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POSITIVE PRODUCT FORMULAS AND HYPERGROUPS ASSOCIATED WITH SINGULAR STURM-LIOUVILLE PROBLEMS ON A COMPACT INTERVAL

BY

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TO PROFESSOR ANTONI ZYGMUND WHOSE LIFE AND WORKS HAVE BEEN A CONSTANT SOURCE OF INSPIRATION

Introduction. A number of authors have given sufficient conditions on a Sturm–Liouville problem so that the eigenfunctions of the problem will be the characters of a measure algebra. In this paper we will bring together known results, with a consistent notation, so that they can be compared. Further, some of these old results are extended and some new results in this direction are proved. In [7] we identify a class of measure algebras on $[0,\pi]$ called Jacobi type hypergroups, such that for each of these measure algebras, the characters are a complete set of eigenfunctions for the Sturm–Liouville problem in one of its self-adjoint forms:

$$(\mathrm{S}) \qquad (\rho^2 y')' + \mu\rho^2 y = 0, \qquad y'(0) = y'(\pi) = 0.$$

In this article we are concerned with the inverse question of when the eigenfunctions of a Sturm–Liouville problem on $[0,\pi]$ are also the characters of a hypergroup. This is equivalent to asking if there are non-negative measures $\sigma_{s,t}$ supported on $[0,\pi]$ such that $\sigma_{0,t}$ is the unit mass concentrated at t and $\operatorname{supp}(\sigma_{s,t})$ shrinks to $\{t\}$ as s decreases to 0 and which satisfy

$$\int_0^\pi y(r)\, d\sigma_{s,t}(r) = y(s)y(t) \qquad (0 \le s,t \le \pi)$$

for every eigenfunction y of (S) normalized to $y(0) = 1$. Equations such as the one above are called *product formulas*. The problem was addressed with some success in [4] (with the notation $\gamma = \alpha + 1/2$) for the perturbed ultraspherical problem

$$(\mathrm{U}) \qquad u''(s) + 2\gamma(\cot s)u'(s) + (\lambda - q)u(s) = 0, \qquad u'(0) = u'(\pi) = 0.$$

We shall also be concerned with a Liouville normal form of the problem:

$$\text{(L)} \qquad w'' + (\nu - Q)w = 0, \qquad w(0) = w(\pi) = 0.$$

It is interesting to note (see [6]) that the only orthogonal polynomials which have a product formula with $\sigma_{s,t}$ as above are the Jacobi polynomials of order (α, β) with $\alpha \geq \beta > -1$ and either $\beta \geq -1/2$ or $\alpha + \beta \geq 0$.

Our approach here shares that of [4] in that it is based on an associated hyperbolic Cauchy problem, but it is technically much simpler and includes some cases not covered by [4]. The advantages of the approach taken in [4] (which is based on the Riemann integration method), is that more detailed information about the product formula for the eigenfunctions is obtained. In the last section we derive improved results by combining the two types of theorems.

There are a number of related results on the infinite interval about the eigenfunctions of analogous Sturm–Liouville problems; see [3] for (S), [2] and [8] for (U), and [10, p. 53] for (L). The proofs of these theorems can be adapted to $[0, \pi]$ if symmetric boundary conditions and symmetric coefficients are required. These results are brought together here. Theorems 3 and 4 for (U) are given in [4] and a more limited version of Theorem 1 can be found in [1]. The results in Theorem 5 for (L) are new, and by a simple change of variables produce an extension of the results in [4].

We conclude this section with some notation. C will denote the space of continuous functions on $[0, \pi]$ with the supremum norm $\|f\|_\infty$. C_c consists of the functions in C which are supported on a closed subset of $(0, \pi)$; C_0 consists of the functions in C which vanish at 0 and at π, and C^k consists of those with k derivatives belonging to C. If χ is a measure on $[0, \pi]$, its total variation will be denoted by $\|\chi\|$.

The self-adjoint form. Consider the problem (S) where ρ is required to satisfy the following conditions:

S1. ρ is positive and continuous on $(0, \pi)$ and normalized so that

$$\int_0^\pi \rho^2(s)\, ds = 1.$$

S2. $\rho(\pi - s) = \rho(s)$.

S3. $\rho(s) = (\sin^\gamma s) g(s)$ for some $\gamma \geq 0$ where g is real-analytic at 0 and at π, $g \in C^p$ where p is an integer greater than $\max(\gamma + 1/2, 2)$ and $g(0) > 0$.

S4. $\rho'(s)/\rho(s)$ is a non-increasing function on $(0, \pi)$.

THEOREM 1. *Let ρ satisfy* S1–S4. *For each s and t in $[0, \pi]$, there is a non-negative measure $\sigma_{s,t} = \sigma^\rho_{s,t}$ with the following properties:*

(i) $\int_0^\pi y(r)\,d\sigma_{s,t}(r) = y(s)y(t)$ *where* y *is any eigenfunction of* (S) *normalized so that* $y(0) = 1$.

(ii) $\|\sigma_{s,t}\| = 1$.

(iii) $\operatorname{supp}(\sigma_{s,t}) \subseteq [|s-t|, \pi - |s+t-\pi|]$.

(iv) *If* $f \in C$, *then* $f(s,t) = \int_0^\pi f(r)\,d\sigma_{s,t}(r)$ *is a continuous function on* $[0,\pi] \times [0,\pi]$.

The Hilbert space L^2 associated with (S) is defined by the inner product

$$\langle f, h\rangle = \int_0^\pi f(r)h(r)\rho^2(r)\,dr$$

and the norm $\|f\|_2 = \langle f, f\rangle^{1/2}$. It will be convenient to gather some facts about the eigenvalues and eigenfunctions of (S) into the following

LEMMA 1. (i) *The eigenvalues of* (S) *may be arranged in an increasing sequence* $0 = \mu_0 < \mu_1 < \mu_2 < \ldots$

(ii) $\lim_{k\to\infty} k^2\mu_k = 1$.

(iii) *Corresponding to each* μ_k *is an eigenfunction* y_k *normalized so that* $y_k(0) = 1$.

(iv) y_k *has exactly* k *zeros in* $[0,\pi]$.

(v) $y_k(\pi - s) = (-1)^k y_k(s)$.

(vi) *There is a constant* M *such that* $\|y_k\|_\infty \le M$ *for every non-negative integer* k.

(vii) $\lim_{k\to\infty} k^{2\gamma}\|y_k\|_2^2 = 2^{\gamma-1/2}\Gamma(\gamma - 1/2)g(0)$ (cf. S3).

(viii) $\{y_k\}_{k=0}^\infty$ *is a complete orthogonal system for* L^2.

Proof. The case $\gamma = 0$ is classical, so we assume that $\gamma > 0$. [7] contains a discussion of eigenvalues and eigenfunctions of (S) with $\gamma = \alpha + 1/2$. In that setting, (S) arises from a hypergroup, but many of the results obtained do not depend on that fact. We use some of those results here. For (i), (ii), and (iv) see Lemmas 3.3 and 3.4 of [7]. That the normalization in (iii) is possible follows from the Frobenius method and the requirement that γ and $g(0)$ be non-negative.

To obtain (v) note that if $y(s)$ is an eigenfunction belonging to μ_k, then so is $y(\pi - s)$. Thus there are eigenfunctions $y_e(s) = y(s) + y(\pi - s)$, and $y_o(s) = y(s) - y(\pi - s)$ which are even and odd with respect to $\pi/2$. Each must have k zeros in $(0,\pi)$; if k is odd, it follows that y_e is identically zero, which implies that $y(s) = -y(\pi - s)$ and y is an odd function with respect to $\pi/2$, and similarly if k is even.

(vi) can be obtained by combining (ii) of this lemma with the Hilb type estimates in [7, Theorem 4.1]:

$$y_k(s) = \frac{g(0)s^\gamma}{\rho(s)} 2^\alpha \Gamma(\alpha+1)(m_k s)^{-\alpha} J_\alpha(m_k s) + I_k(s)$$

where $\alpha = \gamma - 1/2$, $m_k^2 = \mu_k$, and J_α is the Bessel function of the first kind of order α. For any $\varepsilon > 0$, there is $C = C_\varepsilon > 0$ such that

$$|I_k(s)| \le C/k \quad (0 \le s \le 1/k),$$
$$|I_k(s)| \le Ck^{-1}(ks)^{-\gamma} \ln k \quad (k^{-1} \le s \le \pi - \varepsilon).$$

(vii) follows from [7, Theorem 4.3(iii)].

To obtain (viii) substitute $w = \rho y$ in (S) to obtain the Liouville normal form (L) with $\nu = \mu + \gamma$ and

$$Q(s) = \gamma(\gamma - 1)(\cot^2 s) + [g''(s) + 2\gamma(\cot s)g'(s)]/g(s),$$

and use the argument in the proof of [4, Lemma 1.1] to show that the eigenfunctions of this problem are a complete orthogonal system on $[0, \pi]$ with respect to Lebesgue measure, and (viii) follows. ■

We define

$$h_k = \|y_k\|_2^{-2}, \qquad \hat{f}_k = \langle f, y_k \rangle, \qquad S(n, f) = \sum_{k=0}^{n} \hat{f}_k h_k y_k.$$

Let $\mathcal{P}$ be the space of finite linear combinations of eigenfunctions and $\mathcal{P}_+$ the non-negative functions in $\mathcal{P}$. If f belongs to $\mathcal{P}$ then $f = S(n, f)$ for some finite n; in this case, let $T^t f(s) = \sum_{k=0}^{n} \hat{f}_k h_k y_k(s) y_k(t)$. We shall see that T^t is a generalized translation in the sense of Levitan [10]. If $0 \le a \le b \le \pi$, let $\Delta(a, b) = \{(s, t) : s \ge t,\ a \le s - t \le s + t \le b\}$ and $\Delta = \Delta(0, \pi)$.

The proof of Theorem 1 requires two lemmas.

LEMMA 2. *$\mathcal{P}$ is dense in $\mathcal{C}$.*

Proof. The argument is inspired by [9]. If $f \in \mathcal{C}_c^{2p}$, then $\hat{f}_k = (-1/\mu_k)(\mathcal{D}f)^\wedge$, where $\mathcal{D}f = \rho^{-2}(\rho^2 y')'$; because of S3, this can be repeated $p - 1$ more times to obtain $\hat{f}_k = (-1/\mu_k)^p (\mathcal{D}^p f)^\wedge$; it follows that $\hat{f}_k = O(k^{-2p})$ so that by Lemma 1(vii), $\sum_{k=0}^{\infty} |\hat{f}_k| h_k < \infty$. It then follows from Lemma 1(vi) that $S(n, f)$ converges absolutely and uniformly to f, so that $\mathcal{P}$ is dense in $\mathcal{C}_c^{2p}$, and also in $\mathcal{C}_0$.

By Lemma 1(iii) and (iv), $y_1(0) = -y_1(\pi) = 1$, so if $f \in \mathcal{C}$, then

$$f_1 = f - f(0)(1 + y_1)/2 - f(\pi)(1 - y_1)/2$$

belongs to $\mathcal{C}_0$. Now if $\varepsilon > 0$, we can choose $f_2 \in \mathcal{P}$ such that $\|f_2 - f_1\|_\infty < \varepsilon$, then $F = f_2 + f(0)(1 + y_1)/2 + f(\pi)(1 - y_1)/2$ belongs to $\mathcal{P}$ and the lemma follows. ■

LEMMA 3. *Let $f \in \mathcal{P}$ and assume $f(s) \ge 0$ for all s in some interval $[a, b] \subseteq [0, \pi]$. Then $f(s, t) \ge 0$ for all (s, t) belonging to $\Delta(a, b)$.*

Proof. Begin by assuming that $f \in \mathcal{P}$ is strictly positive on $[a, b]$. Let $f(s, t) = T^t(f)(s)$, and assume by way of contradiction that $f(s, t)$ is negative at some point of $\Delta(a, b)$. Then it is possible to choose $P = (\xi, \eta)$ in

$\Delta(a,b)$ so that $f(\xi,\eta)=0$, but $f(s,t)>0$ if $(s,t)\in\Delta(a,b)$ and $0\le t<\eta$. Let $W(s,t)=\rho^2(s)\rho^2(t)$. Then $f(s,t)$ solves the hyperbolic Cauchy problem

$$(Wf_s)_s-(Wf_t)_t=0,\quad f(s,0)=f(s),\quad f_t(s,0)=0\quad(0\le s,t\le\pi).$$

Now let $c=\xi-\eta$ and $d=\xi+\eta$. By Green's Theorem

$$\begin{aligned}0&=\iint\limits_{\Delta(c,d)}[(Wf_s)_s-(Wf_t)_t]\,ds\,dt\\&=\oint\limits_{\partial\Delta(c,d)}(Wf_s\,dt+Wf_t\,ds)\\&=-\Big(\int\limits_{CP}+\int\limits_{BP}\Big)W\,df\end{aligned}$$

where $C=(c,0)$ and $D=(d,0)$. Integration by parts yields

$$\begin{aligned}2W(P)f(P)&=W(C)f(C)+W(D)f(D)\\&\quad+\int\limits_{CP}f(W_t+W_s)\,dt+\int\limits_{DP}f(W_t-W_s)\,dt.\end{aligned}$$

Finally, S4 implies $W_t\pm W_s\ge 0$ on $\Delta(a,b)$, so $2W(P)f(P)\ge W(C)f(C)+W(D)f(D)>0$, which contradicts our assumption about $f(P)$.

If now $f(s)\ge 0$ on (a,b), let $\varepsilon>0$ and $f_\varepsilon=f+\varepsilon=f+\varepsilon y_0$. Then, since $y_0(s)=1$, $f_\varepsilon(s,t)=f(s,t)+\varepsilon y_0(s)y_0(t)=f(s,t)+\varepsilon$. Thus by the argument above, $f_\varepsilon(s,t)\ge 0$ on $\Delta(a,b)$; since ε is arbitrary, this establishes the lemma. ∎

Proof of Theorem 1. We begin by assuming $f\in\mathcal{P}_+$. Then Lemma 3 implies that $f(s,t)=T^tf(s)\ge 0$ if $(s,t)\in\Delta$. But $f(s,t)=f(t,s)=f(\pi-t,\pi-s)$ by Lemma 1(v), so $f(s,t)\ge 0$ on all of $[0,\pi]\times[0,\pi]$, or equivalently, $T^tf\in\mathcal{P}_+$.

Now if $f\in\mathcal{P}$, then $h=\|f\|_\infty\pm f\in\mathcal{P}_+$, hence $T^th=\|f\|_\infty\pm T^tf\in\mathcal{P}_+$, from which it follows that $\|T^tf\|_\infty\le\|f\|_\infty$ for every $f\in\mathcal{P}$. Since $\mathcal{P}$ is dense in $\mathcal{C}$ (Lemma 2) it follows that

(a) T^t can be extended to all of $\mathcal{C}$.
(b) For all f in $\mathcal{C}$, $\|T^tf\|_\infty\le\|f\|_\infty$.
(c) If $f\in\mathcal{C}_+$ then $T^tf\in\mathcal{C}_+$.
(d) If $f(s)$ is continuous then $f(s,t)$ is continuous in (s,t).

The Riesz Representation Theorem together with (b) and (c) guarantees the existence of a non-negative measure $\sigma_{s,t}$ with $\|\sigma_{s,t}\|\le 1$ such that

$$T^tf(s)=\int_0^\pi f(r)\,d\sigma_{s,t}(r).$$

$T^t y_0(s) = 1$ for all s, so $\|\sigma_{s,t}\| = 1$. To obtain (i) set $f = y_k$.

Suppose $(s,t) \in \Delta$, and assume $f \in C$ vanishes on $[s-t, s+t]$; then since $f \geq 0$ and $-f \geq 0$ on $[s-t, s+t]$ Lemma 3 implies that $f(s,t) = 0$, hence $\operatorname{supp}(\sigma_{s,t}) \subseteq [s-t, s+t]$. Part (iii) of the theorem follows from symmetry since $\sigma_{s,t} = \sigma_{t,s} = \sigma_{\pi-s,\pi-t}$.

Hypergroups and convolution. It is now easy to associate a measure algebra with the Sturm–Liouville problem (S): define $\chi * \omega$ by

$$(1) \qquad \int_0^\pi f(r)\, d(\chi * \omega)(r) = \int_0^\pi \int_0^\pi \int_0^\pi f(r)\, d\sigma_{s,t}(r)\, d\chi(s)\, d\omega(t),$$

so that $\delta_s * \delta_t = \sigma_{s,t}$.

THEOREM 2 [7, Theorem 3.6]. *The measure algebra described above is a Jacobi type* $(\gamma - 1/2, \gamma - 1/2)$ *hypergroup with characters* $\{y_k\}$.

Thus the measure algebra defined by (1) has all the properties of Jacobi type $(\gamma - 1/2, \gamma - 1/2)$ hypergroups (cf. [4]). In particular,

$$\|y_k\|_\infty \leq y_k(0) = 1,$$

$dm(s) = \rho^2(s)\, ds$ is the Haar measure in the sense that $m * \delta_t = m$ for all t in $[0,\pi]$, and δ_0 is the identity because $\delta_0 * \delta_t = \delta_t$ for all t in $[0,\pi]$.

There is a convolution of functions defined by

$$(f * g)(s) = \int_0^\pi T^t f(s) g(t) \rho^2(t)\, dt$$

which has the usual properties. In particular, let L^p consist of those f that are measurable on $[0,\pi]$ for which

$$\|f\|_p = \left(\int_0^\pi |f(s)|^p \rho^2(s)\, ds \right)^{1/p} < \infty,$$

and let L^∞ and $\|f\|_\infty$ have the usual meaning. Then

$$(f * g)^\wedge_k = \hat{f}_k \hat{g}_k,$$
$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1,$$
$$\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty,$$

or more generally,

$$(2) \qquad \|f * g\|_r \leq \|f\|_p \|g\|_q \qquad \left(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, \qquad 1 \leq r,p,q \leq \infty \right);$$

if $r = \infty$, then $f * g$ is equal almost everywhere to a continuous function; $f * g$ is non-negative if f and g are. The Riesz Representation Theorem, together with (2) for $r = \infty$, shows that T^t operates on L^p and $\|T^t f\|_p \leq \|f\|_p$ for

$1 \le p \le \infty$. At this point it is possible to obtain some results analogous to those of classical Fourier analysis (see [5] for example).

Perturbed ultraspherical form. In [4] product formulas are obtained for the eigenfunctions of the perturbed ultraspherical Sturm–Liouville problem (U): The following assumptions are made on q and γ:

U1. $\gamma > 0$ and $(\sin s)q(s)$ is continuous on $[0,\pi]$ and real-analytic at 0 and π.

U2. $q(\pi - s) = q(s)$.

In order to establish positivity for the product formula, we also need

U3. q is decreasing on $(0, \pi/2)$.

Analogs to Lemma 1(i), (iii)–(v), and (viii) are obtained in [4, Lemma 1.1 and the discussion preceding it]. Let u_k denote the eigenfunction with k zeros in $(0,\pi)$ normalized so that $u_k(0) = 1$. We then obtain the following

THEOREM 3 (cf. [4, Theorem 1.2]). *For each k and for all s and t belonging to* $[0,\pi]$

$$\int_0^\pi u_k(r)D(r,s,t)\,dr = u_k(s)u_k(t).$$

D is a non-negative continuous function supported on $(|s-t|, \pi - |s+t-\pi|)$, and there is a constant A such that for all s and t belonging to $[0,\pi]$

$$\int_0^\pi D(r,s,t)\,dr \le A.$$

A result for a closely related hypergroup is obtained in terms of the functions

$$R_k = \frac{u_k}{u_0} \quad \text{and} \quad K(r,s,t) = \frac{D(r,s,t)}{u_0(r)u_0(s)u_0(t)(\sin^{2\gamma} r)},$$

and the measure $dm(r) = [u_0(r)]^2(\sin^{2\gamma} r)\,dr$:

THEOREM 4 ([4, Theorem 4.3]). *For all k and all s and t in* $[0,\pi]$

$$\int_0^\pi K(r,s,t)R_k(r)\,dm(r) = R_k(s)R_k(t).$$

$K(r,s,t)$ is symmetric in all three variables; it is continuous, non-negative, and supported on $(|s-t|, \pi - |s+t-\pi|)$. For all s and t belonging to $[0,\pi]$

$$\int_0^\pi K(r,s,t)\,dm(r) = 1.$$

In particular, the R_k are characters for a Jacobi type $(\gamma - 1/2, \gamma - 1/2)$ hypergroup with Haar measure $dm(s)$.

Liouville normal form. Many studies of ordinary differential equations, and of the Sturm-Liouville problem in particular, revolve around the Liouville normal form of the problem (L). We assume that Q satisfies:

L1. $Q(s) = b(\csc^2 s) + h(s)$ where $b \geq -1/4$, and $(\sin s)h(s)$ is real-analytic at 0 and π and continuous on $(0, \pi)$.

L2. $Q(\pi - s) = Q(s)$.

L3. Let γ be the larger solution of $\gamma(\gamma - 1) = b$; then $Q \in C^{p-2}$ where p is as in S3.

If we substitute $w(s) = (\sin^\gamma s)u(s)$ in the differential equation (L), we obtain the differential equation in (U) with $q(s) = Q(s) - b(\csc^2 s)$ and $\lambda = \nu - \gamma^2$. Thus q satisfies U1 and U2 so the resulting problem (U) satisfies the analogs to Lemma 1(i), (iii)–(v), and (viii). Hence if $\{\lambda_k\}$ are the eigenvalues of (U) arranged in increasing order and $\{u_k\}$ are the corresponding eigenfunctions, then the eigenvalues of (L) are $\nu_k = \lambda_k + \gamma^2$ with corresponding eigenfunctions $(\sin^\gamma s)u_k(s)$. If

$$\rho(s) = c(\sin^\gamma s)u_0(s) \tag{3}$$

where c is chosen so that $\int_0^\pi \rho^2(s)\,ds = 1$, and if $\{\mu_k\}$ and $\{y_k\}$ are the eigenvalues and eigenfunctions of the corresponding problem (S) (normalized so that $y_k(0) = 1$), then $w_k = \rho y_k$ is an eigenfunction of (L) corresponding to the eigenvalue $\nu_k = \mu_k + \nu_0$.

In order to obtain a product formula for this system of eigenfunctions we must introduce an additional condition on Q so that ρ will satisfy S4:

L4. One of the following hold:

(i) $b \geq 0$, $Q(s) \geq Q(t)$ for $0 < s < t < \pi/2$.

(ii) $b \geq 0$, $Q(s) \geq Q(t)$ for $0 < s < t < s_0$, and $Q(s) \leq \nu_0$ for $s_0 \leq s \leq \pi/2$ where s_0 is the smallest positive solution of $Q(s) = \nu_0$.

(iii) $b \leq 0$, and $Q(s) \leq \nu_0$ for $0 < s < \pi$.

(iv) $(\ln w_0)'' \leq 0$ on $(0, \pi)$.

THEOREM 5. *If Q satisfies* L1–L4, *then there is a product formula*

$$w_k(s)w_k(t) = \int_0^\pi w_k(r)\,d\tau_{s,t}(r)$$

where, if ρ is defined by (3),

$$d\tau_{s,t}(r) = \frac{\rho(s)\rho(t)}{\rho(r)}\,d\sigma^\rho_{s,t}(r).$$

Proof. The theorem will follow by a simple calculation once we show that any of L4(i)–(iii) implies L4(iv); since $\rho = w_0$, this leads directly to S4. We do this first assuming L4(ii).

We begin by showing that

$$\rho'(s) > 0 \quad (s \in (0, s_0)). \tag{4}$$

Choose $s \in (0, s_0)$. Then by two invocations of the Mean Value Theorem there is $s_1 \in (0, s)$ such that $\rho'(s_1) = [\rho(s) - \rho(0)]/s > 0$, and there is $s_2 \in (s_1, s)$ such that

$$\rho'(s) = \rho'(s_1) + (s - s_1)\rho''(s_2) = \rho'(s_1) + (s - s_1)(Q - \nu_0)\rho(s_2)$$

since $\rho = w_0$. Finally, L4 yields (4).

Now, let $v = \ln \rho$. Then S4 will follow if we can show $v''(s) \leq 0$ for $0 < s < \pi/2$. Substitution of $\rho = e^v$ in the differential equation (L) yields

$$v'' + (v')^2 + \nu_0 - Q = 0, \tag{5}$$

so by L4, $v''(s) \leq 0$ for $s_0 \leq s \leq \pi/2$. By (3), $v'(s) = u_0'(s)/u_0(s) + \gamma \cot s$, so $v''(s) < 0$ for s in some interval $(0, \varepsilon)$.

We claim $v''(s) \leq 0$ for $0 < s < \pi/2$. Assume by way of contradiction that

$$v''(s) > 0 \quad \text{for some } s \in (0, \pi/2). \tag{6}$$

Then there are s_1 and s_2, $0 < s_1 < s_2 \leq s_0$, such that $v''(s_1) = v''(s_2) = 0$ and

$$v''(s) > 0 \quad \text{for } s_1 < s < s_2. \tag{7}$$

If now equation (5) is evaluated at $s = s_1$ and $s = s_2$, and the difference taken, we obtain

$$[v'(s_2)]^2 - [v'(s_1)]^2 = Q(s_2) - Q(s_1).$$

Now the Mean Value Theorem yields $s_3 \in (s_1, s_2)$ such that

$$2v'(s_3)v''(s_3)(s_2 - s_1) = Q(s_2) - Q(s_1) \tag{8}$$

and L4 implies that the right side of (8) is negative, but $v'(s_3) > 0$ by (4) and $v''(s_3) > 0$ by (7), hence (6) is false.

The other assumptions in L4 lead to S4 since (i) implies (ii), and if (iii) is assumed, (iv) follows directly from (5). ∎

Comparisons of the results. The problems (S), (L), and (U) can be compared by using appropriate transformations. That is, suppose that ρ satisfies S1–S3, q satisfies U1 and U2, and Q satisfies L1–L3. Then the problems are equivalent if

$$Q(s) = b(\csc^2 s) + q(s) = \frac{\rho''(s)}{\rho(s)} + \nu_0 \tag{9}$$

and the eigenfunctions are related by

$$w_k(s) = g(0)(\sin^\gamma s)u_k(s) = \rho(s)y_k(s).$$

Thus each of Theorems 1, 3, and 5 yields product formulas for all three systems of eigenfunctions.

1. Theorems 3 and 5 are not logically comparable; for example, the fact that q has at most a first order singularity at 0 shows that it is possible for U3 to fail while L4 holds even in the strong sense of Q being a decreasing function on $(0, \pi/2)$.

2. It is shown in [4, Theorem 1.2] that even if U3 fails the conclusions of Theorems 3 and 4 will hold except for the positivity of the kernel. This observation yields the following improvements on the results obtained above.

THEOREM 6. *Theorems* 3 *and* 4 *hold if* U3 *is replaced by*

U3′. $Q(s) = \gamma(\gamma - 1)(\csc^2 s) + q(s)$ is decreasing on $(0, \pi/2)$

(*or even the more general assumption that* $Q(s)$ *satisfies* L4).

THEOREM 7. *If* $\gamma > 0$, *then Theorems* 1 *and* 5 *hold with*

$$d\sigma_{s,t}(r) = D(r, s, t)\, dr$$

where $D(r, s, t)$ *is a non-negative continuous function on* $(|s - t|, \pi - |s + t - \pi|)$.

The reader is referred to [4, Sect. 4] for a discussion of other properties of the resulting hypergroups.

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