ANNALES
POLONICI MATHEMATICI
XXXIII (1976)
CONFERENCE ON ANALYTIC FUNCTIONS

New viewpoints, results and problems in the theory of Phragmén–Lindelöf

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Abstract. By presenting a number of new results and problems we show that the theory of Phragmén–Lindelöf should not be regarded as a field in which further development is out of the question. To make use of Lemma 3.1 conveniently, we consider the half plane H_r particularly favourable for our purpose. We stress that it is natural to impose restrictions on the increase of the analytical function under consideration not only in the whole of H_r, but also at least on one ray in H_r starting from 0. This new viewpoint permits us also to obtain a number of very elementary results.

1. Introduction. Throughout this paper we consider analytic functions in the half plane $H_r = \{s = u+iv\colon\ u > 0\}$. If we include the imaginary axis, we write $\bar{H}_r = \{s\colon\ u \geqslant 0\}$. For the sake of brevity we put

$$\bar{N} = \left\{ f(s)\colon \begin{array}{l} \text{analytic in } H_r, \\ \text{continuous in } \bar{H}_r,\ \ |f(iv)| \leqslant 1 \end{array} \right\},$$

$$\bar{M} = \{f(s) \in \bar{N}\colon\ |f(s)| \leqslant 1 \text{ in } H_r\}.$$

For all results which are based on Lemma 3.1 – they will be called *non-elementary* – we need slightly more restricted sets, namely

$$N = \{f(s) \in \bar{N}\colon \text{ zeros of } f \text{ have no finite limit point}\},$$

$$M = \bar{M} \cap N.$$

The basic problem in the theory of Phragmén–Lindelöf is to obtain the conclusion "$R \in \bar{N}$ is also $\in \bar{M}$" from appropriate restrictions imposed on the increase of R. This increase can be conveniently described by means of the function

$$M_R(r) = \max_{|\varphi| \leqslant \frac{\pi}{2}} |R(re^{i\varphi})|,$$

which defines the order and type of $R \in \bar{N}$. But also complicated average functions describing the behaviour of $R(re^{i\varphi})$ on semi-circular arcs are sometimes used; see e.g. [2].

At present the theory of Phragmén–Lindelöf seems to be "out of fashion" and it is only rarely that an author tries to develop it. In spite of that there are several new results and possibilities which deserve attention.

In this paper we want to make it clear that it is natural in the theory of Phragmén–Lindelöf to restrict the rate of increase of R not only generally in the whole of H_r. In all theorems of this paper (except Theorem 2.1) we also impose restrictions on the increase of $R(re^{i\varphi})$ $(r\to\infty;\ \varphi$ fixed); in this way we can relax the general restrictions. A few theorems of this kind are not new. The most important of them is that of Pólya–Szegö (see Section 2), but unfortunately only little attention has been paid to it so far. A systematic treatment of this viewpoint seems to be possible and desirable.

2. The Phragmén–Lindelöf Principle with elementary consequences. We will state the following principle in a version which is rather trivial if we consider it as a theorem. But we want to study the restriction for the increase of $R\in\bar{N}$ which is expressed by it. Of course, the principle could prove fruitful only, because there exist very useful sufficient (and necessary) conditions for inequality (2.1).

PRINCIPLE OF PHRAGMÉN–LINDELÖF. *Assume $R\in\bar{N}$ and let there exist a function $f\in\bar{M}$, $f\not\equiv 0$, such that for all $d\in(0,1]$*

$$|f^d(s)R(s)|\leqslant 1,\qquad s\in H_r; \tag{2.1}$$

then $R\in\bar{M}$.

Putting $f(s)=\exp\{-s^q\}$, $0<p<q<1$ we easily obtain the following result from this principle.

THEOREM 2.1. *Assume $R\in\bar{N}$ and let R have at most order $p\in(0,1)$; then $R\in\bar{M}$.*

In textbooks this proposition is often used to prove the famous theorem of Phragmén–Lindelöf ($p=1$, minimal type). Characteristic for this proof is the decomposition of H_r into two quarter planes; we can map Theorem 2.1 to them and then apply it to the auxiliary function $R_\delta(s)=e^{-\delta s}R(s)$ in each quarter plane. It was Pólya and Szegö who recognized that for this method order 1 and minimal type is necessary only for $s=u>0$. From this assumption it follows that $R_\delta(u)$ is bounded and three applications of Theorem 2.1 yield $|R_\sigma(s)|\leqslant 1$ $\forall\delta\in(0,1]$. In this way an important generalization of the theorem of Phragmén–Lindelöf can be obtained, namely the

THEOREM OF PÓLYA–SZEGÖ. *Assume $R\in\bar{N}$ at most of order 1 (intermediate type) and*

$$\overline{\lim_{u\to\infty}}\frac{\log|R(u)|}{u}\leqslant 0; \tag{2.2}$$

then $R\in\bar{M}$.

As shown in [4] this idea yields even the more general

THEOREM 2.2. *Assume* $R \in \bar{N}$ *at most of order* $p \in (0, 2)$ *and* (2.2); *then* $R \in \bar{M}$.

Obviously the procedure described above can be repeated, but this time we make use of Theorem 2.2 instead of Theorem 2.1. Then we obtain a theorem in the assumptions of which 3 rays play an outstanding role [4].

THEOREM 2.3. *Assume* $R \in \bar{N}$ *at most of order* $p \in (0, 4)$, (2.2), *and*

$$\overline{\lim_{r\to\infty}} \frac{\log |R(re^{\pm i\pi/4})|}{r^2} \leqslant 0; \tag{2.3}$$

then $R \in \bar{M}$.

Now we are in a position to improve Theorem 2.2 slightly; namely Theorem 2.3 immediately yields the following result, in the assumptions of which it is only one ray which plays a role.

COROLLARY 2.1. *Assume* $R \in \bar{N}$ *at most of order* $p = 2$ (*minimal type*) *and* (2.2); *then* $R \in \bar{M}$.

This idea can of course be extended to an arbitrary odd number of rays. If we use an even number of rays, the results are more complicated, see [3]. Let us for instance consider the case of the rays $re^{\pm i\pi/6}$, $r > 0$. Then we can again consider $R_3(s)$ and apply Corollary 2.1 to the angle $-\pi/6 < \varphi < \pi/2$; but for the angle $-\pi/2 < \varphi < -\pi/6$ – if we do not want to decompose it – the best theorem we have at hand is Theorem 2.2. Hence we need different assumptions on the increase of R in the two angles and on the two rays as well, namely order 1 on $s = re^{-i\pi/6}$ and order 3/2 on $s = re^{i\pi/6}$ (intermediate types).

But these are not the only possibilities of making an elementary use of the Phragmén–Lindelöf principle. As an example we give a theorem of Ostrowski ([6], Lemma 5.2.1) slightly adapted to our representation. For its proof H_r is also decomposed once. We expect that interesting generalizations of it will be possible.

THEOREM 2.4. *Assume* $R \in \bar{N}$ *and* (2.2); *moreover*,

$$|R(s)| \leqslant e^{u^2}, \quad s = u + iv \in H_r;$$

then $R \in \bar{M}$.

Interesting in our context is Beurling's theorem, for which we obtain two different generalizations. In both of them the assumption (2.2) plays an important role.

BEURLING'S THEOREM (elementary version). *Suppose that* $f(s)$ *is analytic and at most of order* 2 (*minimal type*) *in* H_r; *furthermore let* (2.2) *be satisfied. Suppose that* $\Phi(r) > 0$ *is continuous for* $r > 0$ *and unbounded* *as* $r \to \infty$; *moreover, let be* $0 < \varrho < 1$ *and*

$$\overline{\lim_{r\to\infty}} \frac{\log \Phi(r)}{r^{\varrho}} = 0.$$

If $|f(\pm iv)| \leqslant \Phi(v)$ *for* $v > 0$, *then there exists a sequence*

$$0 < R_1 < R_2 < \ldots, \qquad R_n \to \infty,$$

such that

$$\log M_f(R_n) < \sec \frac{\pi\varphi}{2} \log \Phi(R_n).$$

The proof is literally the same as in [1]; we only use Corollary 2.1 instead of the theorem of Phragmén–Lindelöf.

3. Non-elementary theorems with restrictions on rays. In Section 2 the importance of the asymptotic behaviour of $R(re^{i\varphi})$ for one or several $\varphi \in (-\pi/2, \pi/2)$ was a consequence of the decompositions of H_r, which play a considerable role in the proofs. Now we show that it is not only this simple trick which leads to the necessity of directing attention to certain rays.

In [1], Chapter 6, the theory of "functions of exsponential type" is presented. We generalize this concept in the following way.

DEFINITION 3.1. Let the function $f(s)$ be analytic in H_r. It is of exponential type in the larger sense (i.l.s.) if its increase is restricted in the following way. For every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that for a certain number $C > 0$ — independent of ε — we have the inequality

$$\log |f(s)| < C_\varepsilon + \frac{\varepsilon r}{\cos\varphi} + (C+\varepsilon) r, \qquad s = re^{i\varphi} \in H_r. \tag{3.1}$$

We next quote a lemma from [10]; without the additional remark "i.l.s." it can be found in [1] and (slightly less general) in [5]. Its basic importance for the Phragmén–Lindelöf theory has not been recognized yet.

LEMMA 3.1. *Let* $g \in N$ *belong to the exponential type i.l.s. Then we have in* H_r *the representation*

$$\log |g(s)| = \frac{u}{\pi} \int_{-\infty}^{\infty} \frac{\log |g(it)|}{(t-v)^2 + u^2}\, dt + \log |h_g(s)| + k_g u, \tag{3.2}$$

where $h_g(s)$ *is analytic and* $|h_g(s)| \leqslant 1$ *in* H_r,

$$k_g = \overline{\lim_{u\to\infty}} \frac{\log |g(u)|}{u}. \tag{3.3}$$

This lemma is essential in our context, as no simpler proofs of the following Theorems 3.1 and 3.2 can be expected than those which are based on (3.2). But the lemma is also interesting for another reason, namely it makes apparent the essential role played by certain rays in the Phragmén–Lindelöf theory. In the assumptions of the lemma no ray is mentioned. But, as shown in (3.2), the asymptotic behaviour of $g(u)$ $(u > 0)$ is essential, because it is just the constant (3.3) which decides if $g \in N$ is bounded or not.

But the positive real half-line is not the only possible ray. From [5] (V. § 4, Theorem 7) we infer

COROLLARY 3.1. $k_g \leqslant 0$ *if and only if for some fixed* φ

$$\overline{\lim_{r\to\infty}} \frac{\log|g(re^{i\varphi})|}{r} \leqslant 0, \quad |\varphi| < \frac{\pi}{2}.$$

THEOREM 3.1. *Assume that* $R \in N$ *belongs to the exponential type i.l.s.,* $k_R \leqslant 0$; *then* $R \in M$.

Remark. The proof is obvious from (3.2). Note that Theorem 3.1 is a generalization of the theorem of Pólya–Szegö obtained without a decomposition of H_r. Probably it is the most general result which can be obtained without decomposition.

THEOREM 3.2(¹). *Assume* $R \in N$, *and let there exist* $f \in M$, $f(s) \not\equiv 0$, *with* $k_f = 0$ *that*

$$|f(s)R(s)| \leqslant 1 \quad \text{in } H_r; \tag{3.4}$$

then $R \in M$.

Remark. In this case it is the assumption $k_f = 0$ that implies a condition on a ray. Without this assumption the theorem is wrong, as is shown by the example $f(s) = e^{-s}$, $R(s) = e^{s}$. Note that $k_f = 0$ is a substitute for the assumption "for all $d \in (0,1]$" in the principle of Phragmén–Lindelöf; in (3.4) we have only $d = 1$.

In [10] we proved the following counterpart of Theorem 3.2.

THEOREM 3.3. *Assume* $R \in N$ *and* (2.2), *and let there exist a function* $f \in M$, $f(s) \not\equiv 0$, *such that* (3.4) *holds in* H_r; *then* $R \in M$.

Finally we return to Beurling's theorem. Replacing the theorem of Phragmén–Lindelöf by Theorem 3.1 in the proof given in [1], we obtain

BEURLING'S THEOREM (non-elementary version). *In the elementary version we can raplace the assumption "f at most of order two" by "f is the exponential type i.l.s.".*

4. Applications of the Principle and Theorem 3.2. The Principle and Theorem 3.2 have a common feature, namely the assumption "There

(¹) The intimate relation of this theorem to the Phragmén–Lindelöf theory was analyzed first by W. Tutschke in [12].

exists a function $f \in M$ ...". In spite of that they admit applications of quite different types.

As for the Principle, several of its immediate consequences have been stated in this paper, namely Theorems 2.1 and 2.4; they contain explicit restrictions for the increase of $R \in \bar{N}$ from which $R \in \bar{M}$ follows. It is characteristic for the applications of the Principle that very special $f \in \bar{M}$ are used which seem to be appropriate; often it is the essence of such proofs to find an appropriate $f \in \bar{M}$. But the use of a special $f \in \bar{M}$ implies a loss of information.

Theorem 3.2, too, has already been applied to very different problems, particularly in probability theory and statistics. We give here an example to indicate a quite different type of results which can be proved by it; cf. [9].

THEOREM 4.1. *Let $r(x)$ be a real function of bounded variation in $0 \leqslant x \leqslant \infty$, $r(x) = 0$ $(x < 0)$. Furthermore, let $F(x)$ be a distribution function with the property $F(0) = 0$, $F(\varepsilon) > 0$ $\forall \varepsilon > 0$, the Laplace–Stieltjes transform of which*

$$\int_0^\infty e^{-sx} dF(x) = f(s), \qquad s \in \bar{H}_r,$$

having no zeros in $\bar{H}_r$. If

$$\int_{-0}^{\infty} F(u-x)\, dr(u) = 0, \qquad \forall x \geqslant t \geqslant 0 \tag{4.1}$$

(for fixed $t \geqslant 0$), then $r(x) =$ const *for $x > t$.*

Remark 1. It is basic for the use of Laplace–Stieltjes Transforms in Theorem 3.2 that "$F(0) = 0$, $F(\varepsilon) > 0$" is equivalent to $k_f = 0$.

Remark 2. The point is that by means of Theorem 3.2 we get rid of the unknown function F in (4.1) and have a statement about $r(x)$. In this way we actually make the best of the fact that $f(s) \in M$ in Theorem (3.2) is arbitrary.

Remark 3. If zeros of $f(s)$ are admitted, the theorem is wrong

Quite a different application is made in the following proposition, which is apparently new. Note that by Lemma 3.1 every $g \in M$ can be represented by

$$g(s) = \varphi(s) h_g(s) e^{-k_g s}, \tag{4.2}$$

where $\varphi(s) \neq 0$ in H_r has the property $k_\varphi = 0$; moreover, we have $h_g(s) \equiv 1$, if $g(s) \neq 0$ in H_r; see [8].

THEOREM 4.2. *Assume* $f, g \in M$, $g(iv) \neq 0$,

$$Q(s) = \frac{f(s)}{g(s)}, \quad where \ |Q(iv)| \leqslant 1. \tag{4.3}$$

Then $R(s) = Q(s)h_g(s)e^{-k_g s}$ *is bounded in* H_r.

Proof. By assumption we have $R \in N$; from (4.2) and (4.3) we obtain

$$R(s)\varphi(s) = f(s) \in M.$$

Hence Theorem 3.2 yields $R \in M$, and this is the desired result.

CONSEQUENCE 4.1. *Suppose that* $g \in M$ *has no zeros in* H_r *and that*

$$0 < a \leqslant |g(iv)| \leqslant 1.$$

If, moreover, $k_g = 0$, *then* $a \leqslant |g(s)| \leqslant 1$ *in* $\bar{H}_r$.

Proof. Put $f(s) \equiv a$ in Theorem 5.2; then it follows that $Q \in M$.

5. Application of Theorem 2.3 to entire functions. Obviously we can now improve many well-known results whose proves are based on the theorem of Phragmén–Lindelöf; we need only replace this theorem by a more general one.

We give an example from [4] which immediately follows from Theorem 2.3 by conformal mapping of the plane to a half plane.

THEOREM 5.1. *Let* $R(s)$ *be entire at most of order* 2 (*intermediate type*); *if*

$$|R(u)| \leqslant 1 \quad (u \geqslant 0), \tag{5.1}$$

$$\overline{\lim_{v \to \pm\infty}} \frac{\log|R(iv)|}{v} \leqslant 0, \tag{5.2}$$

$$\overline{\lim_{u \to -\infty}} \frac{\log|R(u)|}{|u|^{1/2}} \leqslant 0, \tag{5.3}$$

then $R(s) \equiv \text{const}$.

This theorem contains two special cases, which are well known.

a) The first case is due to S. Bernstein, who required $|R(u)| \leqslant 1$ for all u and at most order 1 (minimal type). This theorem is "sharp" because of the example $R(s) = \frac{1}{2}(e^{is} + e^{-is})$. Note that in our theorem this example is excluded by (5.2).

b) The second case (Theorem 3.1.5 of [1]) requires (5.1) and at most order $\frac{1}{2}$, minimal type.

Of course we could obtain an even better result if we imposed more assumptions on appropriate rays.

6. Vanishing functions. There exists a group of theorems stating not only that $R \in N$ is also $\in M$, but even that $R \equiv 0$. In our context the following classical result is of interest because of assumption (6.1); see [2], Theorem 24.

THEOREM 6.1. *Suppose that $R \in N$ is at most of order* 1 *and*

$$\overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\varphi})|}{r} = -\infty \tag{6.1}$$

for some fixed φ, $|\varphi| < \pi/2$; then $R \equiv 0$.

For the proof it is shown that $|R(s)e^{Bs}| \leqslant 1$ for all $B > 0$. In view of Theorems 2.2 and 2.3 the assumption "order 1" can be relaxed for this purpose.

But we want to state another theorem from [3] in which the assumptions of Theorem 6.1 are replaced by much weaker ones, but on the other hand a supposition on $R(iv)$ $(v > 0)$ must be added; this combination of assumptions on $s = iv$ $(v > 0)$ and $s = u$, $u > 0$ is a new type of proposition.

THEOREM 6.2. *Suppose that $R \in \bar{N}$ belongs at most to the order* 2 (*minimal type*) *and*

$$\overline{\lim_{u\to\infty}} \frac{\log|R(u)|}{u^q} \leqslant 0, \quad 1 < q < 2;$$

if, moreover,

$$|R(iv)| < e^{-v^q}, \quad \forall v > 0, \tag{6.2}$$

then $R(s) \equiv 0$.

In the proof all the assumptions are used to show, that for $R \not\equiv 0$ and $\delta > 0$

$$R_\delta(s) = R(s)\exp[\delta s^q e^{i\pi/2}(1+q)]$$

belongs to $\bar{M}$; hence $R \in \bar{M}$. But for $R \in \bar{M}$ the integral

$$\int_1^\infty \frac{\log|R(iv)|}{v^2}\,dv$$

exists, see [1] or [8]. This contradicts (6.2).

7. Final remarks.

1. In this exposition we want to convince the reader that additional restrictions on the rate of increase of $R \in N$ imposed at least on one ray in H_r are something natural in the theory of Phragmén–Lindelöf. The reason is twofold: Firstly the simple trick of decomposing H_r has proved extremely fruitful; secondly Lemma 3.1 makes the importance of the

constant (3.3) obvious. We suppose that every theorem stating that $R \in N$ is also $\in M$ can be generalized in the same way as that used in the proof of Theorem 2.2 by means of Theorem 2.1. Condition (2.2) must be satisfied anyway, hence $R_\delta(u) = e^{-\delta u}R(u)$ will be bounded. Now conformal mapping of the given theorem to quarter planes will yield new conditions under which $|R_\delta(s)| \leqslant 1$ can be inferred.

2. The theorem of Phragmén–Lindelöf can be proved in quite different ways, which we have not mentioned so far, e.g. by means of Carleman's principle of harmonic majorization. We would obtain a deeper insight into the theory if we succeeded in proving also the theorem of Pólya–Szegö in this way.

3. Theorem 3.2 has proved very useful not only in the examples given in Section 4. More important is the question whether finite limit points of zeros can be admitted for the functions f and R in this theorem. In some applications the assumption under consideration is satisfied by itself, in others it is a nuisance.

4. Finally we give a number of criteria to stress important differences between the theorems quoted above.

a) In our view the following distinction for the proofs of the propositions presented in this paper is essential. Elementary results are the theorems of Section 2 and Section 5, because they are immediate consequences of the Principle. Contrary to them, the others require (for an easy proof) Lemma 3.1, which is distinctly deeper than the "elementary results". It is because of this lemma that we consider the domain H_r particularly favourable for the theory of Phragmén–Lindelöf.

b) In Section 7.1 we supposed that every theorem stating that $R \in N$ is also $\in M$ can be generalized by means of a decomposition of H_r. Therefore it is an interesting question, which results can be proved without any decomposition of H_r; note that Lemma 3.1 itself does not require such a decomposition for its proof. Of particular interest seems to be the question whether Theorem 3.1 can be improved without decomposition. It is also of interest to know which propositions require exactly one decomposition, like Ostrowski's Theorem 2.4.

c) In some theorems (Pólya–Szegö, 3.1, 3.2, 3.3) the rays playing an important role in the assumptions can be turned about 0 without changing the other assumptions. This seems to be a criterion that the theorem can be generalized. On the other hand, if we turn the ray in Theorem 2.2 for instance, the other restrictions on the increase of R must also be changed, see [4].

Acknowledgement. My thanks are due to I. Lehmann and M. Riedel for reading a draft of this paper and helpful discussions.

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Remark in proof-reading. A successful study concerning our remarks 7.2 and 7.3 was undertaken by M. Riedel in his paper: M. Riedel, *Some theorems of the Phragmén–Lindelöf theory for subharmonic functions*, Math. Nachrichten, in print.