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Asymptotic relationships between the solutions of two second order differential equations*

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Some asymptotic relationships between the solutions of the second order linear differential equation

$$y'' = a(t)y \qquad (t \geqslant t_0) \tag{1}$$

and the second order differential equation

$$x'' = a(t)x + f(t, x) \qquad (t \geqslant t_0) \tag{2}$$

are studied. Conditions are found that lead to an equivalence between certain of the solutions of the linear equation (1) and certain of the solutions of the non-linear (in general) equation (2).

A closely related result which is valid whenever $f(t, x) = b(t)x$ in (2), is due to P. Hartman and A. Wintner. This result may be found in [4], p. 379, Theorem 9.1. There are distinct differences in the approaches and in the conclusions between the two results. For example, Hartman and Wintner require that the differential equation (1) be non-oscillatory; we do not need this hypothesis.

The basic assumptions that will be made about equation (1) include the following: Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be solutions of (1) with the Wronskian of y_1 and y_2 equal to -1. Suppose that there exist positive continuous functions $y_i^* = y_i^*(t)$, $i = 1, 2$, which satisfy the inequality

$$|y_i(t)| \leqslant y_i^*(t) \qquad (i = 1, 2;\ t \geqslant t_0). \tag{3}$$

In equation (2), we will assume that $f = f(t, x)$ is continuous on $I \times R$, $I = [t_0, \infty)$. Furthermore, let f satisfy the inequality

$$|f(t, x)| \leqslant \omega(t, |x|), \tag{4}$$

where $\omega(t, r)$ is a continuous function defined on $I \times R^+$ (R^+ denotes the non-negative real numbers) that is non-decreasing for $r \in R^+$ for each fixed $t \in I$.

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THEOREM 1. *Let conditions* (3) *and* (4) *be satisfied. Suppose that*

$$\int_{t_0}^{\infty} y_1^*(s)\,\omega\big(s,\, ky_2^*(s)\big)\,ds < \infty \tag{5}$$

for some constant $k,\ k > 1$; *and that*

$$\gamma(t)\int_{t_0}^{t} \gamma^{-1}(s)\,y_1^*(s)\,\omega\big(s,\, ky_2^*(s)\big)\,ds = o(1) \qquad (t \to \infty), \tag{6}$$

where $\gamma(t) \equiv y_1^*(t)[y_2^*(t)]^{-1}$. *Then, there exists a solution* $x = x(t)$ *of equation* (2) *such that*

$$|x(t) - y_2(t)| = o\big(y_2^*(t)\big) \qquad (t \to \infty). \tag{7}$$

In addition, if $x_2 = x_2(t)$ *is any solution of* (2) *that satisfies the inequality* $|x_2(t)| \leqslant ky_2^*(t)$, *then there exists a solution* $y = y(t)$ *of* (1) *such that*

$$|y(t) - x_2(t)| = o\big(y_2^*(t)\big) \qquad (t \to \infty). \tag{8}$$

Proof. Consider the subset $\mathfrak{F}$ of $C(I)$, the set of continuous functions on I, defined by

$$\mathfrak{F} = \{x \in C(I)\colon |x(t)| \leqslant ky_2^*(t),\ t \in I\}.$$

For $x \in \mathfrak{F}$, define $\mathscr{I}x$ by the equation

$$\mathscr{I}x(t) = y_2(t) + y_1(t)\int_{t_0}^{t} y_2(s) f\big(s, x(s)\big)\,ds + y_2(t)\int_{t}^{\infty} y_1(s) f\big(s, x(s)\big)\,ds. \tag{9}$$

By virtue of conditions (5) and (6) we can assume that t_0 is sufficiently large so that

$$\int_{t_0}^{\infty} y_1^*(s)\,\omega\big(s,\, ky_2^*(s)\big)\,ds < (k-1)/2 \tag{10}$$

and

$$\gamma(t)\int_{t_0}^{t} \gamma^{-1}(s)\,y_1^*(s)\,\omega\big(s,\, ky_2^*(s)\big)\,ds < (k-1)/2 \qquad (t \geqslant t_0). \tag{11}$$

An application of quantitites (3), (4), (10) and (11) in (9) shows that $\mathscr{I}\mathfrak{F} \subset \mathfrak{F}$.

Next, we will verify that the transformation $\mathscr{I}$ is continuous; let x_n $(n = 1, 2, \ldots)$ and x be in $\mathfrak{F}$ with $\{x_n\}$ converging uniformly to x on every compact subinterval of I. Consider any compact interval of the form $[t_0, T]$, $T > t_0$. Let $\varepsilon > 0$ be given and choose t_1 $(t_1 \geqslant T)$ such that

$$\int_{t_1}^{\infty} y_1^*(s)\,\omega\big(s,\, ky_2^*(s)\big)\,ds < \varepsilon/4\mathrm{M}, \tag{12}$$

where $M = \max[\max_{t\in[t_0,T]} y_1^*(t),\ \max_{t\in[t_0,T]} y_2^*(t)]$.

Since f is continuous and the sequence $\{x_n\}$ converges uniformly to x on $[t_0, t_1]$, there exists a positive constant N such that if $n \geqslant N$, then

(13) $$y_2^*(t)\,|f(t, x_n(t)) - f(t, x(t))| < \varepsilon/4M(t_1 - t_0) \qquad (t \in [t_0, t_1]).$$

Using (12) and (13), we obtain

$$|\mathscr{I}x_n(t) - \mathscr{I}x(t)| \leqslant y_1^*(t) \int_{t_0}^{t} y_2^*(s)|f(s, x_n(s)) - f(s, x(s))|\,ds + \\ + y_2^*(t) \int_{t}^{t_1} y_2^*(s)|f(s, x_n(s)) - f(s, x(s))|\,ds + \\ + 2y_2^*(t) \int_{t_1}^{\infty} y_1^*(s)\,\omega(s, ky_2^*(s))\,ds < \varepsilon \qquad (n \geqslant N,\ t \in [t_0, T]).$$

Therefore, the mapping $\mathscr{I}$ is continuous on $\mathfrak{F}$.

From equation (9) we obtain the estimate

$$|(\mathscr{I}x)'(t)| \leqslant |y_2'(t)| + |y_1'(t)| \int_{t_0}^{t} y_2^*(s)\,\omega(s, ky_2^*(s))\,ds + \\ + |y_2'(t)| \int_{t}^{\infty} y_1^*(s)\,\omega(s, ky_2^*(s))\,ds.$$

For t in any compact subinterval of I, the right-hand side of the above inequality is bounded by a constant independent of $x \in \mathfrak{F}$. Thus $\mathscr{I}\mathfrak{F}$ is equicontinuous on every finite subinterval of I.

The Schauder-Tychonoff fixed point theorem (see, for example, Coppel [2], p. 9) implies that the transformation $\mathscr{I}$ has a fixed point $x = x(t)$ in $\mathfrak{F}$. It can be shown directly that x is a solution of (2). To verify that the order relation (7) is satisfied, we use (9) with $\mathscr{I}x = x$ to obtain

$$|x(t) - y_2(t)| \leqslant y_1^*(t) \int_{t_0}^{t} y_2^*(s)\,\omega(s, ky_2^*(s))\,ds + \\ + y_2^*(t) \int_{t}^{\infty} y_1^*(s)\,\omega(s, ky_2^*(s))\,ds.$$

The right-hand side of this inequality is $o(y_2^*(t))$ as $t \to \infty$ by virtue of hypotheses (5) and (6).

To complete the proof of the theorem, the opposite relationship between the solutions of (1) and (2) must be established. Suppose that $x_2 = x_2(t)$ is a solution of (2) that satisfies the inequality $|x_2(t)| \leqslant ky_2^*(t)$ on I. The function $y = y(t)$ defined by the equation

$$y(t) = x_2(t) - y_1(t) \int_{t_0}^{t} y_2(s) f(s, x_2(s))\,ds - y_2(t) \int_{t}^{\infty} y_1(s) f(s, x_2(s))\,ds$$

is easily seen to be a solution of (1) that satisfies the order relation (8).

In the next theorem we replace the order relation (6) by some different conditions.

THEOREM 2. *Let conditions (3), (4), and (5) be satisfied. Suppose that*

$$\int_{t_0}^{\infty} y_2^*(s)\,\omega\big(s, ky_2^*(s)\big)\,ds < \infty \tag{13}$$

and

$$\gamma(t)\int_{t_0}^{\infty} \gamma^{-1}(s)\,y_1^*(s)\,\omega\big(s, ky_2^*(s)\big)\,ds = o(1) \qquad (t \to \infty). \tag{14}$$

Then, there exists a solution $x = x(t)$ of equation (2) such that (7) is satisfied. In addition, if $x_2 = x_2(t)$ is any solution of (2) that satisfies the inequality $|x_2(t)| \leqslant ky_2^*(t)$, then there exists a solution $y = y(t)$ of (1) such that (8) is satisfied.

Proof. The proof is similar to the proof of Theorem 1; for this reason the details will be omitted. The main difference in the proofs is that the transformation $\mathscr{I}_1$, defined on $\mathfrak{F}$ by

$$\mathscr{I}_1 x(t) = y_2(t) - y_1(t)\int_{t_0}^{\infty} y_2(s) f\big(s, x(s)\big)\,ds + y_2(t)\int_{t}^{\infty} y_1(s) f\big(s, x(s)\big)\,ds$$

is used instead of (9).

Remark 1. The reason the order relations (6) and (14) are stated in terms of the function $\gamma(t) = y_1^*(t)[y_2^*(t)]^{-1}$ lies in the following comments. Let $g = g(t)$ be non-negative on $[t_0, \infty)$ and suppose that $\int_{t_0}^{\infty} g(t)\,dt < \infty$.

(a) If $c = c(t)$ is a continuous, non-decreasing, positive function on I such that $c(t^a)[c(t)]^{-1} = o(1)$ as $t \to \infty$ for some a, $0 < a < 1$, then

$$c^{-1}(t)\int_{t_0}^{t} c(s)g(s)\,ds = o(1) \qquad (t \to \infty).$$

(b) If $h = h(t)$ is a continuous, non-increasing, positive function on I, then

$$h^{-1}(t)\int_{t}^{\infty} h(s)g(s)\,ds = o(1) \qquad (t \to \infty).$$

The proof of (a) may be found in [3]. The proof of (b) is trivial. A simple test which implies that $c(t^a)[c(t)]^{-1} = o(1)$ as $t \to \infty$ is that $\lim_{t\to\infty} c(t) = \infty$ and $c'(t)$ be non-decreasing on I. This is a consequence of the inequalities

$$\limsup_{t\to\infty} \frac{c(t^a)}{c(t)} \leqslant \limsup_{t\to\infty} \frac{c'(t^a)\,at^{a-1}}{c'(t)} \leqslant \limsup_{t\to\infty} at^{a-1} = 0.$$

The first inequality above is obtained by using the boundedness form of L'Hôspital's rule.

Remark 2. It is clear from the nature of hypotheses (6) and (14) that Theorems 1 and 2 are independent in the sense that they are generally concerned with solutions that have a different asymptotic behavior. It is easy to give examples that illustrate this statement; consider, for instance,

$$y'' = 0 \tag{15}$$

and

$$x'' = b(t)x^r \quad (r > 0), \tag{16}$$

where $b(t)$ is continuous on I and $\int_{t_0}^{\infty} t|b(t)|dt < \infty$. To obtain a solution $x_1 = x_1(t)$ of (16) with the asymptotic behavior $x_1(t) = 1+o(1)$ as $t \to \infty$, we take $y_1(t) = y_1^*(t) = t$ and $y_2(t) = y_2^*(t) = 1$. It is easily seen that (5), (13), and (14) are satisfied; hence, Theorem 2 implies the existence of the solution x_1 of (16).

For the case $r = 1$, we let $y_1(t) = -1$, $y_1^*(t) = 1$, and $y_2(t) = y_2^*(t) = t$. We use (a) of Remark 1 to see that (6) is satisfied. Theorem 1 implies that there exists a solution $x_2 = x_2(t)$ of (16) such that $x_2(t) = t+o(t)$, $t \to \infty$. The above results are well-known; for the case $r = 1$, see Bellman ([1], p. 114). When $r > 1$, a stronger integrability condition than the one used above is necessary to obtain a solution like x_2.

Remark 3. To demonstrate that the above theorems are applicable when equation (1) is oscillatory, we present another simple example. The equation

$$y'' - \mu t^{-2} y = 0 \quad (\mu > \tfrac{1}{4}, t \geqslant 1) \tag{17}$$

has linearly independent solutions given by

$$y_1(t) = t^{1/2}\cos(\mu - \tfrac{1}{4})^{1/2}\log t,$$

and

$$y_2(t) = t^{1/2}\sin(\mu - \tfrac{1}{4})^{1/2}\log t.$$

We take $y_1^*(t) = y_2^*(t) = t^{1/2}$. In the equation

$$x'' = \mu t^{-2}x + b(t)x^r \quad (r > 0, t \geqslant 1) \tag{18}$$

we suppose that $b(t)$ is continuous and that $\int_{t_0}^{\infty} t^{(r+1)/2}|b(t)|dt < \infty$. Since $\gamma(t) = 1$, Theorem 2 may be applied to yield the solution $x_1 = x_1(t)$ and $x_2 = x_2(t)$ of (17) such that

$$x_1(t) = y_1(t) + o(t^{1/2}) \quad (t \to \infty)$$

and

$$x_2(t) = y_2(t) + o(t^{1/2}) \quad (t \to \infty).$$

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