COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 1

DISCRETE ORBITS IN $\beta N - N$

BY

RODNEY NILLSEN (SWANSEA)

0. Introduction. Let N denote the set of natural numbers, including 0, and let βN be the Stone-Čech compactification of N. A function σ from N into N will be called a *motion* if it is an injection having no periodic points. The motion $n \to n+1$ will be denoted by τ. Throughout the paper, σ will denote a given motion. σ may be extended to a continuous function from βN into βN whose restriction to $\beta N - N$, which we also denote by σ, is a homeomorphism into $\beta N - N$ having no periodic points (cf. [9]). A subset of $\beta N - N$ is said to be *σ-invariant* if it is non-void, closed and mapped into itself by σ. A σ-invariant set is said to be *σ-minimal* if it properly contains no σ-invariant set. Let A^σ be the union of all σ-minimal sets. Let M_σ be the set of all regular Borel probability measures on $\beta N - N$ which are invariant under σ. Let K^σ be the closure of $\bigcup\{\operatorname{supp}\mu\}$, where this union is taken over $\mu \in M_\sigma$. For $\omega \in \beta N - N$ let $O_\sigma(\omega)$ be the set $\{\sigma^i\omega : i \in N\}$. The set $O_\sigma(\omega)$ is called the *σ-orbit* of ω, and if it is discrete as a subspace of $\beta N - N$, then ω is said to be *σ-discrete*. Let D^σ denote the set of all σ-discrete points of $\beta N - N$.

In the situation described above, the following results are known:

$A^\sigma \subset K^\sigma$ and $A^\sigma \neq K^\sigma$ [1];

there is an extreme point of M_σ whose support is not σ-minimal [1];

K^σ is nowhere dense in $\beta N - N$ ([1] and [9]);

the interior of $D^\sigma - K^\sigma$ is dense in $\beta N - N$ (see [1]), and in 1.4 of [11] Rudin has virtually proved that $D^\sigma \cap A^\sigma = \emptyset$.

Among the results proved here are the following:

$D^\sigma \cap K^\sigma$ is dense in K^σ;

$D^\sigma \cap \overline{A^\sigma}$ is dense in $\overline{A^\sigma}$;

there is an extreme point of M_σ whose support is not σ-minimal although it is contained in $\overline{A^\sigma}$;

any σ-minimal set is homeomorphic to each set in an uncountable family of σ-minimal sets and A^σ is nowhere dense in K^σ.

1. Preliminaries. Let l_∞ be the space of all bounded real-valued functions on N and l'_∞ its dual. $\mu' \in l'_\infty$ is called a *mean* if $\|\mu'\| = 1$ and $\mu'(1) = 1$, in which case $\mu' \geqslant 0$. Let M' be the set of all means. We shall regard βN as all multiplicative means on l_∞ and identify N with its natural injection into βN. If $\mu' \in M'$, there is a unique regular Borel probability measure μ on βN for which

$$\mu'(f) = \int_{\beta N} \bar{f} d\mu \quad \text{for } f \in l_\infty,$$

where $\bar{f}$ is the continuous extension of f to βN. The mean $\mu' \in M'$ is *σ-invariant* if $\mu'(f \circ \sigma) = \mu'(f)$ for $f \in l_\infty$. Let M'_σ be the set of all σ-invariant elements of M'. Then $M'_\sigma = \{\mu' : \mu \in M_\sigma\}$.

Let N^* denote the set of all non-zero elements of N. Then if $m \in N$ and $n \in N^*$, define $T_\sigma(m, n) \in M'$ by

$$T_\sigma(m, n)(f) = (1/n)\Big(\sum_{i=0}^{n-1} f(\sigma^i m)\Big) \quad \text{for } f \in l_\infty. \tag{1.1}$$

The following lemma is then well known (see [7]):

LEMMA 1.1. *Let I be a directed set and, for $a \in I$, let $m_a \in N$ and $n_a \in N^*$ be given. Let*

$$\mu' = \lim_a T_\sigma(m_a, n_a),$$

where this limit is assumed to exist in the weak topology. Then $\mu' \in M'_\sigma$ whenever* $\lim_a n_a = \infty$.

If $\mu' \in M'$ and $A \subseteq N$, we write $\mu'(A)$ for $\mu'(\chi_A)$, where χ_A is the characteristic function of A. We then define $\bar{d}_\sigma(A)$ by

$$\bar{d}_\sigma(A) = \sup_{\lambda' \in M'_\sigma} \lambda'(A).$$

It has been shown in [8] that

$$\bar{d}_\sigma(A) = \limsup_{n \to \infty} \big(\sup_m T_\sigma(m, n)(A)\big). \tag{1.2}$$

For $m \in N$ and $n \in N^*$ let $I_\sigma(m, n) = \{\sigma^i m : 0 \leqslant i \leqslant n-1\}$ and $I_\sigma(m, 0) = \emptyset$. Let $|A|$ be the cardinality of a finite subset A of N. Then, by (1.1),

$$T_\sigma(m, n)(A) = (1/n)\big(|A \cap I_\sigma(m, n)|\big). \tag{1.3}$$

2. Discrete σ-orbits in K^σ. For $A \subseteq N$ let $\hat{A} = \bar{A} \cap (\beta N - N)$, where $\bar{A}$ is the closure of A in βN. The sets $\hat{A}$ form a base for the topology of $\beta N - N$. For this and other properties of the sets $\hat{A}$, see [10].

LEMMA 2.1. *Let $A_0, A_1, A_2, \ldots$ be a sequence of subsets of N such that*

(2.1) $\sigma^m A_m \cap \sigma^n A_n$ *is finite if* $m \neq n$.

Then $\bigcap_{k=0}^{\infty} \hat{A}_k$ *is contained in* D^σ.

Proof. For $n \in N$ let

$$V_n = \sigma^n \hat{A}_n \quad \text{and} \quad \omega \in \bigcap_{k=0}^{\infty} \hat{A}_k.$$

Then V_n is a neighbourhood of $\sigma^n \omega$ for each n and, in view of (2.1), $V_m \cap V_n = \emptyset$ if $m \neq n$. Hence $O_\sigma(\omega)$ is discrete and $\omega \in D^\sigma$.

LEMMA 2.2. *Let $A \subseteq N$. Then there are sequences (p_n) and (q_n) in N such that*

$$\lim_{n\to\infty} q_n = +\infty, \tag{2.2}$$

$$\lim_{n\to\infty} (1/q_n)\big(|A \cap I_\sigma(p_n, q_n)|\big) = \bar{d}_\sigma(A), \tag{2.3}$$

and

(2.4) *the sets* $\big(I_\sigma(p_n, q_n)\big)_{n\in N}$ *are pairwise disjoint.*

Proof. Suppose $p_0, p_1, \ldots, p_{n-1}$ and $q_0, q_1, \ldots, q_{n-1}$ have been chosen so that

(a) $q_i > i$ for $0 \leqslant i \leqslant n-1$,

(b) $(1/q_i)\big(|A \cap I_\sigma(p_i, q_i)|\big) > \bar{d}_\sigma(A) - (1/i+1)$, and

(c) the sets $\big(I_\sigma(p_i, q_i)\big)_{0\leqslant i \leqslant n-1}$ are pairwise disjoint.

In view of (1.2) and (1.3) this can be done for $n = 1$.

Now let

$$X = \bigcup_{i=0}^{n-1} I_\sigma(p_i, q_i)$$

and choose $r_0, r_1, \ldots, r_k$ and $s_0, s_1, \ldots, s_k$ in N so that if some set $I_\sigma(p, q)$ meets X, there is a unique j, $0 \leqslant j \leqslant k$, such that $I_\sigma(p, q) \cap X \subseteq I_\sigma(r_j, s_j)$.

By (1.2) and (1.3), sequences (m_p) and (n_p) in N can be found so that

$$\lim_{p\to\infty} (1/n_p)\big(|A \cap I_\sigma(m_p, n_p)|\big) = \bar{d}_\sigma(A).$$

For each p, let

$$I_\sigma(m_p, n_p) = I_\sigma(m_p, u_p) \cup I_\sigma(\sigma^{u_p} m_p, v_p - u_p) \cup I_\sigma(\sigma^{v_p} m_p, n_p - v_p),$$

where $I_\sigma(m_p, u_p)$ and $I_\sigma(\sigma^{v_p} m_p, n_p - v_p)$ are disjoint from X and $I_\sigma(\sigma^{u_p} m_p, v_p - u_p)$ is contained in some $I_\sigma(r_j, s_j)$, $0 \leqslant j \leqslant k$. We see that

$$\lim_{p\to\infty} \big(1/(u_p + n_p - v_p)\big)\big(|A \cap I_\sigma(m_p, u_p)| + |A \cap I_\sigma(\sigma^{v_p} m_p, n_p - v_p)|\big) = \bar{d}_\sigma(A),$$

so that either

$$\limsup_{p\to\infty}(1/u_p)\big(|A\cap I_\sigma(m_p,u_p)|\big)\geqslant \bar{d}_\sigma(A)$$

or

$$\limsup_{p\to\infty}\big(1/(n_p-v_p)\big)\big(|A\cap I_\sigma(\sigma^{v_p}m_p,n_p-v_p)|\big)\geqslant \bar{d}_\sigma(A).$$

In either case it is now clear that p_n and q_n can be found so that (a), (b) and (c) hold with n in place of $n-1$, so that (p_n) and (q_n) have been defined for all n and (2.2) and (2.4) hold. By (b),

$$\liminf_{n\to\infty}(1/q_n)\big(|A\cap I_\sigma(p_n,q_n)|\big)\geqslant \bar{d}_\sigma(A)$$

and (1.1), (1.2), (1.3) and Lemma 1.1 ensure that

$$\limsup_{n\to\infty}(1/q_n)\big(|A\cap I_\sigma(p_n,q_n)|\big)\leqslant \bar{d}_\sigma(A).$$

Hence (2.3) holds.

Definition 2.1. Let $A=\{\sigma^{n_0}m,\sigma^{n_1}m,\ldots,\sigma^{n_{p-1}}m\}$, where $m\in N$ and $0\leqslant n_0<n_1<\ldots<n_{p-1}$. Assuming that $s\leqslant p$, a map $\gamma\colon I_\tau(r,s)\to A$ is defined by letting $\gamma(r+i)$ be $\sigma^{n_i}m$ for $0\leqslant i\leqslant s-1$. The map γ is called the *σ-injection* of $I_\tau(r,s)$ into A.

PROPOSITION 2.1. *For each motion σ, $D^\sigma\cap K^\sigma$ is dense in K^σ.*

Proof. This is divided into a number of steps.

I. For each $n\in N^*$ let $t_{1n},t_{2n},\ldots,t_{nn}$ be given in N^*. Then subsets $B(1,n)$, $B(2,n),\ldots,B(n,n)$ of N are defined inductively as follows:

$$B(1,n)=\{2i\colon 0\leqslant i\leqslant t_{11}-1\}.$$

If $2\leqslant k\leqslant n$ and $B(1,n),B(2,n),\ldots,B(k-1,n)$ have been defined, then

$$B(k,n)=\Big\{\Big(\sum_{j=1}^{k-1}(j+1)t_{jn}\Big)+(k+1)i\colon 0\leqslant i\leqslant t_{kn}-1\Big\}.$$

Then $|B(k,n)|=t_{kn}$. Moreover, if $0\leqslant j$ and $k\leqslant n$ with $k\neq j$, then

$$\tau^pB(j,n)\cap\tau^qB(k,n)=\emptyset\quad\text{for all } 0\leqslant p\leqslant j \text{ and } 0\leqslant q\leqslant k.$$

For $n,s\in N^*$ and $1\leqslant k\leqslant n$ we now let

$$B(k,n,s)=\bigcup_{i=0}^{s-1}\Big[\Big(\bigcup_{j=k}^{n}B(j,n)\Big)+i\Big(\sum_{j=1}^{n}(j+1)t_{jn}\Big)\Big].$$

The significance of the sets $B(k,n,s)$ lies in the facts that if $1\leqslant j$, $k\leqslant n$ and $j\neq k$, then for all s

$$\tau^jB(j,n,s)\cap\tau^kB(k,n,s)=\emptyset \tag{2.5}$$

and

(2.6) $$B(j, n, s) \subseteq B(k, n, s) \quad \text{if } j \geqslant k.$$

For future reference let us remark that if we take $t_{in} = 2^{n+1-i}$ for $1 \leqslant i \leqslant n$, then

$$\frac{\sum_{i=k}^{n} t_{in}}{\sum_{i=1}^{n}(i+1)t_{in}} = \frac{2^{-k+1} - 2^{-n}}{3 - 3\cdot 2^{-n} - n\cdot 2^{-n}},$$

so that

(2.7) $$\lim_{n\to\infty} \frac{\sum_{i=k}^{n} t_{in}}{\sum_{i=1}^{n}(i+1)t_{in}} = (1/3)\cdot 2^{-k+1}.$$

II. Let $A \subseteq N$ and suppose $\hat{A}$ meets K^{σ}. Then $\bar{d}_{\sigma}(A) > 0$ and, by Lemma 2.2, sequences (p_n) and (q_n) may be found in N so that (2.2), (2.3) and (2.4) hold. For $n \in N$ let

(2.8) $$|A \cap I_{\sigma}(p_n, q_n)| = u_n\Big(\sum_{i=1}^{n}(i+1)t_{in}\Big) + v_n,$$

$$\text{where } u_n, v_n \in N \text{ and } 0 \leqslant v_n < \sum_{i=1}^{n}(i+1)t_{in}.$$

In view of (2.2) and (2.3), $|A \cap I_{\sigma}(p_n, q_n)| \to \infty$ as $n \to \infty$, and we may assume that

(2.9) $$u_n \geqslant 1 \quad \text{and} \quad \lim_{n\to\infty} v_n/\big(|A \cap I_{\sigma}(p_n, q_n)|\big) = 0.$$

By virtue of (2.8) and (2.9) we see that $B(k, n, u_n)$ is contained in

$$I_{\tau}\Big(0, u_n \sum_{i=1}^{n}(i+1)t_{in}\Big)$$

and that we may let γ_n be the σ-injection of this latter set into $A \cap I_{\sigma}(p_n, q_n)$. Then define A_k for $k \in N^*$ by

$$A_k = \bigcup_{n=k}^{\infty}\Big(\gamma_n\big(B(k, n, u_n)\big)\Big)$$

and let $A_0 = A_1$. From the definitions and (2.4), (2.5) and (2.6) it follows that

(2.10) $$\sigma^j A_j \cap \sigma^k A_k = \emptyset \quad \text{if } j \neq k$$

and

(2.11) $$A \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$$

Now let $\lambda'_n = T_\sigma(p_n, q_n)$. Then, by (1.3),

$$\lambda'_n(A_k) = (u_n/q_n)\sum_{i=k}^{n} t_{in} \quad \text{for } n \geqslant k.$$

Together with (2.8) this gives

$$\lambda'_n(A_k) = \frac{\sum_{i=k}^{n} t_{in}}{\sum_{i=1}^{n}(i+1)t_{in}}\left(1-\frac{v_n}{|A\cap I_\sigma(p_n, q_n)|}\right)\frac{|A\cap I_\sigma(p_n, q_n)|}{q_n}.$$

If we now choose the t_{in} so that (2.7) holds and use (2.3) and (2.9), then

$$\lim_{n\to\infty}\lambda'_n(A_k) = (1/3)\cdot 2^{-k+1}\bar{d}_\sigma(A) > 0.$$

Hence, if λ' is a limit point of (λ'_n), then $\lambda' \in M'_\sigma$ by Lemma 1.1 and

$$\lambda'(A_k) > 0 \quad \text{for } k \in N. \tag{2.12}$$

III. In view of (2.11) and (2.12), there is

$$\omega \in (\operatorname{supp}\lambda)\cap\left(\bigcap_{0}^{\infty}\hat{A}_k\right),$$

so that $\omega \in K^\sigma \cap \hat{A}$. However, by (2.10) and Lémma 2.1, $\omega \in D^\sigma$. Hence, if $\hat{A}$ meets K^σ, then $\hat{A}$ meets $D^\sigma \cap K^\sigma$ and the proof is complete.

3. Discrete σ-orbits in $\bar{A}^\sigma$. For $C \subseteq N$ and $k \in N^*$, C is called a *k-chain* if $|m-n| \leqslant k$ whenever m and n are successive elements of C.

LEMMA 3.1. *The following are equivalent conditions on a point ω in $\beta N - N$:*

(1) $\omega \in A^\sigma$, *and*

(2) *given a neighbourhood U of ω in $\beta N - N$, there is $k \in N^*$ for which $\{i: i \in N \text{ and } \sigma^i\omega \in U\}$ is a k-chain having an infinite number of elements.*

Proof. This is a straightforward adaptation of either Lemma 2.4 of [1] or Proposition 3.1 of [4].

PROPOSITION 3.1 (cf. Remark 1.4 of [11]). *For each motion σ, we have $D^\sigma \cap A^\sigma = \emptyset$.*

Proof. If $\omega \in D^\sigma$, there is a neighbourhood V of ω in βN, so that $V \cap O_\sigma(\omega) = \{\omega\}$. By Lemma 3.1, $\omega \notin A^\sigma$.

LEMMA 3.2. *Let $A \subseteq N$. Then the following conditions are equivalent:*

(1) $\bar{d}_\sigma(A) = 1$,

(2) *given $n \in N$, there is $m \in N$ for which $I_\sigma(m, n) \subseteq A$, and*

(3) *there are sequences (p_n) and (q_n) in N such that $\lim_{n\to\infty} q_n = \infty$, the sets $\big(I_\sigma(p_i, q_i)\big)_{i\in N}$ are pairwise disjoint and $I_\sigma(p_i, q_i) \subseteq A$ for each i.*

Proof. (1) implies (3) is a consequence of Lemma 2.2. (3) implies (2) is obvious, and (2) implies (1) is immediate from (1.2) and (1.3).

LEMMA 3.3. *Let $A \subseteq N$. Then the following conditions are equivalent*:

(1) $\hat{A} \cap A^{\sigma} \neq \emptyset$,

(2) $\hat{A} \cap \bar{A}^{\sigma} \neq \emptyset$, *and*

(3) *for some* $r \in N^*$, $\bar{d}_{\sigma}(A \cup \sigma A \cup \ldots \cup \sigma^{r-1} A) = 1$.

Proof. The equivalence of (1) and (2) is obvious as $\hat{A}$ is open. If (3) holds, $\hat{A} \cup \sigma\hat{A} \cup \ldots \cup \sigma^{r-1}\hat{A}$ must contain some σ-minimal set which will then meet $\hat{A}$, and (1) follows. If (3) does not hold, it is straightforward to adapt the argument of [1], p. 784, to prove the result — the idea is to use Lemma 3.1 instead of Lemma 2.4 of [1].

LEMMA 3.4. *Let $A \subseteq N$, $m \in N$ and $r, n \in N^*$ with $r \leqslant n$. If*

$$I_{\sigma}(m, n) \subseteq A \cup \sigma A \cup \ldots \cup \sigma^{r-1} A,$$

then $\{i\colon 0 \leqslant i \leqslant n-1$ and $\sigma^i m \in A\}$ is an r-chain whose first element is inclusive between 0 and $r-1$.

The proof is easy and is omitted.

PROPOSITION 3.2. *For each motion σ, $D^{\sigma} \cap \bar{A}^{\sigma}$ is dense in $\bar{A}^{\sigma}$.*

Proof. Let $A \subseteq N$ and suppose $\hat{A}$ meets $\bar{A}^{\sigma}$. By Lemma 3.3 we may choose $r \in N^*$ so that $\bar{d}_{\sigma}(A \cup \sigma A \cup \ldots \cup \sigma^{r-1} A) = 1$. By Lemmas 3.2 and 3.4, choose sequences (p_n) and (q_n) in N so that (2.2) and (2.4) hold; if

$$J_n = \{i\colon 0 \leqslant i \leqslant q_n - 1 \text{ and } \sigma^i p_n \in A\},$$

then

(3.1) J_n is an r-chain,

and

$$|J_n| \geqslant \sum_{i=1}^{n} (i+1) t_{in}. \tag{3.2}$$

Here the t_{in} are as in the proof of Proposition 2.1 and we choose them so that

$$\lim_{n\to\infty} t_{kn} = \infty \quad \text{for } k \in N^*. \tag{3.3}$$

In view of (3.2), we may let γ_n be the σ-injection of $I_{\tau}\big(0, \sum_{i=1}^{n} (i+1) t_{in}\big)$ into $A \cap I_{\sigma}(p_n, q_n)$. Then define C_k for $k \in N^*$ by

$$C_k = \bigcup_{n=k}^{\infty} \big(\gamma_n\big(B(k, n, 1)\big)\big)$$

and let $C_0 = C_1$. By (2.4), (2.5) and (2.6), we have

$$\sigma^j C_j \cap \sigma^k C_k = \emptyset \quad \text{if } j \neq k, \tag{3.4}$$

and

$$A \supseteq C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots \tag{3.5}$$

Now each set $B(k, n, 1)$ contains $B(k, n)$ which is a $(k+1)$-chain having t_{kn} elements, so that, by (3.1), the set

$$\{i\colon 0 \leqslant i \leqslant q_n - 1 \text{ and } \sigma^i p_n \in \gamma_n(B(k, n, 1))\}$$

must contain an $r(k+1)$-chain having t_{kn} elements. Hence there is $p'_n \in \gamma_n(B(k, n, 1))$, so that

$$I_\sigma(p'_n, t_{kn}) \subseteq C_k \cup \sigma C_k \cup \ldots \cup \sigma^{r(k+1)-1} C_k.$$

In view of (3.3) and Lemma 3.2 we can deduce that

$$\bar{d}_\sigma(C_k \cup \sigma C_k \cup \ldots \cup \sigma^{r(k+1)-1} C_k) = 1.$$

By Lemma 3.3, $\hat{C}_k$ meets $\overline{A^\sigma}$ for all $k \in N$. By (3.4), (3.5) and Lemma 2.1, we may choose $\omega \in D^\sigma \cap \overline{A^\sigma} \cap \hat{A}$ and the result is established.

The following corollary is not surprising, but to the author's knowledge has not been proved before.

COROLLARY 3.1. *A^σ is not closed.*

The proof is immediate from Propositions 3.1 and 3.2.

In [1], Chou has proved that there are extreme points of M_τ whose support is not τ-minimal. None of these extreme points he constructs are supported by $\overline{A^\tau}$, although the following proposition shows that this does happen.

PROPOSITION 3.3. *For each motion σ, there is an extreme point of M_σ whose support is not σ-minimal but which is contained in $\overline{A^\sigma}$.*

Proof. Let $\omega \in D^\sigma \cap \overline{A^\sigma}$ by Proposition 3.2. Let $\gamma\colon N \to O_\sigma(\omega)$ be $\gamma(i) = \sigma^i \omega$. The map γ extends to a homeomorphism, which we also denote by γ, from βN onto $\beta(O_\sigma(\omega))$. By 6.9 and 14N of [5], $\beta(O_\sigma(\omega)) = \overline{O_\sigma(\omega)}$. Moreover, $\gamma \circ \tau = \sigma \circ \gamma$ on βN. By 2.5 of [1], let $\lambda \in M_\tau$ be an extreme point of M_τ whose support is not τ-minimal. Then $\lambda \circ \gamma^{-1}$ is an extreme point of M_σ whose support is not σ-minimal. However, the support of $\lambda \circ \gamma^{-1}$ is in $\overline{O_\sigma(\omega)}$. Now $\omega \in \overline{A^\sigma}$, so that $\overline{O_\sigma(\omega)} \subseteq \overline{A^\sigma}$ and the result follows.

4. Homeomorphic minimal sets. Here we show that any σ-minimal set is homeomorphic to uncountably many other σ-minimal sets.

LEMMA 4.1. *Let $B \subseteq N$ and assume that $\bar{d}_\sigma(B) = 1$. Then there is $\omega \in D^\sigma$ for which $\overline{O_\sigma(\omega)} \subseteq \hat{B}$.*

Proof. Let t_{kn} be numbers as in the proof of Proposition 2.1. Let

$$q_n = \sum_{i=1}^{n} (i+1) t_{in}.$$

Then, by Lemma 3.2, there is a sequence (p_n) in N, so that (2.2) and (2.4) hold and $I_\sigma(p_n, q_n) \subseteq B$ for all n. Let γ_n be the σ-injection of $I_\tau(0, q_n)$ into $I_\sigma(p_n, q_n)$ and, for $k \in N^*$, let

$$A_k = \bigcup_{n=k}^{\infty} \big(\gamma_n(B(k, n, 1))\big) \quad \text{and} \quad A_0 = A_1.$$

Then (2.10) and (2.11) hold for the sets A_k and $\sigma^k(A_k) \subseteq B$ for all $k \in N$. Let

$$\omega \in \bigcap_{k=0}^{\infty} \hat{A}_k.$$

Then $\omega \in D^\sigma$ by Lemma 2.1. Since $\sigma^k \omega \in \sigma^k \hat{A}_k \subseteq \hat{B}$ for all k, $\overline{O_\sigma(\omega)} \subseteq \hat{B}$.

LEMMA 4.2. *Any τ-minimal set in $\beta N-N$ is homeomorphic to each set of an uncountable family of τ-minimal sets in $\beta N-N$.*

Proof. Let M be a τ-minimal set in $\beta N-N$. By Corollary 3.1 there is an infinite family of τ-minimal sets, so that we may choose $A \subseteq N$ with $M \subseteq \hat{A}$ and $\bar{d}_\tau(N-A) = 1$. By (1.9) of [1], there is an uncountable family (B_a) of subsets of $N-A$ such that $\bar{d}_\tau(B_a) = 1$ for all a and $B_a \cap B_\beta$ is finite if $a \neq \beta$. By Lemma 4.1 choose $\omega_a \in D^\tau$ so that $\overline{O_\tau(\omega_a)} \subseteq \hat{B}_a$. Now let ψ_a: $\beta N \to \overline{O_\tau(\omega_a)}$ be the homeomorphism obtained by extending $i \to \tau^i \omega_a$ on N to βM (since $\beta(O_\tau(\omega_a))$ is $\overline{O_\tau(\omega_a)}$ by [5], 6.9 and 14N). Then $\psi_a \circ \tau = \tau \circ \psi_a$ on βN, so that $\psi_a(M)$ is τ-minimal for each a. Since $(\hat{B}_a)$ is a disjoint family and $\psi_a(M) \subseteq \overline{O_\tau(\omega_a)}$, we infer that $(\psi_a(M))$ is an uncountable family of τ-minimal sets to each of which M is homeomorphic.

PROPOSITION 4.1. *Let σ_1 and σ_2 be motions. Then each σ_1-minimal set in $\beta N-N$ is homeomorphic to each set in an uncountable family of σ_2-minimal sets in $\beta N-N$.*

Proof. In view of [2], Theorem 4.8, and [9], Theorem 2.17, there is no loss of generality in assuming that $\{\sigma_1^i m: i \in N\} = N$ for some $m \in N$. Let φ: $\beta N \to \beta N$ be the homeomorphism obtained by extending the map $i \to \sigma_1^i m$ from N to βN. Then $\sigma_1 \circ \varphi = \varphi \circ \tau$ on βN, so that if M is a σ_1-minimal set, $\varphi^{-1}(M)$ is τ-minimal. By Lemma 4.2, there is an uncountable family (M_a) of τ-minimal sets all homeomorphic to $\varphi^{-1}(M)$. By arguing now with σ_2 in place of σ_1, there is a homeomorphism ψ: $\beta N \to \beta N$ such that $\psi(M_a)$ is σ_2-minimal for each a. Then M is homeomorphic to $\psi(M_a)$ for each a, and the result is established.

5. Nowhere denseness of A^σ in K^σ. Let us recall a construction due to Chou in [1]. Let $A_1 = \{1, 2, 3\}$. If $A_1, A_2, \ldots, A_n$ have been defined, let

$$A_{n+1} = A_n \cup \{\sup A_n + n + A_n\} \cup \{2 \sup A_n + 2n + A_n\}.$$

This defines $A_1, A_2, \ldots$ by induction. It is easy to see that

(5.1) any k-chain in A_n has at most $|A_k|$ elements.

PROPOSITION 5.1. *For each motion σ, A^σ is nowhere dense in K^σ.*

Proof. Let $A \subseteq N$ and suppose $\hat{A}$ meets A^σ. By Lemmas 3.2, 3.3 and 3.4, there is an $r \in N^*$ and sequences (p_n) and (q_n) in N exist, so that (2.2) and (2.4) hold and, for each n,

(5.2) $\{i\colon 0 \leqslant i \leqslant q_n - 1 \text{ and } \sigma^i p_n \in A\}$ is an r-chain

having $\sup A_n$ elements, whose first element is 0 and whose last element is $q_n - 1$, where A_n is as in (5.1). Hence we may let γ_n be the σ-injection of $I_\tau(1, \sup A_n)$ into $A \cap I_\sigma(p_n, q_n)$ and let

$$B = \bigcup_{n \in N^*} \gamma_n(A_n).$$

Then $T_\sigma(p_n, q_n)(B) = |A_n|/q_n$. Now

$$\lim_{n\to\infty} |A_n|/(\sup A_n) = 2/3$$

(see [1]), and (5.2) ensures that $(\sup A_n)/q_n \geqslant 1/r$. Hence, if λ' is a limit point of $\big(T_\sigma(p_n, q_n)\big)$, $\lambda' \in M'_\sigma$ by Lemma 1.1, and $\lambda'(B) > 0$. Hence $\hat{B}$ meets K^σ.

It now follows from (5.1) that if $m \in N$ and $k, n \in N^*$, then any k-chain contained in $\{i\colon i \in N \text{ and } \sigma^i m \in \gamma_n(A_n)\}$ has at most $|A_k|$ elements. From this it is easy to see that any k-chain contained in $\{i\colon i \in N \text{ and } \sigma^i m \in B\}$ has at most $|A_1| + |A_2| + \ldots + |A_{k-1}| + 3|A_k|$ elements. In view of Lemmas 3.2, 3.3 and 3.4, we deduce that $\hat{B} \cap \overline{A^\sigma} = \emptyset$. Since $\hat{B} \subseteq \hat{A}$ and $\hat{B}$ meets K^σ, the result is established.

COROLLARY 5.1. *If $\omega \in D^\sigma \cap \overline{A^\sigma}$, then $\overline{A^\sigma} - \overline{O_\sigma(\omega)}$ contains an infinite family of σ-minimal sets.*

Proof. Let us assume that $M_1, M_2, \ldots, M_p$ are all σ-minimal sets in $\overline{A^\sigma} - \overline{O_\sigma(\omega)}$. Then

$$\overline{A^\sigma} - \overline{O_\sigma(\omega)} = M_1 \cup M_2 \cup \ldots \cup M_p.$$

Let γ be the homeomorphism from βN onto $\overline{O_\sigma(\omega)}$ constructed in the proof of Proposition 3.3. Then

$$\gamma(A^\tau) = A^\sigma - (M_1 \cup M_2 \cup \ldots \cup M_p),$$

which, by Proposition 5.1, must be nowhere dense in $\overline{A^\sigma}$. This is clearly impossible.

6. Remarks.

6.1. If $\omega \in \beta N - N$, then the induced topology on $O_\sigma(\omega)$ leads to a topology T_ω on N by identifying N with $O_\sigma(\omega)$ by means of the map $i \to \sigma^i \omega$. The topology T_ω is Hausdorff and τ is T_ω-continuous. If $\omega \in A^\sigma$, then every open set in T_ω will be a k-chain for some $k \in N$ (by Lemma 3.1). So far there do not seem to be examples of points of $\beta N - N$ which belong neither to D^σ nor to A^σ. If such a point ω does exist, the topology T_ω will be non-discrete and will have open sets which are k-chains for no $k \in N^*$. Such topologies on N are possible.

6.2. In view of Proposition 4.1 it would be interesting to have an example of two non-homeomorphic σ-minimal sets or to prove that this is not possible. (**P 938**)

6.3. Assuming the continuous hypothesis, Rudin [10] has constructed homeomorphisms of $\beta N - N$ which arise from no motion on N. I do not know what results of this paper are valid for such homeomorphisms. (**P 939**)

REFERENCES

[1] C. Chou, ***Minimal sets and ergodic measures for $\beta N - N$***, Illinois Journal of Mathematics 13 (1969), p. 777-788.

[2] D. Dean and R. Raimi, ***Permutations with comparable sets of invariant means***, Duke Mathematical Journal 27 (1960), p. 467-479.

[3] R. Ellis, ***Topological dynamics***, New York 1969.

[4] L. Fairchild, ***Extreme invariant means without minimal support***, Transactions of the American Mathematical Society 172 (1972), p. 83-93.

[5] L. Gillman and M. Jerison, ***Rings of continuous functions***, Princeton 1960.

[6] G. H. Hardy and J. E. Littlewood, ***Some problems of Diophantine approximation***, Acta Mathematica 37 (1914), p. 155-191.

[7] M. Jerison, ***The set of all generalised limits of bounded sequences***, Canadian Journal of Mathematics 9 (1957), p. 79-89.

[8] R. Raimi, ***Invariant means and invariant matrix methods of summability***, Duke Mathematical Journal 30 (1963), p. 81-94.

[9] — ***Homeomorphisms and invariant measures for $\beta N - N$***, ibidem 33 (1966), p. 1-12.

[10] W. Rudin, ***Homogeneity problems in the theory of Čech-compactifications***, ibidem 23 (1956), p. 409-419.

[11] — ***Averages of continuous functions on compact spaces***, ibidem 25 (1958), p. 197-204.

UNIVERSITY COLLEGE OF SWANSEA
WALES

Reçu par la Rédaction le 15. 3. 1974