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On a generalized logarithmic kernel and its potentials

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Abstract. In order to give an integral representation to a broader class of superharmonic functions in the plane we use instead of the usual logarithmic kernel a new Λ-kernel, already considered by M. Heins, equal to $\log(1/|x-y|)$ for $|y| \leqslant 1$ and to $\log(|y|/|x-y|)$ for $|y| > 1$. The corresponding potentials, with the logarithmic or Λ-kernel (up to a harmonic or a constant function), are compared and characterized. The study of the case of a harmonic minorant outside a compact set is deepened. This suggests an extension to harmonic spaces without positive potentials and has close relations with the Arsove functions of potential type, the pseudo-potentials of Anandam and the equivalence notion of J. Guillerme.

I. INTRODUCTION

1. In the potential theory in the plane, one considers the logarithmic kernel

$$L(x, y) = \log \frac{1}{|x-y|}$$

and its corresponding potential with respect to a positive measure λ:

$$U_L^\lambda(x) = \int \log \frac{1}{|x-y|} \, d\lambda(x)$$

defined as

$$\int \log^+ \frac{1}{|x-y|} \, d\lambda(y) - \int \log^- \frac{1}{|x-y|} \, d\lambda(y)$$

at every point, where it is well defined (finite or not).

Actually, it is rather trivial but perhaps not well known that iff U_L^λ is defined and finite at one point or iff

$$\int_{|y|>r} \log \frac{1}{|y|} \, d\lambda(y)$$

is finite (for some $r > 0$), then U_L^λ is defined everywhere on $\boldsymbol{R}^2$ and superharmonic, and called a (*true*) *L-potential.* Moreover, for a superharmonic function, with the so-called *associated measure* μ (given by the local F. Riesz representation), u is equal to a certain U_L^λ, up to a harmonic function iff U_L^μ is a (true) L-potential and then $\lambda = \mu$. See other developments in [5].

But in order to get an integral representation for a larger class of superharmonic functions, one is led to introduce a new kernel in $\boldsymbol{R}^2$:

$$A(x, y) = \begin{cases} \log \dfrac{1}{|x-y|} & \text{if} \quad |y| < 1, \\ \log \dfrac{|y|}{|x-y|} & \text{if} \quad |y| \geqslant 1. \end{cases} \tag{1}$$

Actually the kernel $\log(|y|/|x-y|)$ was introduced in an auxiliary way by M. Heins [10]; he proved that if $u(x)$ is a subharmonic function of order less than one in $\boldsymbol{R}^2$ and is harmonic in the neighbourhood of the origin, then there exists a Radon measure $\mu \geqslant 0$ in $\boldsymbol{R}^2$ such that

$$u(x) = u(0) + \lim_{r \to \infty} \int_{|y| < r} \log \frac{|x-y|}{|y|} \, d\mu(y). \tag{2}$$

A little later, M. Brelot, in a study [7] on the behaviour of a subharmonic function in the neighbourhood of a singular point, improved the previous result by replacing $\lim \int$ by a true Radon integral and the growth condition by the weaker one that the mean $\mathfrak{M}_{u^+}^r$ of u^+ in the circle $|x| = r$ satisfies:

$$\int_{r_0}^{\infty} \frac{\mathfrak{M}_{u^+}^r}{r^2} \, dr < +\infty \qquad (r_0 > 0).$$

See in [7] many developments with the kernel $\log(|y|/|x-y|)$, other ones and similar ones in $\boldsymbol{R}^n$, but with a measure 0 in the neighbourhood of the origin. We shall use it often.

2. In this paper we consider only the kernel A in $\boldsymbol{R}^2$, but with other details and aims. Such a study will actually interfere (sometimes being even included), with previous papers like [10], [3] or [4] on various questions of potential theory, often inspired by the theory of entire functions. Therefore we present in chiefly as an introduction and a tool for an easier development of these questions.

In Chapter II, instead of the Riesz type representation of superharmonic function, by using the logarithmic potential, we consider the same problem with respect to the kernel A. We give a representation by a A-potential, up to a harmonic function, for a larger class of superharmonic functions (with an associated measure $\mu \geqslant 0$) characterized by some equivalent conditions like

$$\int_{|y| > r} \frac{d\mu(y)}{|y|} < +\infty \qquad (r > 0),$$

obviously weaker than those of the first representation.

In Chapter III we consider the integral representation in the case where the additional harmonic function is a constant. This will imply the improved Heins' result. We shall mean by Λ^*- (or L^*-) potentials the true Λ- (or L-) potentials up to a constant, the symbols Λ^* (or L^*) denoting the families of such potentials, and we shall try to characterize Λ^*-potentials.

In Chapter IV we recall some old results and recent notions that will be completed and used, first of all those concerning admissible superharmonic functions (Anandam [1]) (i.e. functions having a harmonic minorant outside a compact set). We know that L-potentials are admissible [5], but it is not always true for Λ-potentials. We recall that two admissible superharmonic functions are said to be *equivalent* (Guillerme [8] and [9]) if the difference between their greatest harmonic minorants outside a disc (equivalently, a non-polar compact set) is bounded at infinity (outside a compact set). We shall characterize admissible Λ^*-potentials. They form an equivalence class in the set of all admissible superharmonic functions and are identical to the pseudo-potentials in $\boldsymbol{R}^2$ (introduced by Anandam in an axiomatic theory [2]). Let us also emphasize that a function of "potential type" (Arsove [3]) is the difference of two admissible Λ-potentials up to a constant and that any admissible Λ^*-potential is a function of potential type.

We continue the study in Chapter V by developing the relations between the admissible Λ^*- and L^*-potentials. Thus any L- or L^*-potential is a Λ^*-potential and any admissible Λ^*-potential is equivalent to an L- or L^*-potential.

We hope, and it was our main purpose, that this elementary paper will suggest some research of integral representation in a harmonic space without a positive potential.

II. Λ-POTENTIALS AND INTEGRAL REPRESENTATION

3. We have defined the Λ-kernel. It may be useful to remark that, for $|y| \geq 1$, $\Lambda(x, y)$ is the limit, as $r \to \infty$, of the difference $G_y^r(x) - G_y^r(0)$, where G_y^r is the Green function of the disc B_0^r with its pole at y. That can be seen from the expression of G_y^r.

Definition II. 1. For a given measure $\lambda \geq 0$, let us first define $U_\Lambda^\lambda(x)$, or $\int \Lambda(x, y)\, d\lambda(y)$, as

$$\int \Lambda^+(x, y)\, d\lambda(y) - \int \Lambda^-(x, y)\, d\lambda(y)$$

on the set of x for which this difference is well defined (finite or not).

Note that U_Λ^λ is identical to

$$\int_{|y|<1} \log \frac{1}{|x-y|}\, d\lambda(y) + \int_{|y|\geq 1} \log \frac{|y|}{|x-y|}\, d\lambda(y)$$

for every x for which each part and the sum have a meaning. The same remark is true if we decompose the last integral into the sum of

$$\int_{1 \leqslant |y| \leqslant r} \quad \text{and} \quad \int_{|y| > r} \quad (r > 1).$$

THEOREM II.2. *The following statements are equivalent about this* U_A^λ:

(i) U_A^λ *is defined and finite a.e. on some open set* ω *of* $\boldsymbol{R}^2$.

(ii) U_A^λ *is defined and finite at two different points not collinear with* 0.

(iii) $\displaystyle\int_{|y|>r} \frac{d\lambda(y)}{|y|}$ *is finite for some* r.

(iv) U_A^λ *is defined everywhere and is superharmonic on* $\boldsymbol{R}^2$.

Such a U_A^λ *is then called a "(true) A-potential".*

Proof: The implication (i) $\Rightarrow$ (ii) is obvious.

Now we prove that (ii) $\Rightarrow$ (iii). Suppose that U_A^λ is finite at the points x_1 and x_2 not collinear with 0. Let I be the middle point of $(0, x_1)$ and l the straight line through I perpendicular to $(0, x_1)$; let δ and δ' be two angular domains with a small angle θ, situated in the opposite sides of $(0, x_1)$, having vertex at I and being bounded by two semi-lines through I having l as their bisector. The complement of $\bar{\delta} \cup \bar{\delta}'$ consists of two domains δ'' and δ''', one containing 0 and the other x_1.

Now, $|y|-|x_1-y|$ is of constant sign in each of the domains δ'' and δ''' and its absolute value majorizes a fixed $\varepsilon > 0$. Since, for small real t, $|\log(1+t)| \geqslant t/2$, the quantity

$$\log\left(\frac{|y|}{|x_1-y|}\right) \quad \text{equal to} \quad -\log\left(1+\frac{|x_1-y|-|y|}{|y|}\right)$$

is of constant sign and its absolute value majorizes $\varepsilon/2|y|$ in $\delta_\varrho'' = \delta''\backslash B_0^\varrho$, $\delta_\varrho''' = \delta'''\backslash B_0^\varrho$ for ϱ large enough.

The finiteness of U_A^λ implies the same property for the integrals over the two subsets of $\{|y| > 1\}$, where $\log(|y|/|x_1-y|)$ has a constant sign, and therefore $\int (d\lambda(y)/|y|)$ is finite when taken on δ_ϱ'' and δ_ϱ'''.

We can now proceed with a similar construction relative to x_2 in such a manner that the parts of δ and δ' outside a disc B_0^r are contained in the regions analogous to δ_ϱ'' and δ_ϱ''' where $\int (d\lambda(y)/|y|)$ is finite. Consequently, $\int_{|y|>r} (d\lambda(y)/|y|)$ is finite.

Finally, we show that (iii) $\Rightarrow$ (iv) (which obviously implies (i)). When $\mu(B_0^1) = 0$, the proof is contained in [7], theorem 1'(α), p. 145. For this important point we give here a direct proof.

For a fixed x and $|y| \geqslant r > |x|$, $\log(|y|/|x-y|)$ is of a constant sign on the semi-spaces $|y| < |x-y|$ and $|y| > |x-y|$ and is majorized in absolute value by $(|x-y|-|y|)/|y|$ on the first semi-space, by $(|y|-|x-y|)/|x-y|$ on the second one, because $\log|1+z| \leqslant z$ for $z > 0$.

Further, since $|y| \leqslant |x|+|x-y|$, we have

$$\log \frac{|y|}{|x-y|} \leqslant \log\left(1+\frac{|x|}{|x-y|}\right) \leqslant \frac{|x|}{|x-y|} \leqslant \frac{|x|}{|y|-|x|},$$

and so

$$\Big|\int_{|y|\geqslant r} \log \frac{|y|}{|x-y|}\, d\lambda(y)\Big| \leqslant |x| \int_{|y|\geqslant r} \frac{d\lambda(y)}{|y|-|x|}.$$

Let us now consider x to be variable, with $|x| < \varrho < r$ (ϱ, r fixed) and with the previous conventions, the integrals $\int \Lambda(x, y)\, d\lambda(y)$ and

$$\int_{|y|<1} \log \frac{1}{|x-y|}\, d\lambda(y) + \int_{1\leqslant y<r} \log \frac{|y|}{|x-y|}\, d\lambda(y) + \int_{|y|\geqslant r} \log \frac{|y|}{|x-y|}\, d\lambda(y)$$

making sense and being equal at the same x. The sum of the first two integrals in the latter expression is for all x in B_0^ϱ a superharmonic function and has λ as its associated measure in B_0^ϱ, and the third integral is, for every $x \in B_0^\varrho$, the limit of the integral taken on $\{r \leqslant |y| < r'\}$ as $r' \to \infty$, as it can be seen by considering $\log^+$ and $\log^-$. Moreover, the convergence is uniform on B_0^ϱ. Since this integral is harmonic for any r', the limit is harmonic.

Consequently, U_Λ^λ has a meaning for any x and is superharmonic in $\boldsymbol{R}^2$ with λ as its associated measure.

II.3. Important equivalent criterion for λ to define a true Λ-potential.

Recall that the previous condition (iii) on any measure $\lambda \geqslant 0$ on $\boldsymbol{R}^2$:

$$\int_{|y|>r} \frac{d\lambda(y)}{|y|} < +\infty \qquad \text{for some } r > 0$$

is equivalent to:

$$\int_{r_0}^{\infty} \frac{\lambda(B_0^r)}{r^2}\, dr < +\infty \qquad (r_0 > 0);$$

Same result with

$$\int \frac{d\lambda}{|y|^\alpha} \quad \text{and} \quad \int \frac{\lambda(B_0^r)}{r^{\alpha+1}}\, dr \qquad (\alpha > 0)$$

which implies $\lambda(B_0^r) = o(r^\alpha)$.

(See [7], Lemma 2′, C with a correction of the misprint (read $\overline{OP}^{-\alpha}$ instead of $\overline{OP}^{\alpha}$) obvious by a transposition of Lemma 2, C.)

4. Integral representation. It is now easy to prove:

Theorem II.4. *A superharmonic function u in $\boldsymbol{R}^2$ with the associated* *measure μ is a true U_Λ^λ-potential, up to a harmonic function, iff U_Λ^μ is a true Λ-potential; and then $\lambda = \mu$.*

Another criterion is that $\int_{r_0}^{\infty} r^{-2}\mathfrak{M}_u^r\,dr$ be finite ($r_0 > 0$), and then $\mathfrak{M}_u^r = o(r)$ ($r \to \infty$).

As regards the second statement, we have only to prove that it is equivalent to:

$$\int_{r_0}^{\infty} \frac{\mu(B_0^r)}{r^2}\,dr \quad \text{is finite.}$$

More generally, let us see that for any $\alpha > 0$, the condition

$$\int_{r_0}^{\infty} r^{-1-\alpha}\mathfrak{M}_u^r\,dr$$

is equivalent to:

$$\int_{r_0}^{\infty} r^{-1-\alpha}\mu(B_0^r)\,dr < +\infty.$$

By replacing in the disc $\{|x| < 1\}$ u by its Poisson integral we get another superharmonic function with the same μ and $\mathfrak{M}_u^r$ for $r > 1$. Therefore we may suppose $u(0)$ to be finite ($r > 1$). By the Riesz representation theorem in the disc B_0^r we get

$$u(0) = \mathfrak{M}_u^r + \int_{|y|<r} \log\frac{r}{|y|}\,d\mu(y) \geqslant \mathfrak{M}_u^r + \int_{|y|<r/2} \log\frac{r}{|y|}\,d\mu(y)$$

and that is

$$\geqslant \mathfrak{M}_u^r + \log r\,\mu(B_0^{r/2}).$$

Consequently,

$$r^{-1-\alpha}\mathfrak{M}_u^r \leqslant -r^{-1-\alpha}\mu(B_0^{r/2})\log 2 + r^{-1-\alpha}u(0).$$

Hence, if $\int_{r_0}^{\infty} r^{-1-\alpha}\mathfrak{M}_u^r\,dr$ is finite, then $\int_{r_0}^{\infty} r^{-1-\alpha}\mu(B_0^r)\,dr$ is finite ($r_0 > 1$).

Conversely, suppose $\int_{r_0}^{\infty} r^{-1-\alpha}\mu(B_0^r)\,dr < +\infty$. The left-sided derivative $(d\mathfrak{M}_u^r/dr)^-$ at $t = \log r$ is equal to $-\mu(B_0^r)$. Hence $r(d\mathfrak{M}_u^r/dr)^- = -\mu(B_0^r)$ and $\int_{r_0}^{\infty} r^{-\alpha}(d\mathfrak{M}_u^r/dr)^-\,dr$ is finite. But this integral is equal to the Stieltjes integral $\int_{r_0}^{\infty} r^{-\alpha}\,d\mathfrak{M}_u^r \cdot \mathfrak{M}_u^r$ is decreasing, and therefore always $\geqslant 0$ or < 0 for $r > r_0$ large enough. We may apply [7], Lemma 1. Then $\int_{r_0}^{\infty} \mathfrak{M}_u^r\,d(r^{-\alpha})$ or $-\alpha\int_{r_0}^{\infty} r^{-1-\alpha}\mathfrak{M}_u^r\,dr$ is finite and $r^{-\alpha}\mathfrak{M}_u^r \underset{r\to\infty}{\longrightarrow} 0$.

COROLLARY II.5. *If a superharmonic function v majorizes, at infinity, u, a true Λ-potential + a harmonic function, then it is itself a true Λ-potential up to a harmonic function.*

Indeed, $\mathfrak{M}_v^r$ is upper bounded ($r > r_0$) and majorizes $\mathfrak{M}_u^r$.

Remark.

APPROXIMATION PROPERTY II.6. *Every true L- (resp. Λ-) potential u of a measure $\mu \geqslant 0$ is the pointwise limit of a sequence $u_n + v_n$ of C^∞-functions where u_n is a true C^∞ L- (resp. L^*-) potential with compact support and v_n is a function that tends to 0 at any point and is harmonic in every fixed disc if n is large enough.*

We decompose $u = u_n^0 + v_n^0$, then sum of the potentials of the restrictions of μ to $B_0^{R_n}$ and $CB_0^{R_n}$ ($R_n \uparrow \infty$). The convolution with a usual function like $C_n \cdot e^{1/(|x|/\varrho_n - 1)}$ on $B_0^{\varrho_n}$ continued by 0 (C_n being such that the dx-integral be equal to 1) gives for $\varrho_n \to 0$ the wanted u_n and v_n.

III. INTEGRAL REPRESENTATION UP TO A CONSTANT

5. THEOREM III.1. *Let u be a superharmonic function with the associated measure $\mu \geqslant 0$. If it is a Λ^*-potential (i.e. a true Λ-potential, in fact, U_Λ^μ up to a constant), then $\mathfrak{M}_{u^-}^r = o(r)$ and even $u^-(x) = o(|x|)$ ($|x| \to \infty$).*

If $\int_{r_0}^{\infty} \frac{1}{r^2} \mathfrak{M}_{u^-}^r\, dr$ ($r_0 > 0$) is finite, then u is a true Λ-potential U_Λ^μ + const.

Proof. Since $(a+b)^- \leqslant a^- + b^-$ when a or b is finite, we consider the true Λ-potential $v = U_\Lambda^\lambda$ and prove that $v^- = o(|x|)$. Now,

$$v(x) = \int_{|y| \leqslant 1} \log \frac{1}{|x-y|}\, d\lambda(y) + \int_{|y|>1} \log \frac{|y|}{|x-y|}\, d\lambda(y).$$

The first term v_1 is obviously equal to $\log \frac{1}{|x|} \lambda(\bar{B}_0^1) + \varepsilon(x)$ ($\varepsilon(x) \to 0$ for $|x| \to \infty$); it is < 0 at infinity and $o(|x|)$ ($|x| \to \infty$).

As regards the second term, denote it by r_2, we have $v_2^-(x) = o(|x|)$ as a particular case ($s = 1$) of Theorem 1′, β(1) in [7] when using Section 3, last line. Hence the wanted property of v^- being majorized by $v_1^- + v_2^- = o(|x|)$.

Let us now start with u satisfying:

$$\int_{r_0}^{\infty} r^{-2} \mathfrak{M}_{u^-}^r\, dr \text{ finite}.$$

(1) Let us mention a misprint in this theorem. In the first line of β, instead of "qui entrainerait α" read "qu'entrainerait α".

Since $\mathfrak{M}_u^r \leqslant -\mathfrak{M}_{u^-}^r$, it follows that for $\mathfrak{M}_u^r$ upper bounded, $\int_{r_0}^{\infty} r^{-2}\mathfrak{M}_u^r\,dr$ is finite. Hence u = a potential U_Λ^λ $(= w)+a$ harmonic function h (Theorem II.4).

Now $h^- \leqslant w^+ + u^-$. This is obvious at any x where $u(x) = +\infty$, because $w(x) = w^+(x) = +\infty$. Suppose $u(x)$ is finite; then so is $w(x)$. We get at x:

$$h = u - w, \quad h^- \leqslant u^- + (-w)^- = u^- + w^+,$$
$$h^- \leqslant u^- + w - w^-.$$

We know that $\mathfrak{M}_w^r = o(r)$ (Theorem II.4); we just saw that $\mathfrak{M}_{w^-}^r = o(n)$; the hypothesis implies $\mathfrak{M}_{u^-}^r = o(r)$ (use for instance II.4 for $\inf(u, 0)$). Hence $\mathfrak{M}_{h^-}^r = o(r)$, which implies that h = const.

6. Order of a superharmonic function. In agreement with the definition of the order of a subharmonic function, we choose the following:

DEFINITION III.2. The *order* of a superharmonic function in $\boldsymbol{R}^2$ is the infimum of the set of all real $\varrho = 0$ such that $\liminf_{r\to+\infty}(\sigma_u^r/r^\varrho) > -\infty$ where σ_u^r is the infimum of u on the circle Γ_r: $|x| = r$.

THEOREM III.3. *Let u be a superharmonic function in $\boldsymbol{R}^2$.*

If $u \in \Lambda^*$, u is of order $\leqslant 1$. If the order of u is < 1, then $u \in \Lambda^*$.

Suppose $u \in \Lambda^*$. We know that $u^- = o(|x|)$, which implies $\sigma_{-u^-}^r > \lambda r$ for a λ real finite < 0 and r large enough ($r > r_0$). Hence $\sigma_u^r/r > \lambda$; 1 is a ϱ of the definition and the order is $\leqslant 1$.

Conversely, suppose that the order is < 1. There exists ϱ $(0 \leqslant \varrho < 1)$ such that $\sigma_u^r/r^\varrho > \lambda$ (λ finite < 0) ($r >$ some r_0). The minimum of u on Γ_r is attained at a point x_0. If $u(x_0) \geqslant 0$, $-u^- = 0$ on Γ_r. If $u(x_0) < 0$, $\inf(u, 0)$ majorizes $u(x_0)$ on Γ_r, i.e. $\sigma_{-u^-}^r = u(x_0) = \sigma_u^r$. In any case,

$$\sigma_{-u^-}/r^\varrho > \lambda \ (r > r_0); \quad \mathfrak{M}_{u^-}^r/r^\varrho < -\lambda;$$

$$r^{-2}\mathfrak{M}_{u^-}^r < (-\lambda) r^{\varrho-2}; \quad \int_{r_0}^{\infty} \frac{\mathfrak{M}_{u^-}^r}{r^2}\,dr \text{ is finite.}$$

Then $u \in \Lambda^*$.

Application. The second part of the theorem implies the original (improved) form of the Heins theorem. Indeed: when u is harmonic in a neighbourhood of 0, we have

$$u(x) = \int_{R^2} \log \frac{|y|}{|x-y|}\, d\mu(y) - \int_{|y|\leqslant 1} \log|y|\, d\mu(y) + k;$$

hence

$$u(0) = -\int_{|y|\leqslant 1} \log|y|\, d\mu(y) + k,$$

and the Heins' result.

Remark. The order τ of u is equal to $\limsup_{r\to\infty} \log|\sigma_u^r|/\log r$, just as for u subharmonic.

The proof is the same. Obvious if $u \geqslant 0$, which implies u = constant. If not, for r large enough, $\sigma_u^r < 0$. Then for $\varrho > \tau$, $\sigma_u^r/r^\varrho > \lambda$ ($\lambda < 0$) implies

$$\limsup_{r\to\infty} \frac{\log|\sigma_u^r|}{\log r} < \varrho.$$

Hence this lim sup is $\leqslant \tau$. If this lim sup were $< \tau$, then σ_u^r/r^ϱ (for ϱ intermediate) would be lower bounded and ϱ would majorize the order τ. Contradiction.

IV. CASE OF ADMISSIBLE FUNCTIONS

7. From the study of the behaviour of a subharmonic function in the neighbourhood of a point (the point itself excluded) [4], we shall extract some results and transform them to statements involving a superharmonic function given on $\boldsymbol{R}^2$ or only outside a disc. (2)

Let u be superharmonic outside a disc. The mean $\mathfrak{M}_u^r$ on $|x| = r$ is a concave function of $t = \log r$. The quotient by t has a limit A for $t \to +\infty$, i.e. $r \to \infty$ (A finite or $-\infty$). The derivatives (right-sided or left-sided) of that function of t have the same limit A, but they may be interpreted as two outer generalized fluxes of Riesz, up to the factor $1/2\pi$ (see [4], p. 26, or [5], p. 304).

When u is defined in $\boldsymbol{R}^2$, these fluxes are equal, up to -2π, to the associated measure of the disc $\{|x| < r\}$ or $\{|x| \leqslant r\}$; and if $r \to \infty$, we get finally $A = -\int d\mu$ (see [5], p. 307).

We know also that for u superharmonic outside a disc the *existence of a harmonic minorant at infinity* (i.e. outside a disc) is equivalent to the condition: A is finite (see [4], p. 32 or [5], Theorem 2). The "special case" where there is such a minorant of the form: $k\log|x| +$ a bounded function (k finite), is characterized by the condition $\lim_{r\to\infty} (\mathfrak{M}_u^r\text{-}/\log r)$ (or equivalently: $\lim_{r\to\infty} (\mathfrak{M}_{|u|}^r/\log r)$) is finite [4]. Then the greatest harmonic minorant outside a disc has the form $A\log|x| +$ a bounded function. Moreover, in this case, $u(x)/\log|x|$ has a $\liminf_{|x|\to\infty}$ equal to $A = \lim_{r\to\infty} (\mathfrak{M}_u^r/\log r)$. An example of this special case is given by any L-potential (see [5], p. 307, where in line 18, α_u has to be read: $-\alpha_u$).

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(2) In the following considerations such a closed disc may be generally replaced by a non-polar compact set.

8. As a particular case of a notion introduced by Anandam [1] in a harmonic space without a positive potential, a superharmonic function in $\boldsymbol{R}^2$ is said to be *admissible* if it has a harmonic minorant outside a disc or some compact set.

We have recalled above a criterion that we repeat in the following theorem with a direct proof.

THEOREM IV.1. *A superharmonic function u in $\boldsymbol{R}^2$ with the associated measure $\mu \geqslant 0$ is admissible iff $\|\mu\|$, i.e. $\int d\mu$, is finite.*

Proof. We apply an inversion having the origin 0 as its pole and transforming u into u' and μ into μ' associated to u'. The hypothesis on u implies that u' admits a harmonic minorant, and a greatest one, denoted h', on some $B_0^r\backslash 0$. Then

$$u'(x)-h'(x) = \int G(x,y)\,d\mu'(y),$$

where $G(x,y)$ is the Green function of $B_0^r\backslash 0$ and also of B_0^r. Let $x \neq 0$ be fixed such that $u'(x)-h'(x)$ is finite. $G(x,y) > \varepsilon$ for y in a neighbourhood N of 0. Hence $\mu'(N)$ is finite; thus so is the μ-measure of a neighbourhood of the point at infinity, i.e. $\|\mu\|$ is finite.

Conversely, if $\|\mu\|$ is finite, μ' may be continued in the neighbourhood of 0 (with $\mu'(0) = 0$) and there is a Green potential of μ' in a B_0^ϱ or $B_0^\varrho\backslash\{0\}$; u' will be this potential up to a harmonic function which will be a minorant and we get a harmonic minorant for u outside a disc.

FIRST CONSEQUENCES. IV.2. *A true L-potential (of $\lambda \geqslant 0$) is admissible, because the property of $\int_{|y|>r} \log|y|\,d\lambda(y)$ being finite implies the finiteness of $\lambda(\complement B_0^r)$ or $\|\lambda\|$* (see [5], p. 307).

IV.3. *A true Λ-potential may be not admissible.*

EXAMPLE. Consider the measure μ given by the density: $|x|^{-1/2}$ if $|x| > 1$, 0 if $|x| \leqslant 1$. Then

$$\int_{|y|>1} \frac{d\mu}{|y|} = 2\pi \int_1^\infty r^{-3/2}\,dr \text{ is finite}$$

and

$$\int_{|y|>1} d\mu(y) = 2\pi \int_1^\infty r^{-1/2}\,dr \text{ is infinite.}$$

IV.4. *Any superharmonic admissible function with an associated measure $\mu \geqslant 0$ is equal to U_Λ^μ up to a harmonic function.* (Because $\int_{|y|>r} (d\mu/|y|) \leqslant 1/r \int d\mu$ finite.)

9. Characterizations of admissible Λ-potentials. It is interesting and useful to extract from the above-mentioned results the following property.

LEMMA IV.5. *Let u be a harmonic function outside a disc, majorizing a function $\alpha \log |x| + $constant outside some disc. Then u is of the form: $\beta \log |x|$ + a bounded harmonic function at infinity. In particular, if u is harmonic in R^2, then u = const.*

Proof. The final property is obvious by considering the mean on $|x| = r$, which implies $\beta = 0$, u bounded, therefore constant.

The first statement may be proved directly by using an inversion with pole at 0. We get u'; then $u' - \alpha \log |x|$ is lower bounded in the neighbourhood of 0, therefore can be continued as a superharmonic function, harmonic outside 0; therefore it is of the form: $\beta \log (1/|x|)$ + a harmonic function. Hence the conclusion for u.

LEMMA IV.6. *Let u be an admissible Λ-potential with an associated measure μ. Then $u(x) \geqslant -\|\mu\| \log (1+|x|)$.*

Proof. Indeed, by defifiition

$$u(x) = \int\limits_{|y|<1} \log \frac{1}{|x-y|} d\mu(y) + \int\limits_{|y|\geqslant 1} \log \frac{|y|}{|x-y|} d\mu(y) = I_1 + I_2.$$

If we set $|x| = \varrho$, then $|x-y| \leqslant \varrho + |y|$ and the first integral majorizes

$$\int\limits_{|y|<1} \log \frac{1}{1+\varrho} d\mu(y) = -\mu(B_0^r) \log (1+\varrho).$$

Now, applying a circular projection of the measure μ onto a line through the origin, we get a new measure characterized by the increasing function $v(r) = \mu(B_0^r)$. Then

$$I_2 \geqslant I = \int_1^\infty \log \frac{r}{r+\varrho} dv(r) = \left[v(r) \log \frac{r}{r+\varrho} \right]_{1^-}^\infty - \int_{1^-}^\infty v(r) \frac{\varrho}{r(r+\varrho)} dr$$
$$\geqslant -(\|\mu\| - \mu(B_0^1)) \log (1+\varrho).$$

Consequently, $u(x) \geqslant -\|\mu\| \log (1+|x|)$.

We get now with the Λ-potential a "special" type of a minorant at infinity.

PROPOSITION IV.7. *The greatest harmonic minorant of an admissible Λ-potential u outside any disc is of the form $\alpha \log |x| + a$ harmonic function bounded at infinity (i.e. in a neighbourhood of the point at infinity). (Moreover, $\alpha \leqslant 0$.)*

Proof. This is a consequence of the previous lemma where

$$-\|\mu\| \log (1+|x|) = -\|\mu\| \log (x) + \text{ a bounded function at infinity}$$

and of a recalled result of Section 7.

For a direct proof, let h be the greatest harmonic minorant of u in $(|x| > \varrho)$. We know that h is of the form $\gamma \log |x| + v$ harmonic in $R^2 +$

a bounded function. The function h, just like u, majorizes $-\|\mu\| \log|x|+$ constant at infinity; therefore v majorizes some function of the type $\beta \log|x|+$ constant at infinity. Hence, by Lemma 5, $v = \text{const.}$

Remark. For a superharmonic function u with the associated measure μ, in R^2, we have recalled that $\mathfrak{M}_u^r/\log r \to -\|\mu\|$. When $\|\mu\|$ is finite, this is equivalent, owing to IV.4, to the same property for U_A^μ.

It is possible to give a direct proof of this property of U_A^μ, rather similar to that of IV.6.

We may now summarize some characterizations of admissible Λ^*-potentials.

THEOREM IV.8. *Let u be a superharmonic function in R^2 with the associated measure μ. Suppose that u is admissible, i.e.* $A = \lim_{r\to\infty} (\mathfrak{M}_u^r/\log r) = -\|\mu\|$ *is finite* ($\mathfrak{M}_u^r$ *standing for the mean of u on* $|x| = r$). *Then the following statements are equivalent.*

(i) $u \in \Lambda^*$;

(ii) *outside a disc (or compact set), u majorizes a harmonic function of the form* $\alpha \log|x| + \text{const}$;

(iii) $\liminf_{|x|\to\infty} u(x)/\log|x|$ *is finite (then equal to A) or*

(iii′) *the order of u is* 0;

(iv) $\mathfrak{M}_{u^+}^r/\log r$ *or equivalently* $\mathfrak{M}_{u^-}^r/\log r$ *or* $\mathfrak{M}_{|u|}^r/\log r$ *has a finite limit as* $r \to \infty$.

For such a $u \in \Lambda^*$, *the greatest harmonic minorant of u in a set* $|x| > \varrho$ *with ϱ large enough is* $-\|\mu\| \log|x| +$ *a bounded function at infinity.*

Proof. (i) implies (ii) in view of Lemma IV.6. (ii) obviously implies (iii). The recalled results of Section 7 give the equality with A, the equivalence of (ii) and (iv) and the final assertion of the theorem. Obviously, (iii) $\Rightarrow$ (iii)′ $\Rightarrow$ (i) (see III.3). We show directly that (iii) $\Rightarrow$ (i).

Let $u > \alpha \log|x|$ at infinity. Since u is admissible, we get, according to IV.4,

$$u = \text{an admissible } \Lambda\text{-potential } v + \text{ a harmonic function } w.$$

Consequently, by using the greatest harmonic minorant of v outside a disc, we see that w majorizes at infinity a function $\beta \log|x| +$ a constant; hence it is a constant. Thus $u \in \Lambda^*$.

Note that, regarding (i), (ii), (iii) and the final assertion, one could avoid the recalled results of Section 7, giving direct proofs based on the previous properties established in this paper.

COROLLARY IV.9. *Any superharmonic admissible function minorizing or majorizing a Λ^*-potential at infinity is a Λ^*-potential. If u, v are admissible Λ^*-potentials, then so is* $\inf(u, v)$, *f.i.* $\inf(u, 0) = -u^-$.

These are easy consequences of (iv) and Proposition IV.7.

10. Relations with functions of potential type (Theorem IV.10). According to [3], Theorem (16), the definition of a function u of "potential type" is equivalent to: *u equals the difference of 2 admissible Λ-potentials plus a constant*; therefore also: *u equals the difference of 2 admissible superharmonic functions of order 0 plus a constant.* And thus, any *admissible Λ^*-potential is a function of potential type.*

We have therefore at our disposal, regarding admissible Λ^*-potentials, all the properties stated in [3] for the Arsove functions of potential type. This paper [3] contains, in a different setting and wording, several results of our paper that are here deduced from older properties; e.g. IV.1, IV.2, IV.4, IV.8, (iii), (iii'), (iv), IV.9, etc.

11. Coincidence with the pseudo-potentials. Let u be an admissible superharmonic function in $\boldsymbol{R}^2$ for which h is the greatest harmonic minorant outside a disc. As a particular case of a notion introduced in the axiomatic theory [2], we say that u is a *pseudo-potential* if, for some α, $h-\alpha\log|x|$ is bounded at infinity. (This does not depend on the disc, as can be seen directly or as a consequence of the following.)

THEOREM IV.11. *The admissible Λ^*-potentials are precisely the pseudo-potentials in $\boldsymbol{R}^2$.*

Proof. If u is a Λ^*-potential, it is a pseudo-potential according to the final property of the previous theorem. Conversely, if u is a pseudo-potential, it majorizes at infinity some function of the form: $\alpha\log|x|+$constant, and thus, according to (ii), it is a Λ^*-potential.

12. Notion of equivalence. Let us recall that two admissible superharmonic functions are said to be *equivalent* if the difference between their greatest harmonic minorants outside a disc (equivalently, a non-polar compact set) is bounded at infinity (Guillerme [8] or [9]).

THEOREM IV.12. *If u is an admissible superharmonic function equivalent to $v\in\Lambda^*$, then $u\in\Lambda^*$. In other words, Λ^* is an equivalence class in the set of all admissible superharmonic functions.*

Proof. By Theorem IV.8 the greatest harmonic minorant of v outside a disc B_0^r is $-\|\mu\|\log|x|+$ constant, where μ is the measure associated with v. Hence u equivalent to v majorizes $-\|\mu\|\log|x|+$constant at infinity. Hence $u\in\Lambda^*$.

V. RELATIONS BETWEEN L OR L^* AND Λ OR Λ^*-POTENTIALS

13. THEOREM V.1. (a) *Any true L-potential U_L^λ is a Λ^*-potential $U_L^\lambda+$* const.

(b) *Any true Λ-potential U_Λ^λ is an L^*-potential $U_L^\lambda+$* const *iff* $\int_{|y|>r}\log|y|\,d\lambda(y)$ *is finite (in particular, if the support of λ is compact).*

Let u be an admissible superharmonic function; then the following statements are equivalent:

(i) $u \in \Lambda^*$;

(ii) *u majorizes an* L^**-potential;*

(iii) *u is equivalent to an L-potential.*

Proof. (a) and (b) are easy consequences of the following equality (whenever it has a meaning):

$$\int \Lambda(x, y)\, d\mu(y) = \int L(x, y)\, d\mu(y) + \int_{|y| \geq 1} \log |y|\, d\mu(y).$$

Now, in view of (i), u has outside of a disc a greatest harmonic minorant $\alpha \log |x| +$ a bounded function at infinity ($\alpha \leqslant 0$) (Theorem IV.8). But the function equal to $\alpha \log |x|$ for $|x| > 1$ and to 0 for $|x| \leqslant 1$ is a $U_L^\lambda = p$ (for a suitable measure λ on $|x| = 1$). The function u has the same greatest harmonic minorant as p outside a disc, up to a bounded function at infinity; and u majorizes $p+k$ in $\boldsymbol{R}^2$ for a suitable constant k. So we get (ii) and (iii).

In turn, (ii) or (iii) implies that u majorizes at infinity an L^*-potential, which is a Λ^*-potential, therefore u is a Λ^*-potential.

14. Approximation of a Λ^*-potential by finite continuous L^*-potentials.

THEOREM V.2. *Any admissible* Λ^**-potential u is the limit of an increasing sequence* u_n *of finite continuous* L^**-potentials with compact supports, where each* u_n *is equivalent to u.*

The greatest harmonic minorant of u outside a disc is of the form $\alpha \log |x|$ ($\alpha \leqslant 0$) + a bounded function at infinity; hence if p is the potential $\alpha \log |x|$ in $\{|x| > 1\}$ extended by 0 on $\{|x| < 1\}$, then $u(x) \geqslant p(x)+\lambda$ in $\boldsymbol{R}^2$ for some constant λ.

Let f_n be an increasing sequence of finite continuous functions tending to u, satisfying the condition $u(x) \geqslant f_n(x) \geqslant p(x)+\lambda$ in $\boldsymbol{R}^2$. Let v_n be the infimum of the family of superharmonic functions majorizing f_n. Then v_n is a finite continuous superharmonic function (see [6], Theorem 2). It is even admissible and equivalent to $p(x)+\lambda$. Hence v_n is an Λ^*-potential.

Let us replace in the annulus $n < R < q$, v_n by the solution of the Dirichlet problem. The boundary values v_n define a superharmonic function v_n^q satisfying

$$p+\lambda \leqslant v_n^q \leqslant v_n \leqslant u.$$

The limit for $q \to \infty$ is a superharmonic function u_n satisfying

$$p+\lambda \leqslant u_n \leqslant v_n \leqslant u,$$

u_n is increasing (like v_n) and is equivalent to u; hence it is also a Λ^*-potential, harmonic for $|x| > n$; it follows that u_n is an L^*-potential.

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