników szeregu Czebyszewa na podstawie równania (1.21); rząd tego związku jest równy liczbie (1.29). W cytowanej pracy zauważono, że w pewnych wypadkach nie jest to związek najniższego możliwego rzędu.

W obecnej pracy przedstawiamy metodę budowy związku rekurencyjnego (1.20) dla współczynników Gegenbauera, w którym $\lambda_0, \lambda_1, \ldots, \lambda_r$ są funkcjami wymiernymi, najniższego rzędu wśród związków wynikających z (1.21). Opisowi i uzasadnieniu tej metody jest poświęcony § 2, który dzieli się na 5 podparagrafów. Po podaniu podstawowych definicji (§ 2.1) wprowadzamy zbiór $\boldsymbol{A}(L)$ operatorów różnicowych i definiujemy operator minimalny w tym zbiorze, a następnie dowodzimy pewnych własności zbioru $\boldsymbol{A}(L)$, które grają zasadniczą rolę w proponowanej metodzie konstrukcji równania (1.20) (§ 2.2). W § 2.3 określamy pewne pomocnicze operatory różnicowe i dowodzimy ich własności, które zostaną wykorzystane w §§ 2.4, 3.1 i 3.2. W § 2.4 podajemy (poprzedzone niezbędnymi definicjami i lematami) twierdzenie 2.1, zawierające zapowiedziany wynik: istnienie i konstrukcja związku (1.20), najniższego rzędu dla współczynników Gegenbauera funkcji spełniającej równanie (1.21). W § 2.5 formalizujemy metodę budowy tego związku (Algorytm I), a także przedstawiamy dwa przykłady jej zastosowania.

W § 3 uogólniamy wspomnianą wyżej metodę Paszkowskiego dla współczynników Gegenbauera dla $\lambda \neq 0$. Wynik ten jest zawarty w twierdzeniach 3.1 (§ 3.1) i 3.2 (§ 3.2). W § 3.1 podajemy sformalizowany opis metody (Algorytm II) i dwa przykłady jej zastosowania.

Wreszcie w § 4 formułujemy konieczny i dostateczny warunek na to, by algorytm Paszkowskiego prowadził do związku rekurencyjnego dla współczynników szeregu Czebyszewa funkcji spełniającej równanie (1.21), najniższego możliwego rzędu (zobacz twierdzenie 4.2 w § 4.2).